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FUZZY NON-HORN KNOWLEDGE BASES: CALCULI, MODELS, INFERENCE

Abstract. This paper investigates inference in knowledge bases with fuzzy fuzzy non-Horn facts and rules. Sequent calculi with one structural, one logical rule, and non-logical axioms representing knowledge base rules and facts serve as a proof theory for these knowledge bases. These knowledge bases are also characterized by constrained real-valued models which are applicable to a variety of truth functions. Inference for fuzzy non-Horn knowledge bases is done by applying a variant of ordered resolution, transforming resolution refuations into sequent calculus derivations, building symbolic expressions from the derivations, and evaluating the symbolic expressions.

§1. Introduction

The languages of logic programs and knowledge bases (KB) are usually based on first-order logic (FOL) [17]. Atoms are expressions $P(t_1, \dots, t_k)$ where P is a predicate and $t_1, \dots, t_k$ are terms. Literals are atoms or their negations. A literal is called ground if it does not contain variables.

In non-Horn KBs, facts are literals. Non-Horn rules are expressions $A \Leftarrow A_1 \wedge \dots \wedge A_k$, where $A, A_1, \dots, A_k$ are literals. The advantages of non-Horn KB over Horn KBs and normal logic programs are discussed in [20]. In fuzzy KBs, the truth values of atoms are real numbers as opposed to boolean values, and thus, KB facts are fuzzy. KB rules are also fuzzy. A real number is associated with every fact or rule.

The principle of Reductio Ad Absurdum (RAA) states that if A is deduced from a hypothesis that is A's complement, then A is derivable. It will be explained later that reasoning by contradiction, i.e. with using RAA, is not quite adequate for KBs with fuzzy predicates. Procedures implementing FOL inference for non-Horn KBs without RAA include an adaptation of ordered resolution [18].

Key words and phrases: resolution, non-Horn rule, truth function, fuzzy logic, sequent calculus, Reductio Ad Absurdum.

We present a set of very simple sequent calculi that characterize inference without reasoning by contradiction for fuzzy non-Horn KBs. These sequent calculi have one structural and one logical rule. Non-logical axioms of the calculi represent KB rules and facts. We also introduce contrained real-valued models that characterize fuzzy non-Horn KBs. These models are applicable to a variety of truth functions. Inference for fuzzy non-Horn KBs is compounded of several steps. Ordered resolution steering clear of RAA is applied to these KBs. Resolution refutations [3] are mapped to derivations in the sequent calculi. Ground symbolic expressions are built from the latter derivations. These expressions are evaluated, yielding the lower bounds of literal truth values. The time complexity of the computations following resolution is linear in the size of the resolution refutations.

§2. Fuzzy Non-Horn Knowledge Bases

A substitution is a finite mapping of variables to terms. Let $\alpha\{b_1 \to \beta_1, ..., b_j \to \beta_m\}$ denote the substitution of term β_i for all occurrences of variable b_i in term α for $i = 1, \dots, m$. The result of applying a substitution to a formula or set of formulas is called its instance.

We consider inference of ground literals, which are called goals, from the facts and rules of fuzzy non-Horn KBs. A KB is called consistent if for no atom A, both A and $\neg A$ are derivable. As usual, it is assumed that all KBs under consideration are consistent. Note that some predicates can be implemented by external means such as neural networks or algorithms in some programming language. All other predicates will be called derivable. Some functions can also be implemented by algorithms.

Fuzzy truth values are usually represented by real numbers from interval $[0, 1]$. For non-Horn KBs, it is more convenient to use interval $[-1, 1]$ for the representation of truth values. One represents true, minus one represents false. Other real numbers from interval $[-1, 1]$ represent fuzzy truth values. It is expected that real numbers lower than one and higher than a certain threshold $h > 0$ are assigned to some facts and rules of a fuzzy KB. These assigned numbers are the lower bounds of the truth values of all instances of the respective rules or facts, i.e. it is assumed that all instances of a given rule or fact have the same lower bound. One is the default truth value for the other KB facts and rules. These assigned numbers will be called truth bounds.

For any function implemented by an external algorithm, its atoms with constant arguments are evaluated as soon as they appear in KB derivations. The same applies to atoms of externally-implemented predicates with constant arguments. This evaluation may not terminate, in which case it is assumed that the truth value is zero. Any complete search strategy for inference from KBs with externally-implemented predicates or functions should continue and-or search [17] simultaneously with the evaluation. Suppose the evaluation of ground atom $A(\dots)$ yields real number r. If r is greater or equal h, then $A(\dots)$ is considered a fact and r is its truth bound. If r is lower or equal $-h$, then $\neg A(\dots)$ is considered a fact and $-r$ is the truth bound of this fact.

Let $|A|$ denote the truth value of logical formula A. Traditionally, the negation truth function for fuzzy KBs is defined as: $|\neg A| = -|A|$. The Godel t-norm is traditionally used as the conjunction truth function for fuzzy KBs: $|A \wedge B| = \min\{|A|, |B|\}$. Other truth functions are used as well, and they may be a better fit for particular KBs [2]. Properties of various truth functions have been extensively investigated [8].

For non-Horn KBs, the use of the negation truth function is limited to the calculation of the truth values of negative literals, and the use of the conjunction truth function is limited to the calculation of the truth values of the bodies of KB rules. In fact, we use truth functions for the calculation of truth bounds as opposed to the calculation of exact values. We assume that the conjunction truth function has a variable number of arguments since KB rule bodies are conjunctions of multiple arguments.

The disjunction truth function is not used here. The implication truth function is not used directly. Instead, we assume that the semantics of fuzzy KB rules is based on the residuum of the conjunction truth function [4], i.e. $|A_0| \geqslant |(A_1 \wedge \dots \wedge A_k) \wedge (A_0 \Leftarrow A_1 \wedge \dots \wedge A_k)|$ for KB rule $A_0 \Leftarrow A_1 \wedge \dots \wedge A_k$. Alternatively, this inequality is understood as fuzzy Modus Ponens [4]. For the Godel t-norm as an example, the semantics of the aforementioned KB rule is expressed as follows: $|A_0| \geqslant \min\{|A_1|, \dots, |A_k|, |A_0 \Leftarrow A_1 \wedge \dots \wedge A_k|\}$.

Reasoning by contradiction seems inappropriate for fuzzy KBs. Consider two KB rules $P \Leftarrow Q$ and $P \Leftarrow \neg Q$. Here is reasoning by contradiction in FOL with boolean truth values. Suppose P is false. The first rule implies that Q is false, and hence P is true by the second rule. Now suppose truth values are fuzzy and $|P| = 0$. If $|Q| = 0$ as well, then both rules are satisfied, but they do not provide any evidence that P is true or $|P| > 0$ at least.

§3. Calculi

Let ^{-}A denote the complement of A, i.e. it is the negation of atom A, and the atom of negative literal A. A sequent is $\Gamma \vdash \Pi$ where Γ is an antecedent and Π is a succedent [14]. We consider calculi in which formulas are limited to literals, antecedents are multisets of formulas, and succedents are single formulas. KB inference and logic programming are concerned about the derivation of literals, i.e. sequents of the form $\vdash A$ where A is a literal. Consider the two following rules. The *swap* rule replaces the two standard negation rules [14].

$$\frac{\Gamma \vdash A \quad A, \Pi \vdash B}{\Gamma, \Pi \vdash B}\, cut \qquad\qquad \frac{A, \Gamma \vdash B}{^{-}B, \Gamma \vdash {}^{-}A}\, swap$$

KB facts and rules can be treated as non-logical axioms [14]. Sequents of the form $\vdash A$ represent facts, and rules are represented by sequents of the form $A_1, \ldots, A_n \vdash A$ where $A, A_1, \ldots, A_n$ are literals. Variables can be replaced by any terms in instances of these axioms. The conclusions of *swap* applied to KB rules are called contrapositives.

Definition 1. *L'_{cs} is the set of sequent calculus instances in which formulas are literals, succedents contain one literal, the structural rule is cut, the logical rule is swap whose premises are axioms, no logical axioms are present, and non-logical axioms represent KB rules and facts.*

Theorem 1. *L'_{cs} is sound and complete with respect to the derivation of ground literals in FOL without RAA.*

Proof. It is proved in [18] that ground literal L is derivable from KB facts and rules in FOL without RAA if and only if ^{-}L is refutable by resolution in which the factoring rule is not used and at least one premise of every resolution step is not ^{-}L or its descendant. Consider such resolution refutation. As usual, the resolution steps that are not ascendants of the endclause are discarded. Let us ground this refutation and then exclude the step that resolves ^{-}L. There is only one such step because at least one premise of every resolution step is not ^{-}L or its descendant. As a result, L is added to every descendant clause of this step including the endclause which becomes L.

Let us traverse this resolution tree bottom-up and map every resolution step to an application of *cut* in L'_{cs}. Sequent $\vdash L$ is the conclusion of the last *cut* in the respective L'_{cs} derivation tree. The premises of every *cut* in this tree are uniquely determined by the resolution step. The succedent of the *cut* conclusion is also the succedent of the second premise, and the succedent of the first premise is the principal formula of this *cut*. Every leaf node in the L'_{cs} derivation tree is an instance of a KB fact, an instance of a KB rule, or a sequent that is the conclusion of *swap* applied to an instance of a KB rule. Hence, the resulting tree is a L'_{cs} derivation.

Now consider a ground L'_{cs} derivation of sequent $\vdash L$. Every application of the *cut* rule in this derivation corresponds to a resolution step, and ground instances of KB rules and facts are used as input clauses in this resolution derivation instead of the rules and facts. The endclause of this resolution derivation is L.

The lifting lemma [3] states that if clause A is an instance of A', B is an instance of B', and C is the resolvent of A and B, then there is such clause C' that C is its instance, and C' is the resolvent of A' and B'. It is well-known that the lifting lemma can be generalized onto arbitrary resolution derivations: If C is the endclause of a resolution derivation with input clauses $A_1, \ldots, A_n$ which are instances of $A'_1, \ldots, A'_n$, respectively, then there is such resolution derivation with input clauses $A'_1, \ldots, A'_n$ and endclause C' that C is an instance of C'. This is proved by a straightforward induction on the depth of resolution derivations.

As a consequence of this generalization of the lifting lemma, there is a resolution tree with the input comprised of KB rules and facts treated as clauses and with such endclause L' that L is its instance. A step resolving L' and ^{-}L is added to this derivation. The resolvent of this step is the empty clause, and ^{-}L occurs in one premise of the last step only. Hence, this resolution refutation corresponds to a FOL derivation without RAA. □

§4. Truth Functions

The Lukasiewicz t-norm and the product t-norm are two other fundamental conjunction truth functions along with the Godel t-norm. The Lukasiewicz t-norm is defined on the domain $[0,1]^2$ as

$$|A \wedge B| = \max\{0, |A| + |B| - 1\}.$$

The product t-norm is defined as $|A \wedge B| = |A||B|$ on the same domain. For KBs containing fuzzy predicates, other truth functions for the conjunctions that are KB rule bodies could give more accurate truth values. These conjunction truth functions do not have to be t-norms [8].

Actually, the Lukasiewicz and product t-norms [8] do not look like a good choice for non-Horn KBs. For example, if $|A| = |B| = 0.6$, the product t-norm is 0.36 and the Lukasiewicz t-norm is 0.2. Linearly projecting all these values onto interval $[-1, 1]$, we would get $|A| = |B| = 0.2$, the product t-norm of these truth values is -0.28 and their Lukasiewicz t-norm is -0.6. The values 0.36 and 0.2 make sense in a probabilistic setting, but the corresponding negative values are useless for non-Horn KB rules. Clearly, the Godel t-norm is more meaningful for non-Horn KBs.

In contrast to the product and Lukasiewicz t-norms, the following conjunction truth function $c : [-1, 1]^k \to [-1, 1]$ is more reasonable in application to the bodies of non-Horn KB rules:

$$c(x_1, \dots, x_k) = \sqrt[k]{(x_1 + 1) \dots (x_k + 1)} - 1$$

where k is the number of literals in the rule body. The arithmetic mean of $x_1, \dots, x_k$ is another example of alternatives to the Godel t-norm for the bodies of non-Horn rules. Multivariate functions give more freedom because they do not have to be associative, which is acceptable for non-Horn rules.

Clearly, a negation truth function is expected to be decreasing. Since the language of non-Horn KBs does not include double-negated literals, it is implicitly assumed the the principle of double negation $\neg\neg A \equiv A$ holds for these KBs. Given that, any negation truth function n should be an involution, i.e. $n(n(x)) = x$ for any x. For the symmetry between the facts that are positive literals and the facts that are negative literals, it is expected that $n(h) = -h$. The above conditions significantly limit choices for negation truth functions. Other negation truth functions than $n(x) = -x$ are not worth considering for fuzzy non-Horn KBs.

If c is the conjunction truth function, then the semantics of KB rule $A_0 \Leftarrow A_1 \wedge \dots \wedge A_k$ is expressed as

$$|A_0| \geqslant c(c(|A_1|, \dots, |A_k|), |A_0 \Leftarrow A_1 \wedge \dots \wedge A_k|).$$

We assume that the semantics of non-Horn KB rules is enforced for positive values of $|A_1|, \dots, |A_k|$. If any of these values is negative or zero, then it is fair to say that the premise of the rule is not satisfied, and thus, this rule in itself does not have to impose constraints on the value of $|A_0|$ in this case.

Actually, enforcing the semantics of KB rules for truth values in interval $[-1, 1]$ is problematic for a variety of conjunction truth functions. No real numbers may satisfy the constraints induced by the residuum of a conjunction truth function. Consider the arithmetic mean as an example and a KB with the following facts and rules: C, D, $A \Leftarrow B \wedge C$, $\neg A \Leftarrow \neg B \wedge D$. Clearly, this simple KB is consistent. Since $|C|, |D| \geqslant h$, the semantics of KB rules leads to the inequality $|A| \geqslant h + |A|$ if it is enforced for truth values in interval $[-1, 1]$. The Godel t-norm seems the best choice in this respect. It is proved in [19] that models satisfying the semantics of KB rules for truth values in interval $[-1, 1]$ exist for consistent KBs in the case of the Godel t-norm.

Enforcing the semantics of non-Horn KB rules for positive values of literals in rule bodies is not sufficient for specifying a set of models that is complete with respect to the L'_{cs} calculi because any KB rule $A_0 \Leftarrow A_1 \wedge \cdots \wedge A_k$ holds in L'_{cs} if and only if its contrapositives hold. Contrapositives can be viewed as implications $^-A_j \Leftarrow ^- A_0 \wedge A_1 \wedge \cdots \wedge A_{j-1} \wedge A_{j+1} \wedge ... \wedge A_k$. Positive values $|^-A_0|$ and $|^-A_j|$ correspond to negative values $|A_0|$ and $|A_j|$, and values $|^-A_j|$ are not constrained by the semantics imposed on positive values of literals in the bodies of KB rules.

Additional contraints on the truth values of literals are necessary in order to align these values with L'_{cs}. One option is to enforce the semantics of KB rules for such truth values that at most one of them is negative. Another option is to extend the semantics onto KB rule contrapositves. We prefer the second option as the simpler one. The second option makes it possible to shrink the domain of conjunction truth functions to $(0, 1]^k$. The first option may lead to complicated conjunction truth functions that behave differently when one argument is negative. Also, the first option makes it difficult to derive properties of ^-A_j from inequality $|A_0| \geqslant c(c(|A_1|, \ldots, |A_k|), |A_0 \Leftarrow A_1 \wedge \cdots \wedge A_k|)$.

For any KB rule $A_0 \Leftarrow A_1 \wedge \cdots \wedge A_k$, the semantics of its contrapositives is expressed as follows:

$$|^-A_j| \geqslant c\big(c(|^-A_0|, |A_1|, \ldots, |A_{j-1}|, |A_{j+1}|, \ldots, |A_k|), \\ |^-A_j \ \Leftarrow \ ^-A_0 \wedge A_1 \wedge \cdots \wedge A_{j-1}, A_{j+1}, \ldots, A_k|\big)$$

for $j = 1, \ldots, k$. It is assumed that contrapositives of a given KB rule have the same truth value as this rule. Similar to KB rules, the semantics of contrapositives is enforced for positive values of $|^-A_0|$, $|A_1|$, $\ldots$, $|A_{j-1}|$, $|A_{j+1}|$, $\ldots$, $|A_k|$ only.

Definition 2. *Conjunction truth function* $c : (0,1]^k \to (0,1]$ *is called proper if* c *is increasing in every argument,* c *is commutative for every pair of arguments, and* $h \leqslant c(h,\dots,h)$.

The condition $h \leqslant c(h,\dots,h)$ is justified by the following. Suppose a KB contains facts $A(a)$ and $B(a)$ and rule $C(x) \Leftarrow A(x) \wedge B(x)$. It is expected that $C(a)$ is a derived fact even if the truth values of $A(a)$ and $B(a)$ are minimal, i.e., they equal h.

It is easy to verify that the Godel t-norm, the arithmetic mean, and the conjunction function defined earlier are proper. The conjunction truth function could be parametrized. For example, it could be parametrized by weights assigned to predicates. Potentially, the conjunction truth function used for contrapositives could be different from the function used for KB rules. Also, custom conjunction truth functions could be defined for particular KB rules as it is done in Sugeno KBs [2].

§5. Models

Models are usually defined by truth functions for logical connectives so that the truth values of ground formulas can be calculated. No other formulas than literals and KB rules are used in KB derivations. Because of this, only the conjunction and negation truth functions along with the residuum of the conjunction truth function are necessary, and models for KB inference can be defined by constraints on truth values in ground instances of literals and rules.

Definition 3. *An assignment of real numbers from interval* $[-1,1]$ *to ground literals as well as to KB rules and their contrapositives is a* $\mathcal{M}_r$ *model if* $|\neg A| = -|A|$ *for any ground atom* A, $|L| \geqslant f$ *for every ground KB fact instance* L *with truth bound* $f \geqslant h$, $|A_0 \Leftarrow A_1,\dots,A_k| \geqslant r$ *for every KB rule/contrapositive instance* $A_0 \Leftarrow A_1,\dots,A_k$ *with truth bound* $r \geqslant h$*, and the semantics is satisfied for all instances of KB rules and their contrapositives in which the truth values of literals in their bodies are positive.*

Definition 4. *Literal* A *is valid regarding* $\mathcal{M}_r$ *models if* $|A'| \geqslant h$ *for all groundings* A' *of* A *in all* $\mathcal{M}_r$ *models.*

Let us define terms $t(\tau)$ recursively for all ground L'_{cs} derivations τ. In the following definition, lower-case letters are variables. These variables correspond to the same named upper-case ground literals. Let c denote the conjunction truth function.

– If τ is ground instance A of a KB fact and f is the truth bound of this fact, then $t(\tau) = f$.

– If τ is ground instance $A_0 \Leftarrow A_1, \dots, A_k$ of a KB rule and r is the truth bound of this rule, then $t(\tau) = c(c(a_1, \dots, a_k), r)$.

– If the last rule of τ is *swap* with the conclusion

$$A_0, A_1, \dots, A_{j-1}, A_{j+1}, \dots, A_k \vdash A_j$$

and r is the truth bound of the source rule, then $t(\tau) = c(c(a_0, , a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_k), r)$.

– If the last rule of τ is *cut* with premises

$$A_1, \dots, A_k \vdash E \quad \text{and} \quad E, C_1, \dots, C_m \vdash D,$$

then $t(\tau) = t(\nu)\{e \to t(\mu)\}$, where μ and ν are the parts of τ whose endsequents are the first and second premise of this *cut*, respectively.

Theorem 2. *If τ is a ground L'_{cs} derivation of literal G, and the conjunction truth function is proper, then $|G| \geqslant t(\tau) \geqslant h$ for all $\mathcal{M}_r$ models.*

Proof. By a straightforward induction of the depth of derivations, the only variables occurring in $t(\tau)$ are the variables corresponding to literals in the antecedent of the endsequent of τ. Consequently, $t(\tau)$ does not contain variables for any derivation τ with endsequent $\vdash G$.

Now we will prove by induction on the depth of L'_{cs} derivations that if $A_1, \dots, A_k \vdash D$ is the endsequent of derivation μ, then $t(\mu)$ is increasing in every variable occurring in it, and $|D| \geqslant t(\mu)\{a_1 \to |A_1|, \dots, a_k \to |A_k|\} \geqslant h$ provided that $|A_1| \geqslant h, \dots, |A_k| \geqslant h$. As a corollary, $|G| \geqslant t(\tau) \geqslant h$.

Base. The depth of derivation μ is zero. If the endsequent of μ is $\vdash D$, then D is an instance of a KB fact, and the above inequalities hold. If the endsequent of μ is KB rule instance $A_1, \dots, A_k \vdash D$, then r is the only constant occurring in $t(\mu)$. Clearly, $t(\mu)$ is increasing in every variable. The inequality $|D| \geqslant t(\mu)\{a_1 \to |A_1|, \dots, a_k \to |A_k|\}$ holds due to the semantics of KB rules. The inequality $t(\mu)\{a_1 \to |A_1|, \dots, a_k \to |A_k|\} \geqslant h$ holds by the definition of proper conjunction truth functions.

Induction step. Suppose the statement under consideration holds for all derivations whose depth is less or equal n. Suppose the depth of μ is $n+1$. If the last rule in μ is *swap* and its endsequent is

$$A_0, A_1, \dots, A_{j-1}, A_{j+1}, \dots, A_k \vdash A_j,$$

then the *swap* premise is ground instance $^-A_0 \Leftarrow A_1, \dots, ^-A_j, \dots, A_k$ of a KB rule, μ does not contain constants except r, and $t(\mu)$ is increasing in every variable. The inequality $|A_j| \geqslant t(\mu)\{a_0 \to |A_0|, a_1 \to |A_1|, \ldots, a_{j-1} \to |A_{j-1}|, a_{j+1} \to |A_{j+1}|, \ldots, a_k \to |A_k|\}$ holds due to the semantics of KB rule contrapositives. The inequality $t(\mu)\{a_0 \to |A_0|, a_1 \to |A_1|, \ldots, a_{j-1} \to |A_{j-1}|, a_{j+1} \to |A_{j+1}|, \ldots, a_k \to |A_k|\} \geqslant h$ holds by the definition of proper conjunction truth functions.

Now let the last rule in μ be *cut*, the first premise of this *cut* be $B_1, \ldots, B_k \vdash C_1$, and the second premise be $C_1, \ldots, C_m \vdash D$. If γ is the derivation ending in $B_1, \ldots, B_k \vdash C_1$ and δ is the derivation ending in $C_1, \ldots, C_m \vdash D$, then both $t(\gamma)$ and $t(\delta)$ are increasing in every variable, $|C_1| \geqslant t(\gamma)\{b_1 \to |B_1|, \ldots, b_k \to |B_k|\} \geqslant h$ and $|D| \geqslant t(\delta)\{c_1 \to |C_1|, \ldots, c_m \to |C_m|\} \geqslant h$ by the induction assumption.

Due to the monotonicity of $t(\delta)$,

$$|D| \geqslant t(\delta)\{c_1 \to t(\gamma)\{b_1 \to |B_1|, \ldots, b_k \to |B_k|\}, c_2 \to |C_2|, \ldots, c_m \to |C_m|\} \geqslant h.$$

By the definition of t, $t(\mu) = t(\delta)\{c_1 \to t(\gamma)\}$. Hence, $|D| \geqslant t(\mu)\{b_1 \to |B_1|, \ldots, b_k \to |B_k|, c_2 \to |C_2|, \ldots, c_m \to |C_m|\} \geqslant h$. Clearly, $t(\mu)$ is increasing in every variable. □

Theorem 3. *If $|G| \geqslant h$ in all $\mathcal{M}_r$ models for ground literal G, the KB is consistent, and the conjunction truth function is proper, then there exists a derivation of G in L'_{cs}.*

Proof. Suppose G is not derivable in L'_{cs} from KB facts and rules. Let us look at model M in which $|B| = 1$ for every ground literal B that is derivable from KB facts and rules (including KB fact instances), $|C| = -1$ for every such ground literal C that ^{-}C is derivable, and $|D| = 0$ for every other ground literal D. Such model M exists for any consistent KB, and $|G| = 0$ in M.

Inequality $|L| \geqslant h$ holds for every ground KB fact instance L because ground instances of facts are derivable. Suppose the semantics of a KB rule is violated for its ground instance $A_0 \Leftarrow A_1 \wedge \cdots \wedge A_k$. In this case, $|A_0| \leqslant 0$ and $|A_i| = 1$ for $i = 1 \ldots k$, and thus all sequents $\vdash A_i$ are derivable in L'_{cs}. Hence, A_0 is derivable from the latter by k applications of *cut* to $A_1, \ldots, A_k \vdash A_0$ and to every $\vdash A_i$ for $i = 1 \ldots k$.

Now suppose the semantics of a KB rule contrapositive is violated for its ground instance $^{-}A_j \Leftarrow^{-} A_0 \wedge A_1 \wedge \cdots \wedge A_{j-1} \wedge A_{j+1} \wedge \cdots \wedge A_k$. In this case, $|^{-}A_j| \leqslant 0$, $|^{-}A_0| = 1$, and $|A_i| = 1$ for $i = 1 \ldots j-1, j+1, \ldots, k$. Sequent $^{-}A_0, A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_k \vdash^{-} A_j$ is derived by applying *swap* to $A_1, \ldots, A_k \vdash A_0$. Literal $^{-}A_j$ is derivable by application of *cut* to this sequent and to $\vdash^{-} A_0$ followed by $k-1$ applications of *cut* using $\vdash A_i$ for

$i = 1...j-1$ and $i = j+1 \ldots k$ as the first premise. Hence, the semantics of KB rules and their contrapositives could not be violated. □

Theorem 2 is a soundness theorem for L'_{cs} and $\mathcal{M}_r$. It shows that if there is a L'_{cs} derivation of G, then G is valid regarding $\mathcal{M}_r$ models. Theorem 3 is a completeness theorem. It establishes that every valid literal is derivable. It is also proved in Theorem 3 that $\mathcal{M}_r$ models exist for every consistent KB: M is such $\mathcal{M}_r$ model.

§6. Inference

Ordered resolution is one of the most efficient inference methods for FOL and for non-Horn KBs in particular [1]. It is used in modern theorem provers [11]. Ordered resolution has been adapted to inference from non-Horn KBs without RAA [18]. It can be used as the first step in inference from fuzzy non-Horn KBs. The second step is the transformation of a resolution refutation into a L'_{cs} derivation. The third step is the construction of t from this derivation τ. And the final step is the evaluation of $t(\tau)$.

In addition to the soundness of of L'_{cs} derivations, Theorem 2 establishes that, if τ is a L'_{cs} derivation of ground literal G, then $t(\tau)$ is a lower bound of the truth value of G. The proof of Theorem 1 shows that resolution refutations without factoring can be transformed into L'_{cs} derivations in a single preorder traversal of the resolution refutations. Therefore, the time complexity of this transformation is linear in the size of the resolution refutations.

It is reasonable to assume that the time complexity of an algorithm implementing the conjunction truth function is linear in the number of function arguments. The construction of $t(\tau)$ can be done in a single postorder traversal of τ. Given the aforementioned assumption about truth function algorithms, the time complexity of evaluating t expressions is linear in the size of these expressions. Consequently, the calculation of a lower bound of $|G|$ takes a linear time of the size of G's derivation in L'_{cs} and also of the size of the corresponding resolution refutation.

§7. Related Work

An overview of KB inference methods including resolution-based methods can be found in [17]. Resolution methods [3] are well suited for inference from non-Horn KBs. Ordered resolution is recognized as one of the most efficient inference methods [1]. It is used in modern theorem provers [11].

Ordered resolution is also relevant to fuzzy KBs and it has a very simple proof-theoretic characterization.

The L'_{cs} calculi are similar to the L_{cs} calculi defined in [19]. The only difference is that the premises of *swap* are not limited to KB rules in L_{cs}. L'_{cs} derivations correspond to normal-form derivations in L_{cs}. These two sets of calculi have the same inference power. The proof of Theorem 1 is similar to the proof of completeness of L_{cs} with respect to the derivation of ground literals in FOL without RAA. The paper [19] investigates inference from non-Horn KBs with fuzzy facts and crisp rules for the Godel t-norm. These two calculus sets also have the same inference power as LK_{-c} [20]. The latter calculi employ standard negation rules and allow multiple literals in succedents.

Preliminary results of this work are presented in [21]. Major differences between the two are the following. In [21], the conjunction truth functions are defined for both negative and positive truth values. It is assumed that these functions are defined as symbolic expressions and constraints related to KB rule contrapositives can be derived by transforming these symbolic expressions. As explained earlier, the existence of models satisfying the semantics of KB rules is not guaranteed in such setting.

Fuzzy KB systems [2] usually employ the concept of fuzzy sets. As a result, they involve fuzzification or defuzzification in addition to inference. Forward chaining normally serves as the inference mechanism for fuzzy KBs [2]. Our method combines resolution-based inference with symbolic and numeric calculations, it does not concern fuzzy sets. Inference without RAA is more powerful than the forward application of Modus Ponens in chaining [18].

Non-Horn KBs with fuzzy predicates are similar to possibilistic logic [6] in the sense that in both of them real numbers are associated with derived ground literals. A survey of fuzzy proof theories in which numbers indicating truthness are attached to FOL formulas is presented in [7]. The major difference of our approach is that literals are the only FOL formulas involved in the KB formalism considered here. Instead of applying fuzzy truth functions to FOL formulas [8], we propagate constraints on the truth values of literals.

Numerous recent research papers are devoted to the implementation of predicates or relations as neural networks [5, 9, 10, 22, 23]. These networks yield the fuzzy truth values of atoms of these predicates with constant arguments. Some KBs and logic programs may include so-called evaluable predicates implemented by algorithms [13]. Clearly, these algorithms may yield real numbers treated as fuzzy truth values.

The neural-symbolic method from [16] utilizes weighted real-valued functions for calculating lower and upper bounds of the truth values of FOL formulas. Inference is implemented as alternating upward and downward passes over the structure of the formulas. Truth value bounds are adjusted during these passes. Modus Ponens and Modus Tollens are used to update truth value bounds. In our work, sequents play the role of premises of Modus Ponens, and the *swap* rule can be viewed as a form of Modus Tollens.

ProbLog [15] extends Prolog by associating probabilities with facts. It is assumed that all ground instances of a non-ground fact are mutually independent and have the same probability. ProbLog engines calculate approximate probabilities for inference goals. Fuzzy non-Horn KBs are not probabilistic, they are based on fuzzy logic [8]. DeepProbLog [12] extends ProbLog by allowing neural networks to be associated with facts instead of probabilities. The probabilities of ground instances of a fact are calculated by the neural network associated with the respective predicate. In contrast, we interpret the output of neural networks as lower bounds of fuzzy truth values of ground facts.

§8. Conclusion and Future Work

Fuzzy KBs enable reasoning in the presence of uncertainty. Besides, fuzzy non-Horn KBs are a mechanism for integrating neural networks or other fuzzy calculators into rule-based systems. Both proof and model theories for fuzzy non-Horn KBs are presented here. This paper shows how to piggyback fuzzy inference on efficient resolution methods by means of symbolic computations. Our inference method is applicable to a variety of truth functions but the practical value of this method is yet to be tested. This method will work even when the conjunction truth function is parametrized by predicates or KB rules.

It is often possible to get multiple derivations of one goal. The truth value bounds calculated from these derivations may vary. The design of algorithms capturing higher truth bounds is an open issue. Investigation of the applicability of non-proper truth functions is another topic for future research. The implication truth function and the semantics of KB rules could be defined without using the notion of residuum. Alternative definitions of the semantics of KB rules deserve an investigation as well.

References

1. B. Leo, G. Harald, *Resolution theorem proving. Handbook of automated reasoning.* — Elsevier (2001), 19–99.
2. Barros, Laécio Carvalho de and Bassanezi, Rodney Carlos and Lodwick, Weldon Alexander, *A first course in fuzzy logic, fuzzy dynamical systems, and biomathematics: theory and applications*, Springer, 2017.
3. Ch. Chin-Liang, L. Richard Char-Tung, *Symbolic logic and mechanical theorem proving.* Academic press, 1973.
4. C. Petr, Hájek, Petr, N. Carles, title=Handbook of Mathematical Fuzzy Logic. — College Publ. **1** (2011).
5. D. Honghua Mao, Jiayuan and Lin, Tian and Wang, Chong and Li, Lihong and Zhou, Denny, *Neural logic machines*, Inter. Conf. Learning Representations, 2019.
6. D. Didier, P. Henri, *Possibilistic Logic-An Overview.* — Computational logic (2014), 197–255.
7. G. Siegfried, *A treatise on many-valued logics*, Research Studies Press, 2001.
8. H. Petr, *Metamathematics of fuzzy logic*, Springer Science & Business Media, 2013.
9. H. Jinyung, P. Theodore, *An insect-inspired randomly, weighted neural network with random fourier features for neuro-symbolic relational learning*, Neural-Symbolic Learning and Reasoning, CEUR-WS.org, 2021.
10. T. Ishihara, K. Hayashi, H. Manabe, M. Shimbo, M. Nagata, *Neural tensor networks with diagonal slice matrices*, Proceedings of the 2018 Conf. North American Chapter of the Association for Comput. Linguistics: Human Language Techn. **1** (2018), 506–515.
11. L. Kovács, A. Voronkov, *First-order theorem proving and Vampire*, International Conference on Computer Aided Verification, Springer, (2013), 1–35.
12. R. Manhaeve, S. Dumančić, A. Kimmig, Th. Demeester, L. De Raedt, *Neural probabilistic logic programming in DeepProbLog.* — Artificial Intelligence **298** (2021), 103–504.
13. W. McCune, *Otter 3.3 reference manual and guide.* Tecnical report, Argonne National Lab., 2003.
14. S. Negri, J. Von Plato, *Structural proof theory*, Cambridge University Press, 2001.
15. L. De Raedt, A. Kimmig, *Probabilistic (logic) programming concepts.* — Machine Learning **100**, No. 1 (2015), 5–47.
16. R. Riegel, A. Gray, F. Luus, N. Khan, N. Makondo, I. Y. Akhalwaya, H. Qian, R. Fagin, F. Barahona, U. Sharma, *Logical neural networks*, `arXiv preprint arXiv:2006.13155`, 2020.
17. S. Russell, P. Norvig, *Artificial Intelligence: A Modern Approach*, Prentice Hall Press, 3rd edition, 2009.
18. A. Sakharov, *Inference Methods for Evaluable Knowledge Bases*, Software Engineering Application in Informatics, Lecture Notes in Networks and Systems, Springer, 499–510, 2021.
19. A. Sakharov, *Calculi and Models for Non-Horn Knowledge Bases Containing Neural and Evaluable Predicates*, Logics for New-Generation AI, College Publications (2022), 24–35.

20. A. Sakharov, *A Logical Characterization of Evaluable Knowledge Bases.* — 14th International Conference on Agents and Artificial Intelligence (2022), 681–688.
21. A. Sakharov, *Symbolic Inference for Non-Horn Knowledge Bases With Fuzzy Predicates.* — Polynomial Computer Algebra (2022), 86–95.
22. A. Santoro, D. Raposo, David G. Barrett, M. Malinowski, R. Pascanu, P. Battaglia, T. Lillicrap, *A simple neural network module for relational reasoning.* — Adv. neural inform.proc.systems **30**, (2017).
23. L. Serafini, Artur S. d'Avila Garcez, *Logic Tensor Networks: Deep Learning and Logical Reasoning from Data and Knowledge*, Neural-Symbolic Learning and Reasoning, CEUR-WS.org, (2016).

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