Записки научных
семинаров ПОМИ
Том 517, 2022 г.

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# COMPARING CLASSICALITY OF QUTRITS FROM HILBERT–SCHMIDT, BURES AND BOGOLIUBOV–KUBO–MORI ENSEMBLES

Abstract. In the report we analyze the indicator/measure of classicality of quantum states defined as the probability to find a state with a positive Wigner function within a unitary invariant random ensemble. The indicators of classicality of three ensembles associated with the Hilbert–Schmidt, Bures and Bogoliubov–Kubo–Mori metrics on the space of quantum states of 3-level system are computed. Their dependence on a moduli parameter of the Wigner function is studied for all strata of a qutrit state space stratified in accordance with the unitary group action.

## §1. Introduction

It is natural to expect that some states of a quantum system are more "quantum" than the others. To transform this intuitive thought into a qualitative concept, we use the conventional statistical interpretation of quantum mechanics. The quasiprobability distribution functions will be regarded as a source of information about the classicality/quantumness of a state. Our consideration is based on the ideas borrowed from the geometric probability theory [1] and a commonly accepted opinion that if quasiprobability functions attain negative values, then it is a certain sign of quantum nature (see [2–5] and [6] with references therein). This observation allows one to specify the notion of "classical states Γ la Wigner" as the states whose Wigner function is positive semidefinite everywhere in the phase space. Based on this definition, several measures of classicality/quantumness have been constructed [7–13]. When dealing with an ensemble of random states, the probability to find a "classical state" among the members of an ensemble is an example of these kind of measures [14–16].

*Key words and phrases*: Wigner function, quasiprobability distribution, state nonclassicality, classicality indicator.

The work of A.K. was supported in part by the Shota Rustaveli National Science Foundation of Georgia, Grant FR-19-034.

In the present article, after the introduction of classicality indicator $\mathcal{Q}$ as the geometric probability, we will compute it for a 3-level quantum system, a qutrit. We will compare the characteristics of classicality of qutrits from three random ensembles: the Hilbert–Schmidt and two other ensembles, associated with the monotone Riemannian metrics – the Bures and the Bogoliubov–Kubo–Mori metrics (cf. [17–19]). To make the presentation self-consistent, in the next sections necessary notions and definitions related to these random ensembles and the Wigner function of a finite-dimensional quantum system will prelude calculations of the corresponding probabilities. Calculating the probabilities for different varieties of states, we analyze the dependence of the classicality measure on the moduli parameter of a qutrit Wigner function.

## §2. Unitary invariant ensembles of qudits

Let us consider a qudit – a quantum system associated with an $N$-dimensional Hilbert space. The quantum state space $\mathfrak{P}_N$ of an $N$-level qudit is defined as:

$$\mathfrak{P}_N = \{\varrho \in M_N(\mathbb{C}) \mid \varrho = \varrho^\dagger,\ \varrho \geqslant 0,\ \operatorname{tr}(\varrho) = 1\}. \tag{1}$$

The unitary $U(N)$ automorphism of the Hilbert space of an $N$-level quantum system induces the adjoint $SU(N)$ transformations of density matrices $\varrho \in \mathfrak{P}_N$:

$$g \cdot \varrho = g\varrho g^\dagger, \qquad g \in SU(N). \tag{2}$$

For a closed system it is assumed that the probability density function of the corresponding ensemble of $N$- dimensional qudits is invariant under (2):

$$P(\varrho) = P(g\varrho g^\dagger), \qquad \forall\ g \in SU(N). \tag{3}$$

Further in the report three ensembles of random states respecting this unitary symmetry will be used for evaluation of the measure of classicality. Namely, we will consider the unitary invariant ensembles associated with the following Riemannian metrics on state space:

- the Hilbert–Schmidt metric $\mathrm{g}_{\mathrm{HS}}$;
- the Bures metric $\mathrm{g}_{\mathrm{B}}$;
- the Bogoliubov–Kubo–Mori metric $\mathrm{g}_{\mathrm{BKM}}$.

Before dealing with a specific ensemble, it is worth drawing attention to a common property of each of these ensembles emerging due to $SU(N)$ invariance (3).

**Stratification and factorization of probability distribution on $\mathfrak{P}_N$.** The invariance property (3) leads to a certain factorization of the probability distribution functions $P(\varrho)$ into two factors, one depending on $SU(N)$-invariants solely, and the other being a universal function of the "angular variables". Moreover, the structure of this factorization is universal for all states whose unitary orbits are characterised by the same isotropy group, $H_\alpha \subset SU(N)$, i.e., belong to a class with the same "orbit type"[1]. Isotropy groups $H_\varrho$ of any point $\varrho \in \mathfrak{P}_N$ are determined by the algebraic degeneracy of the spectrum of $\varrho$ and are in one-to-one correspondence with the Young diagrams of all possible decompositions of $N$ into non-negative integers. Hence, we associate the given partition of $N$ with the *stratum* $\mathfrak{P}_{[H_\alpha]}$, defined as the set of all points of $\mathfrak{P}_N$, whose stabilizer is conjugate to subgroup $H_\alpha$:

$$\mathfrak{P}_{[H_\alpha]} := \big\{x \in \mathfrak{P}_N \,|\, H_x \text{ is conjugate to } H_\alpha\big\}, \tag{4}$$

where $\alpha = 1, 2, \ldots, p(N)$.[2] The union of $\mathfrak{P}_{[H_\alpha]}$ results in the state space $\mathfrak{P}_N$:

$$\mathfrak{P}_N = \bigcup_{\text{orbit types}} \mathfrak{P}_{[H_\alpha]}, \tag{5}$$

with each component of the decomposition (5) consisting of density matrices with a fixed algebraic degeneracy,

$$\mathfrak{P}_{[H_\alpha]} = \bigcup_{\omega \in S_s} \mathfrak{P}_{k_{\omega(1)}, k_{\omega(2)}, \ldots, k_{\omega(s)}}. \tag{6}$$

In (6) $S_s$ is a symmetric group acting on a given partition of $N$ into $s$ natural numbers $k_1, k_2, \ldots, k_s$. Algebraically, $\mathfrak{P}_{k_1,k_2,\ldots,k_s}$ being a set of states with a fixed degeneracy is defined via the characteristic polynomial

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[1]Subgroup $H_x \subset SU(N)$ is the *isotropy group* (*stabilizer*) of point $x \in \mathfrak{P}_N$ and is defined as

$$H_x = \{g \in SU(N) \mid g \cdot x = x\}.$$

If the conjugacy class of $H$ is denoted by $[H]$, then we say that *the type of the orbit is* $[H]$, if the stabilizer $H_x$ of some/any point $x$ in the orbit belongs to $[H]$.

[2]The partition function $p(N)$ gives a number of possible partitions of a non-negative integer $N$ into natural numbers.

of a density matrix:[3]

$$\mathfrak{P}_{k_1,k_2,\dots,k_s} = \left\{ \varrho \in \mathfrak{P}_N, k_i \in \mathbb{Z}_+ \,\middle|\, \det(\varrho - \lambda) = \prod_{i=1}^{s} (r_i - \lambda)^{k_i},\ \sum_{i=1}^{s} k_i = N \right\}. \quad (7)$$

Geometrically, the set $\mathfrak{P}_{k_1,k_2,\dots,k_s}$ with $k_1 = k_2 = \dots = k_N = 1$ represents the interior of an $(N-1)$-dimensional simplex $C_{N-1}$ of eigenvalues:

$$C_{N-1} := \left\{ \boldsymbol{r} \in \mathbb{R}^N \,\middle|\, \sum_{i=1}^{N} r_i = 1,\ 1 \geqslant r_1 \geqslant r_2 \geqslant \dots \geqslant r_{N-1} \geqslant r_N \geqslant 0 \right\}, \quad (8)$$

while for all other admissible tuples $\boldsymbol{k} = (k_1, k_2, \dots, k_s)$ each $\mathfrak{P}_{k_1,k_2,\dots,k_s}$ represents the union of the faces and edges of the $(N-1)$-simplex parameterized by the barycentric coordinates of the following kind:

$$\boldsymbol{r}^{\downarrow}(\varrho) = \{r_1 \overbrace{(1,\dots,1)}^{k_1};\ r_2 \overbrace{(1,\dots,1)}^{k_2};\ \dots;\ r_s \overbrace{(1,\dots,1)}^{k_s}\}. \quad (9)$$

Now, bearing in mind the above described stratification of $\mathfrak{P}_N$, it is easy to show the factorization of $SU(N)$-invariant measures. Indeed, one can be convinced that the Singular Value Decomposition (SVD) of the density matrix from a stratum $\mathfrak{P}_{[H_\alpha]}$ with spectrum of the form (9):

$$\varrho = U \operatorname{diag}(r_1, r_2, \dots r_s)\, U^\dagger, \qquad U \in SU(N)/H_\alpha, \quad (10)$$

reveals the following factorization of the invariant probability distribution (3):

$$\mathrm{P}(\varrho) = P(r_1, \dots, r_s)\, \mathrm{d}r_1 \wedge \dots \wedge \mathrm{d}r_N \wedge \mathrm{d}\mu_{U(N)/H}, \quad (11)$$

where the first factor $P(r_1, \dots, r_s)$ represents a measure on subset $\mathfrak{P}_{k_1,k_2,\dots,k_s}$ of the simplex $\mathcal{C}_{N-1}$, while the second factor is the measure on coset $U(N)/H$.

After a preliminary exposition of this generic property of unitary invariant ensembles, we will now specify the form of the distribution $P(r_1, \dots, r_N)$ for the Hilbert–Schmidt metric and for an important class of the monotone metrics.

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[3]Note that in (7) the condition of summing up the degrees of degeneracy to $N$ means that only the maximal rank states are considered.

**The Hilbert–Schmidt ensemble of qudits.** Let us consider the metric corresponding to the distance between two infinitesimally close matrices $\varrho - \mathrm{d}\varrho$ and $\varrho + \mathrm{d}\varrho$ calculated with respect to the Frobenius norm,

$$\mathrm{g}_{\mathrm{HS}} \propto \mathrm{Tr}\,(\mathrm{d}\varrho \otimes \mathrm{d}\varrho)\,. \tag{12}$$

If a density matrix belongs to the interior of the simplex $C_{N-1}$, i.e., the matrix has $N$ distinct non-zero eigenvalues ($k_1 = k_2 = \cdots = k_N = 1$), then the metric (12) defines the standard *Hilbert–Schmidt ensemble* of random full rank $N$-qudits. A straightforward computation shows that the joint probability distribution of eigenvalues reads

$$P^{\mathrm{HS}}(r_1, \dots, r_N) \propto \delta(1 - \sum_{j=1}^{N} r_j) \prod_{j<k}^{N} (r_j - r_k)^2, \tag{13}$$

and unitary random factors $U$ in SVD decomposition are distributed according to the Haar measure on the coset $U(N)/U(1)^N$.

**Degenerate Hilbert–Schmidt qudits.** If the full rank density matrix has a spectrum of the form (9) with an arbitrary algebraic degeneracy, then the joint probability distribution of eigenvalues is reduced to the following expression:

$$P^{\mathrm{HS}}_{k_1,\dots,k_s}(r_1, \dots, r_s) \propto \delta(1 - \sum_{i=1}^{s} k_i r_i) \prod_{i<j}^{1\dots s} (r_i - r_j)^{2k_i k_j}\,. \tag{14}$$

At the same time the angles in the SVD are distributed according to the Haar measure on the coset $U(N)/U(k_1) \times \cdots \times U(k_s)$.

**Monotone metrics and monotone ensembles of $N$-qudits.** Two of the above-mentioned metrics, the Bures and Bogoliubov–Kubo–Mori ones, are members of a special class of unitary covariant monotone metrics. According to [18], any monotone metric can be written as[4]

$$\mathrm{g}_{D}(X, Y) \propto \mathrm{Tr}\left(X K_D^{-1} Y\right), \tag{15}$$

where $K_D$ is an operator,

$$K_D = R_D^{1/2} f(L_D R_D^{-1}) R_D^{1/2},$$

[4]Due to the unitary covariance of the Riemannian metric, it is sufficient to describe monotone metrics evaluated for diagonal matrices $D$.

constructed out of the left and right multiplication operators $L_D$ and $R_D$, i.e., $L_D X = DX$ and $R_D X = XD$ for $X \in M_N(\mathbb{C})$, and operator monotone function $f$, which is symmetric, i.e., $f(t) = tf(1/t)$, and normalized, $f(1) = 1$.

Assuming that $\varrho$ is a full rank density matrix with a simple spectrum and using the 1-form coordinate basis for simplex, $\mathrm{d}r_i$, and non-coordinate basis on $SU(N)$ group, $w_{ij} := \left(U^\dagger \mathrm{d}U\right)_{ij}$, the non-degenerate monotone metrics can be written as

$$\mathrm{g}_f = \frac{1}{4}\sum_{i=1}^{N} \frac{\mathrm{d}r_i \otimes \mathrm{d}r_i}{r_i} + \frac{1}{2}\sum_{i<j}^{N} c_f(r_i, r_j)(r_i - r_j)^2\, \omega_{ij} \otimes \omega_{ij}. \tag{16}$$

In (16) by $c_f(x, y) = \frac{1}{yf(x/y)}$ we denote the Morozova-Chentsov function corresponding to a monotone function $f(t)$. Note that the Bures and Bogoliubov–Kubo–Mori metrics are associated with the following choice of monotone function:

$$f_{\mathrm{B}}(t) = \frac{1+t}{2}, \qquad f_{\mathrm{BKM}}(t) = \frac{t-1}{\ln t}, \tag{17}$$

and corresponding Morozova-Chentsov functions,

$$c_{\mathrm{B}}(x, y) = \frac{2}{x+y}, \qquad c_{\mathrm{BKM}}(x, y) = \frac{\ln x - \ln y}{x - y}. \tag{18}$$

**Probability measures from monotone metrics.** For an arbitrary monotone metric evaluated for degenerate qudits, calculations of the joint probability distribution of eigenvalues give:

$$P^f_{k_1,\dots,k_s}(r_1, \dots, r_s) \propto \frac{\delta\left(1 - \sum\limits_{i=1}^{s} k_i r_i\right)}{\sqrt{r_1 \cdot r_2 \cdot \dots \cdot r_s}} \prod_{i<j}^{s} c_f^{k_i k_j}(r_i, r_j)\,(r_i - r_j)^{2k_i k_j}. \tag{19}$$

## §3. Wigner function positivity and classicality

Here, for the reader's convenience, before giving a definition of the indicator of classicality, we present the basic settings of the Wigner function of a mixed state of a finite-dimensional quantum system.

**Wigner function settings.** The Wigner quasiprobability distribution $W^{(\boldsymbol{\nu})}_{\varrho}(\Omega_N)$ of an $N$-level qudit is constructed via dual pairing [20, 21],

$$W^{(\boldsymbol{\nu})}_{\varrho}(\Omega_N) = \operatorname{tr}\left[\varrho\, \Delta(\Omega_N \mid \boldsymbol{\nu})\right], \tag{20}$$

of a density matrix $\varrho$ with the Stratonovich-Weyl (SW) kernel $\Delta(\Omega_N \mid \boldsymbol{\nu}) \in \mathfrak{P}^*_N$ from the dual space:

$$\mathfrak{P}^*_N = \{X \in M_N(\mathbb{C}) \mid X = X^\dagger, \quad \operatorname{tr}(X) = 1, \quad \operatorname{tr}\left(X^2\right) = N\}. \tag{21}$$

For $N \geqslant 3$, algebraic equations (21) admit a family of solutions. As a result, the generic Wigner function depends on $N-2$ real parameters $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots, \nu_{N-2})$, (see details in [21]). The structure of phase space $\Omega_N$ depends on the isotropy group of the SW kernel. For any given isotropy group $H \in U(N)$ of the form

$$H = U(k_1) \times U(k_2) \times \cdots \times U(k_{s+1}),$$

we identify the phase-space $\Omega_N$ with the complex flag manifold,

$$\Omega_N \to \mathbb{F}^N_{d_1, d_2, \dots, d_s} = U(N)/H,$$

where $(d_1, d_2, \dots, d_s)$ is a sequence of positive integers with a sum $N$, such that $k_1 = d_1$ and $k_{i+1} = d_{i+1} - d_i$ with $d_{s+1} = N$. After presenting necessary notions, we are ready to introduce the definition of the classicality of states.

**Classical states and classicality indicator.** The "classical states" form the subset $\mathfrak{P}^{(+)}_N \subset \mathfrak{P}_N$ of states whose Wigner function is non-negative everywhere over the phase space:

$$\mathfrak{P}^{(+)}_N = \{\varrho \in \mathfrak{P}_N \mid W_\varrho(z) \geqslant 0, \quad \forall z \in \Omega_N\}, \tag{22}$$

and similarly, the "classical states on a fixed stratum" $\mathfrak{P}_{H_\alpha}$ are defined as:

$$\mathfrak{P}^{(+)}_{H_\alpha} = \mathfrak{P}^{(+)}_N \cap \mathfrak{P}_{H_\alpha}. \tag{23}$$

Based on (22), we define the geometric probability of finding a classical state in an ensemble as

$$\mathcal{Q}_N = \frac{\text{Volume(Classical States)}}{\text{Volume(All States)}}. \tag{24}$$

Here it is assumed that the Riemannian volume is calculated with respect to the measure dictated by the probability distribution function of an ensemble. Probability (24) shall be considered as the *global indicator of* *classicality*. Moreover, for classical states (22) on the fixed stratum $\mathfrak{P}_{H_\alpha}$ we define the *$Q$-indicator of classicality of the stratum*:

$$\mathcal{Q}_N[H_\alpha] = \frac{\text{Volume(Classical States on } \mathfrak{P}_{[\mathrm{H}_\alpha]})}{\text{Volume(All States on } \mathfrak{P}_{[\mathrm{H}_\alpha]})}. \tag{25}$$

According to (6), stratum $\mathfrak{P}_{[H_\alpha]}$ consists from subsets of matrices with a certain degeneracy type. Hence, owing to the unitary covariance of probability distribution functions (3), the $\mathcal{Q}$-indicator depends only on the joint probability distribution of eigenvalues of the density matrix and can be rewritten as:

$$\mathcal{Q}_N[H_\alpha] = \frac{\sum\limits_{\omega\in S_s} \int\limits_{\mathcal{C}^*_{N-1}(H_\alpha)} P^f_{k_\omega(1),\dots,k_\omega(s)}(r_1,\dots,r_s)\,\mathrm{d}r_1\wedge\cdots\wedge\mathrm{d}r_s}{\sum\limits_{\omega\in S_s} \int\limits_{\mathcal{C}_{N-1}(H_\alpha)} P^f_{k_\omega(1),\dots,k_\omega(s)}(r_1,\dots,r_s)\,\mathrm{d}r_1\wedge\cdots\wedge\mathrm{d}r_s}. \tag{26}$$

In (26) the integral in the denominator represents the volume of the orbit space of stratum $\mathfrak{P}_{[H_\alpha]}$. The subset $\mathcal{C}_{N-1}(H_\alpha)$ is a union of faces of the simplex $C_{N-1}$ determined by the isotropy group $[H_\alpha]$. The integration in the nominator of (26) is over the image of $\mathfrak{P}^+_{[H_\alpha]}$ under the canonical quotient map:

$$\mathcal{C}^*_{N-1}(H_\alpha) = \Big\{p(x) \ \mid x \in \mathfrak{P}^{(+)}_{H_\alpha}\Big\}. \tag{27}$$

According to [21], the subset (27) can be identified with a certain cone in $\mathbb{R}^{N-1}$. Having denoted by $\boldsymbol{r} = \{r_1, r_2, \dots, r_N\}$ the eigenvalues of the density matrix $\varrho$ and by $\boldsymbol{\pi} = \{\pi_1, \pi_2, \dots, \pi_N\}$ the eigenvalues of the SW kernel, both arranged in decreasing order, we obtain that $\mathcal{C}^*_{N-1}(H_\alpha)$ is the following dual cone:

$$\mathcal{C}^*_{N-1}(H_\alpha) = \big\{\boldsymbol{\pi} \in \mathbf{spec}\,(\Delta(\Omega_N)) \mid (\boldsymbol{r}^{\downarrow}, \boldsymbol{\pi}^{\uparrow}) \geqslant 0,\ \forall\, \boldsymbol{r} \in \mathcal{C}_{N-1}(H_\alpha)\big\}\,, \tag{28}$$

where $(\boldsymbol{r}^{\downarrow}, \boldsymbol{\pi}^{\uparrow}) = r_1\pi_N + r_2\pi_{N-1} + \cdots + r_N\pi_1$.

## §4. Examples

In this section results of the calculations of classicality indicators for the random ensembles described in Section 2 will be given for qubit and qutrit cases.

**4.1. $N = 2$, Qubit.** For a single qubit the expansion coefficients of the density matrix over the Pauli $\boldsymbol{\sigma}$-matrices are given by components of the 3-dimensional Bloch vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$:

$$\varrho = \frac{1}{2}\left(I + (\boldsymbol{\xi}, \boldsymbol{\sigma})\right). \tag{29}$$

Expressing the eigenvalues of (29) in terms of the length $r \in (0, 1]$ of the Bloch vector

$$r_1 = \frac{1+r}{2}, \quad r_2 = \frac{1-r}{2}, \tag{30}$$

and taking into account that the Wigner function of a qubit is uniquely constructed with the aid of the SW kernel, whose spectrum is:

$$\pi_1 = \frac{1+\sqrt{3}}{2}, \qquad \pi_2 = \frac{1-\sqrt{3}}{2}, \tag{31}$$

we conclude that the Wigner function of a qubit is positive definite inside the Bloch ball of radius $1/\sqrt{3}$. Since all states, except the maximally mixed state $r_1 = r_2 = 1/2$, have the torus $T^2 \in SU(2)$ as their isotropy group, there is only one indicator $\mathcal{Q}_{[T^2]}$ for all possible ensembles of qubits. Hence, for any random ensemble of qubits, characterized by a probability distribution $P(r)$, the expression of the classicality indicator is reduced to the following ratio:

$$\mathcal{Q}_{[T^2]} = \frac{\int_0^{\frac{1}{\sqrt{3}}} P(r)\mathrm{d}r}{\int_0^1 P(r)\mathrm{d}r}. \tag{32}$$

**Hilbert–Schmidt ensemble.** Noting that for the Hilbert–Schmidt ensemble the probability distribution function is $P^{\mathrm{HS}}(r) \propto r^2$, the calculation of (32) gives:

$$\mathcal{Q}^{\mathrm{HS}}_{[T^2]} = \frac{1}{3\sqrt{3}} \approx 0.19245. \tag{33}$$

**Bures ensemble.** Using the probability distribution function of the Bures ensemble $P^{\mathrm{B}}(r) \propto \frac{r^2}{\sqrt{1-r^2}}$, we find the classicality indicator:

$$\mathcal{Q}^{\mathrm{B}}_{[T^2]} = \frac{2}{\pi}\left(\arcsin\frac{1}{\sqrt{3}} - \frac{\sqrt{2}}{3}\right) \approx 0.0917211. \tag{34}$$

**Bogoliubov–Kubo–Mori ensemble.** Analogously, calculations with the Bogoliubov–Kubo–Mori measure $P^{\mathrm{BKM}}(r) \propto \frac{r\left(\ln\frac{1+r}{2} - \ln\frac{1-r}{2}\right)}{\sqrt{1-r^2}}$ result in the classicality indicator:

$$\mathcal{Q}^{\mathrm{BKM}}_{[T^2]} = \frac{2}{\pi}\left(\arcsin\frac{1}{\sqrt{3}} - \sqrt{\frac{2}{3}}\,\mathrm{arcoth}\,\sqrt{3}\right) \approx 0.0495506. \tag{35}$$

**4.2. $\boldsymbol{N = 3}$, Qutrit.** Qutrit state space $\mathfrak{P}_3$ admits the following orbit type decomposition:

$$\mathfrak{P}_3 = \mathfrak{P}_{[T^3]} \bigcup \mathfrak{P}_{[S(U(2)\times U(1))]} \bigcup \mathfrak{P}_{[SU(3)]}. \tag{36}$$

Three strata in (36) are labeled by the isotropy group or directly by the degeneracy of the density matrices. For full rank states, putting eigenvalues of $\varrho$ in decreasing order, $1 > r_1 > r_2 > r_3 > 0$, the components of decomposition (36) are described as follows (see geometrical illustration in Fig. 1):

(1) the regular stratum $\mathfrak{P}_{[T^3]}$ of maximal dimension 6 consists of matrices with a simple spectrum, $1 > r_1 \neq r_2 \neq r_3 > 0$. The corresponding orbit space is the face $F_{123}$ of the ordered 2-simplex, the interior of $\triangle AOB$,

(2) the degenerate 4-dimensional stratum $\mathfrak{P}_{[S(U(2)\times U(1))]}$ with density matrices whose degeneracy is $\boldsymbol{k} = (2,1)$ and $\boldsymbol{k} = (1,2)$, i.e., $1 > r_1 \neq r_2 = r_3 > 0$ and $1 > r_1 = r_2 \neq r_3 > 0$. The corresponding orbit space represents the union of edges $F_{1|23}$ and $F_{12|3}$ of the 2-simplex, two sides of $\triangle AOB$,

(3) the 0-dimensional stratum $\mathfrak{P}_{[SU(3)]}$ of the maximally mixed state with the triple degeneracy $\boldsymbol{k} = (3)$, $r_1 = r_2 = r_3 = 1/3$.

Taking into account the decreasing order of the eigenvalues, $1 \geqslant r_1 \geqslant r_2 \geqslant r_3 \geqslant 0$, the spectrum of qutrit admits the following parameterization:

$$r_1 = \frac{1}{3} - \frac{2r}{\sqrt{3}}\cos\left(\frac{\varphi + 2\pi}{3}\right), \tag{37}$$

$$r_2 = \frac{1}{3} - \frac{2r}{\sqrt{3}}\cos\left(\frac{\varphi + 4\pi}{3}\right), \tag{38}$$

$$r_3 = \frac{1}{3} - \frac{2r}{\sqrt{3}}\cos\left(\frac{\varphi}{3}\right), \tag{39}$$

with $r \in [0, 1/\sqrt{3}]$ and the angle $\varphi \in [0, \pi]$. If $r$ and $\varphi$ are treated as the polar coordinates on a plane, $(r\cos\varphi, r\sin\varphi)$, then geometrically the

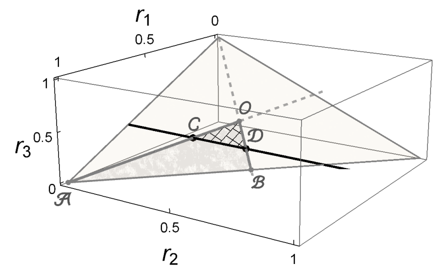



Figure 1. The ordered 2-simplex of qutrit eigenvalues is represented by $\triangle AOB$, and the hatched region, $\triangle COD$, corresponds to the classical states. Edges $AO/\{A\}$ and $BO/\{B\}$ are locus of degenerate states $F_{12|3}$ and $F_{1|23}$, while their parts $CO/\{O\}$ and $DO/\{O\}$ represent the degenerate classical states $F^+_{12|3}$ and $F^+_{1|23}$.

formulae (37)-(39) can be interpreted as a map between the ordered simplex $\mathcal{C}_2$ and the domain of the upper half-plane outlined by the Maclaurin trisectrix:

$$r(\varphi, 1/\sqrt{3}) = \frac{1}{2\sqrt{3}\cos(\varphi/3)}. \tag{40}$$

More precisely, under transformations (37)-(39), the ordered simplex of eigenvalues $\mathcal{C}_2$ maps to the domain (see Fig. 2)

$$F_{123} =: \left\{ r \geqslant 0, \varphi \in [0, \pi] \;\middle|\; \cos\left(\frac{\varphi}{3}\right) \leqslant \frac{1}{2\sqrt{3}r} \right\}. \tag{41}$$

**Wigner function of a qutrit.** The master equations (21) for eigenvalues of the Stratonovich-Weyl kernel of a qutrit,

$$\pi_1 + \pi_2 + \pi_3 = 1, \qquad \pi_1^2 + \pi_2^2 + \pi_3^2 = 3, \tag{42}$$

define a one-parametric family of the Wigner functions. Due to the permutation symmetry of (42), the corresponding moduli space is a unit circle factorised by the symmetric group $S_3$. Let $\mu_3$ and $\mu_8$ be Cartesian coordinates of this arc with a polar angle from the interval $\zeta \in [0, \frac{\pi}{3}]$,

$$\mu_3 = \sin\zeta, \qquad \mu_8 = \cos\zeta, \tag{43}$$

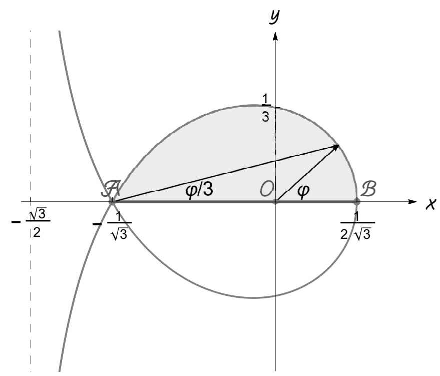



Figure 2. The image of the ordered simplex $\mathcal{C}_2$ on the plane $x = r\cos\varphi$, $y = r\sin\varphi$ under the mapping (37)–(39).

then, providing the decreasing order of the SW kernel eigenvalues, $\pi_1 \geqslant \pi_2 \geqslant \pi_3$, one can represent the whole class of solutions to (42) as:

$$\pi_1 = \frac{1}{3} + \frac{2}{\sqrt{3}}\,\mu_3 + \frac{2}{3}\,\mu_8, \quad \pi_2 = \frac{1}{3} - \frac{2}{\sqrt{3}}\,\mu_3 + \frac{2}{3}\,\mu_8, \quad \pi_3 = \frac{1}{3} - \frac{4}{3}\,\mu_8. \tag{44}$$

**Classical states of qutrit.** The image of classical states from the regular stratum $\mathfrak{P}_{[T^3]}$ to the unitary orbit space is the interior $F^+_{123}$ of a cone which is cut out from the simplex $\mathcal{C}_2$ by the line (see Fig. 1)

$$L_{\boldsymbol{\pi}}(\boldsymbol{r}) : \quad r_1\pi_3 + r_2\pi_2 + r_3\pi_1 = 0, \tag{45}$$

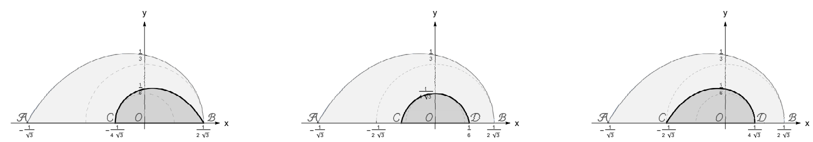
while the orbit space of classical states from the stratum $\mathfrak{P}_{S(U(2)\times U(1))}$ consists of two pieces: $F^+_{1|23}$ and $F^+_{12|3}$, corresponding to the matrices of degeneracy types $(2,1)$ and $(1,2)$ respectively. Using the polar form of parameterization of the spectrum of a density matrix (37)-(39) and expressions (44) for the SW kernel eigenvalues, the cone of classical states on a regular stratum reads:

$$F^+_{123} : \left\{ r > 0, \varphi \in (0,\pi) \;\middle|\; \cos\left(\frac{\varphi}{3} + \zeta - \frac{\pi}{3}\right) \leqslant \frac{1}{4\sqrt{3}r} \right\}, \tag{46}$$

while the cone of classical states on the degenerate stratum $\mathfrak{P}^{(+)}_{[S(U(2)\times U(1))]}$ is:

$$F^{+}_{1|23} = \left\{ \varphi = 0,\ r \in (0, \frac{1}{2\sqrt{3}}) \,\middle|\, \cos\left(\zeta - \frac{\pi}{3}\right) < \frac{1}{4\sqrt{3}r} \right\}, \tag{47}$$

$$F^{+}_{12|3} = \left\{ \varphi = \pi,\ r \in (0, \frac{1}{\sqrt{3}}) \,\middle|\, \cos\left(\zeta\right) < \frac{1}{4\sqrt{3}r} \right\}. \tag{48}$$



$\zeta = 0$ $\quad$ $\zeta = \pi/6$ $\quad$ $\zeta = \pi/3$

Figure 3. The orbit space $F_{123}$ (the region enclosed by the outer solid curve) and its subspace $F^{+}_{123}$ (the region enclosed by the inner solid curve) for different values of the moduli parameter: $\zeta = 0, \pi/6, \pi/3$.

**$\mathcal{Q}_3$-indicator for Hilbert–Schmidt ensemble of qutrits from regular stratum.** The regular stratum $\mathfrak{P}_{[T^3]}$ consists of density matrices with a simple spectrum. The expression $\mathcal{Q}_{[T^3]}$ comprises the integrals over the face $F_{123}$ and its subset $F^{+}_{123}$:

$$\mathcal{Q}^{\mathrm{HS}}_{[T^3]} = \frac{\mathrm{vol}_{\mathrm{HS}}(F^{+}_{123})}{\mathrm{vol}_{\mathrm{HS}}(F_{123})}. \tag{49}$$

In (49) the expression $\mathrm{vol}_{\mathrm{HS}}(X)$ denotes the Riemannian integral over a region $X$ taken with the measure induced on $X \in \mathcal{C}_2$ from the Hilbert–Schmidt on $\mathfrak{P}_3$,

$$\mathrm{vol}_{\mathrm{HS}}(X) = \int\limits_X P^{\mathrm{HS}}_{1,1,1}(r_1, r_2, r_3)\, \mathrm{d}r_1 \wedge \mathrm{d}r_2 \wedge \mathrm{d}r_3. \tag{50}$$

Taking into account the expression (12) for the Hilbert–Schmidt measure and the polar form of the parameterization of the qutrit orbit space (41) and of (46), we obtain the indicator of classicality as a function of the moduli parameter $\zeta$:

$$\mathcal{Q}^{\mathrm{HS}}_{[T^3]}(\zeta) = \frac{20\cos^2(\zeta - \pi/6) + 1}{128\,(4\cos^2(\zeta - \pi/6) - 1)^5}. \tag{51}$$

**$\mathcal{Q}_3$-indicator for Hilbert–Schmidt ensemble of qutrits from degenerate stratum.** The stratum $\mathfrak{P}_{[S(U(2)\times U(1))]}$ has two pieces, $F_{1|23}$ and $F_{12|3}$, associated with density matrices with degenerate eigenvalues $r_1 = r_2 \neq r_3$ and $r_1 \neq r_2 = r_3$, respectively. Hence, the $\mathcal{Q}_3$-indicator for the degenerate stratum of a qutrit reads:

$$\mathcal{Q}^{\mathrm{HS}}_{[S(U(2)\times U(1))]} = \frac{\mathrm{vol}_{\mathrm{HS}}(F^+_{1|23}) + \mathrm{vol}_{\mathrm{HS}}(F^+_{12|3})}{\mathrm{vol}_{\mathrm{HS}}(F_{1|23}) + \mathrm{vol}_{\mathrm{HS}}(F_{12|3})}, \tag{52}$$

where we keep the notation previously used for the regular stratum (50), noticing only that the dimension of integration over the degenerate orbit state strata has decreased by one:

$$\mathrm{vol}_{\mathrm{HS}}(F_{1|23}) = \int\limits_{F_{1|23}} P^{\mathrm{HS}}_{2,1}(r_1, r_2)\,\mathrm{d}r_1 \wedge \mathrm{d}r_2. \tag{53}$$

The evaluation of all integrals in (52) gives:

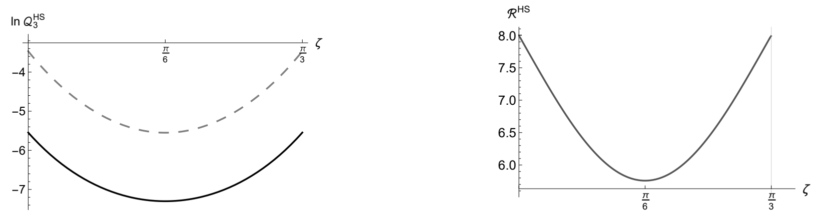
$$\mathcal{Q}^{\mathrm{HS}}_{[S(U(2)\times U(1))]}(\zeta) = \frac{1}{1056}\left(\csc^5\left(\zeta + \frac{\pi}{6}\right) + \sec^5(\zeta)\right). \tag{54}$$

The functional dependence of the indicator $\mathcal{Q}^{\mathrm{HS}}_3$ for the regular (51) and degenerate (54) strata is depicted in Fig. 4a. Apart from this, in Fig. 4b we present the ratio

$$\mathrm{R}^{\mathrm{HS}}(\zeta) = \frac{\mathcal{Q}^{\mathrm{HS}}_{[S(U(2)\times U(1))]}(\zeta)}{\mathcal{Q}^{\mathrm{HS}}_{[T^3]}(\zeta)} \tag{55}$$

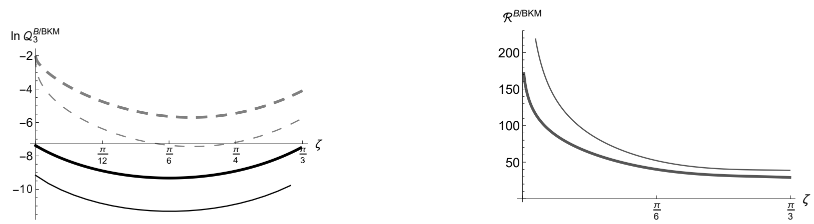
as a certain measure of the relation between the symmetry of a state and its classicality.

**$\mathcal{Q}_3$-indicator for Bures ensemble of qutrits from regular stratum.** Using the generic expressions for the joint probability distributions of eigenvalues for monotone metrics (19) and the technique developed above, we compute the $\mathcal{Q}_3$-indicators for the Bures and Bogoliubov–Kubo–Mori ensembles of qutrits. The results of our calculations are presented in Fig. 5a.




(a) (b)

Figure 4. (A) $\mathcal{Q}_3$-indicators of a Hilbert–Schmidt qutrit as functions of $\zeta$ for the regular (solid curve) and degenerate (dashed curve) strata. The absolute minimum of both indicators is attained at $\zeta = \pi/6$. (B) The ratio of degenerate to regular $\mathcal{Q}_3$-indicators.



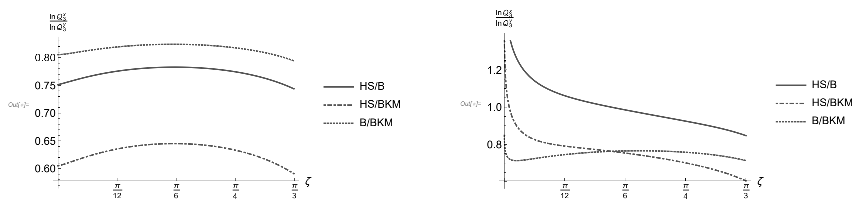

(a) (b)

Figure 5. (A) The plot of $\mathcal{Q}_3$ for the Bures (bold curves) and BKM (thin curves) ensembles of qutrits from the regular (solid curves) and degenerate (dashed curves) strata. (B) The ratio $R$ of degenerate to regular $\mathcal{Q}_3-$indicators for the Bures (bold curve) and the BKM (thin curve) ensembles.

## §5. Summary

Bearing in mind the results of the calculations of $\mathcal{Q}_3$, we will summarize with a few comments. The indicator of classicality $\mathcal{Q}_3$, being a functional of the ensemble probability distribution function, at the same time depends on two characteristics of the SW kernel: its isotropy group $H_\alpha$ and the moduli parameter $\zeta$. Our studies of the $\mathcal{Q}_3$-indicator reveal several interesting peculiarities concerning their interrelations:

- There is a certain coherence between the classification of states according to their classicality and their symmetry properties. In particular, it turns out that the states with a "larger" symmetry are more classical, cf. Fig. 4 and Fig. 5. This observation demands further study and we plan to formalize it in forthcoming publications;
- The character of the dependence of $\mathcal{Q}_3$ on the type of the ensemble is monotone, i.e., the values of $\mathcal{Q}_3$ for all strata are ordered in correspondence with the order of the ensembles, see Fig. 6;
- The $\mathcal{Q}_3(\zeta)$-indicator of the Hilbert–Schmidt ensemble is a symmetric function with respect to the global minimum point, $\zeta = \pi/6$, see Fig. 4a;
- For monotone metrics the symmetry possessed by the Hilbert–Schmidt ensemble is broken. Data specifying the range of violation is given in Table 1.




(a) (b)

Figure 6. Pairwise ratios of $\mathcal{Q}_3$-indicators of different ensembles for the regular **(A)** and for the degenerate **(B)** stratum.

| Global $\mathcal{Q}_3$-indicator vs. moduli parameter | | | |
|---|---|---|---|
| **Ensemble** | min $\mathcal{Q}_3(\zeta)$ | $\zeta_{\min}$ | $\mathcal{Q}_3(0) - \mathcal{Q}_3(\pi/3)$ |
| Hilbert–Schmidt | 0.0006751 | $\pi/6 \approx 0.523599$ | 0 |
| BKM | 0.0000121609 | 0.527798 | 0.0000216102 |
| Bures | 0.0000891011 | 0.525096 | 0.0000472609 |

Table 1. Data on symmetry properties of $\mathcal{Q}_3$-indicators.

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Поступило 12 сентября 2022 г.

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