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COLOURINGS OF EXCEPTIONAL UNIFORM POLYTOPES OF TYPES E_6 AND E_7

Abstract. We compute the cycle indices of the Weyl group $W(\mathrm{E}_6)$ in its action on the vertices of the Schläli polytope (E_6, ϖ_1) and of the Weyl group $W(\mathrm{E}_7)$ in its action on the vertices of the Hesse polytope (E_7, ϖ_7). This is done purely by hand using the following visual aids – weight diagrams of the corresponding representations to encode the action of the Weyl groups on the polytopes, and the enhanced Dynkin diagrams of the corresponding root systems to encode the conjugacy classes of the Weyl groups themselves, in the style of Carter and Stekolshchik.

The present note is based on the Diploma paper of the first-named author written under the supervision of the second-named author. The goal of the project was to review the structure and geometry of the Gosset–Elte uniform polytopes in dimensions 6 through 8, of exceptional symmetry types E_6, E_7 and E_8.

These polytopes were extensively studied by Coxeter, Conway, Sloane, Moody, Patera, McMullen, and many other remarkable mathematicians. We develop a new easier approach towards their combinatorial and geometric properties. In particular, we propose a new way to describe the faces of these polytopes, and their adjacencies, inscribed subpolytopes, compounds, independent subsets, foldings, and the like. Our main tools – weight diagrams, description of root subsystems and conjugacy classes of the Weyl group – are elementary and standard in the representation theory of algebraic groups.

But we believe their specific use in the study of polytopes might be new, and considerably simplifies computations. As an illustration of our methods that seems to be new, we calculate the cycle indices for the actions of the Weyl groups on the faces of these polytopes. With our tools, this can be done by hand in the easier cases, such as the Schläfli and Hesse polytopes

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for E_6 and E_7. Nevertheless, the senior polytopes and the case of E_8 require the use of computers anyway, even after all possible simplifications.

Since the actual proofs are mostly of rather technical and/or computational nature, here we merely state the results, explaining part of the background and the basic ideas of our approach. We refer to the extended abstract/slides [36] of the talk at the PCA-2021

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for more background and references, and to the full text of the Diploma paper [35] for the actual details of calculations. One of the more conceptual steps of the proof, description of the conjugacy classes of the Weyl groups in terms of exhanced Dynkin diagrams, was published in our previous paper [37]. Theorem 1 was previously announced in [36] but Theorem 2 is published here for the first time.

§1. Gosset–Elte polytopes

Marcel Berger [3], pp. 39–40, attributes to René Thom the division of mathematical structures into

• **rich** = rigid, that become progressively scarce in higher dimensions, orders, ranks, etc., and

• **poor** = soft, that abound in higher sizes, and that eventually become impossible to classify.

One of the classical examples of this phenomenon are regular polytopes and their kin, such as semiregular and other strictly uniform polytopes. They abound in dimension 2, are quite freakish in dimension 3, proliferate in dimension 4, and then eventually crystallise to very few possible shapes that self-reproduce throughout all dimensions.

Essentially everything that is of earnest mathematical interest takes place in dimensions 3 through 8, and is closely related to quaternions, octonions[1] and exceptional root systems of types H_3, D_4, F_4, H_4, D_5, E_6, E_7 and E_8.

For obvious reasons we cannot discuss this subject at large here, the bibliography in the current note is set to an absolute minimum and only includes works directly cited in the text, the ones that motivated us and

[1] Even professional mathematicians seldom realise that the fact that in dimension $n = 3$ regular tetrahedron can be vertex embedded into a cube is just another manifestation of the existence of quaternions, and that the next dimension, where the same happens, is $n = 7$, see [13], or [1, 2].

that we used in the proofs. Already [36] gives a broader picture and a more extensive bibliography. However, we highly recommend the reference book by Peter McMullen and Egon Schulte [34] on combinatorial polytopes, and especially the recent book by McMullen [33] on geometric polytopes, which contain *systematic* bibliographies.

For **regular** polytopes their symmetry group acts transitively on flags (a vertex, an edge containing this vertex, a 2-face containing this edge, etc.). Starting with dimension $n \geqslant 5$ regular polytopes become exceedingly dull, they are all classical – simplices, hyperoctahedra, and hypercubes. This means that to get further fascinating examples in dimensions other than 3 and 4 one has to relax the transitivity condition.

• A polytope is called **uniform** if its symmetry group is **vertex transitive** and its **facets** (= the faces of codimension 1) are themselves *uniform*.

• A uniform polytope is called **semiregular** if its facets are *regular*. This is Gosset's definition. Elte would define semiregularity inductively and allow the facets themselves to be *semiregular*.

• In 1900 Thorold Gosset published a list of 7 semiregular polytopes, 3 of them in dimension $n = 4$ and one in dimensions $n = 5, 6, 7, 8$ each, the four remarkable semiregular polytopes of symmetry types D_5, E_6, E_7 and E_8. In Coxeter notation these are the **Clebsch polytope** 1_{21}, the **Schläfli polytope** 2_{21}, the **Hesse polytope** 3_{21}, and the **Gosset polytope** 4_{21}. Oftentimes all of these polytopes are collectively called **Gosset polytopes**.

• In 1912 Emanuel Elte rediscovered those, relaxed the notion of semiregularity, and constructed further exceptional polytopes of symmetry types E_6, E_7 and E_8, the **Elte polytopes** 1_{22}, 2_{31}, 1_{32}, 2_{41} in Coxeter notation.

The uniqueness of Gosset polytopes makes them *extremely* interesting. Most laymen – initially including ourselves! – believe that the combinatorial structure of the Gosset polytopes was known to Gosset and Elte more than a century ago and that the classification of semiregular polytopes in *all* dimensions was completed by Coxeter not later than 1948. Both claims are *outrageous* oversimplifications!

• A polytope is called **regular-faced** if all of its faces are regular. In dimension 3 such polytopes are called **Johnson solids**. Since they may have very low symmetry, their classification is a *highly* non-trivial problem of metric geometry. It was accomplished by Victor Zalgaller, see [24] and references there. On the other hand, in dimension 4 there are hundreds of millions of such similar creatures, whose classification is a *highly* non-trivial problem of combinatorics, see, for instance, [18]. But it all stops there.

In dimensions $d \geqslant 5$ Gerd and Roswitha Blind have completed classification of regular-faced polytopes up to isomorphism sometime before 1980. There, nothing unexpected occurs, just the regular and semiregular polytopes, pyramids and bipyramids. As a spin-off of their classification, in 1991 they obtained a first conclusive completeness proof for the above Gosset list in dimensions $d \geqslant 5$, see [4].

• The same applies to the combinatorial structure of these polytopes. Gosset himself has not given a complete combinatorial description of the polytopes, just their *facets* and *some* incidence properties of the following type: a $(d-3)$-face of the d-dimensional polytope is contained in two $(d-1)$-hyperoctahedra and one $(d-1)$-simplex, etc.

The detailed proofs of such a description announced by Coxeter in 1940–1948 were never published before 1988–1992, by Coxeter himself, Conway, Sloane, Moody, and Patera, see [12,16,38], with some circumstantials being clarified long after that.

§2. Gosset–Elte polytopes and Weyl orbits

The second-named author became genuinely interested in these matters in the process of his work with Alexander Luzgarev on the explicit equations defining the exceptional Chevalley groups of types E_6, E_7 and E_8, see, in particular, [29, 49, 52, 53], and references therein.

There, the polynomial equations themselves and/or the occurring monomials would correspond to the faces of the Schläfli, Hesse and Gosset polytopes, and their kin, with some weird coincidences and kinky symmetries.

Thus, for instance, the highest Weyl orbit of equations on the orbit of the highest weight vector consists of

• 27 Borel–Freudenthal equations defining the projective octave plane E_6/P_1 for (E_6, ϖ_1);

• 126 Freudenthal equations defining the 27-dimensional Freudenthal variety E_7/P_7 for (E_7, ϖ_7);

• 270 quadratic equations in the adjoint representation (E_6, ϖ_2);

• 756 quadratic equations in the adjoint representation (E_7, ϖ_1);

• 2160 quadratic equations in the adjoint representation (E_8, ϖ_8);

and similar results for other orbits. In the two senior cases the explanation of these numbers in terms of the embeddings $\mathrm{A}_7 \subseteq \mathrm{E}_7$ and $\mathrm{D}_8 \subseteq \mathrm{E}_8$ were not immediate to us, and required separate clarification, see [50]. They are now, in terms of the faces of the corresponding Gosset–Elte polytopes!

Our approach is based on an interpretation of the Gosset–Elte polytopes as **permutation polytopes** for the Weyl group action on weights of the corresponding root systems.

To be more specific, we have to recall some notation concerning root systems, Weyl groups, and weights, see [6,27]. In particular, Φ is a reduced irreducible root system of rank l, whereas $W = W(\Phi)$ is its Weyl group. For a root $\alpha \in \Phi$ we denote by $w_\alpha \in W$ the corresponding root reflection. Here, we are only interested in the simply laced systems, in which case the roots are usually normalised so that $(\alpha, \alpha) = 2$.

The Weyl groups of senior exceptional types have the following orders

$$|W(\mathrm{E}_6)| = 51840 = 72 \cdot 6! = 2^7 \cdot 3^4 \cdot 5,$$
$$|W(\mathrm{E}_7)| = 2903040 = 72 \cdot 8! = 2^{10} \cdot 3^4 \cdot 5 \cdot 7,$$
$$|W(\mathrm{E}_8)| = 696729600 = 192 \cdot 10! = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7,$$

these orders occur as the denominators in the expressions of the corresponding cycle indices.

We fix an order on Φ, and let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be the corresponding set of fundamental roots, Φ^+ and Φ^- be the corresponding sets of positive and negative roots, respectively. Usually we denote the fundamental root reflection w_{α_i} simply by w_i. The Weyl group is generated already by the fundamental reflections, $W = \langle w_1, \dots, w_l \rangle$.

Further, we denote by $Q(\Phi)$ the **root lattice**, generated by $\alpha_1, \dots, a_l$, and by $P(\Phi)$ the [dual] **weight lattice**[2] generated by the fundamental weights $\varpi_1, \dots, \varpi_l$. Recall that $(\varpi_i, \alpha_j) = \delta_{ij}$. The cone of **dominant weights** $P(\Phi)_{++}$ consists of non-negative integer linear combinations of $\varpi_1, \dots, \varpi_l$.

Now we take a weight $\omega \in P(\Phi)_{++}$ and consider the Weyl orbit $W\omega \subseteq P(\Phi)$. The most interesting exceptional polytopes can be interpreted as the convex hulls of $W\omega$, usually we refer to such a polytope as the **polytope of type** (Φ, ω). In representation theoretic terms, $W\omega$ are the extremal weights of the representation of $G(\Phi, \mathbb{C})$ with highest weight ω, so that

[2]In the textbooks on lattices and sphere packings the root lattices $Q(\mathrm{E}_l)$ are usually denoted simply by E_6, E_7 and E_8, whereas the weight lattices $P(\mathrm{E}_l)$ are denoted by E_6^* and E_7^*. The lattice $P(\mathrm{E}_8)$ is unimodular and self-dual, $\mathrm{E}_8^* = \mathrm{E}_8$.

the polytope of type (Φ, ω) is the convex hull of the weights $\Lambda(\omega)$ of that representation.

The three series of **classical** regular polytopes that exist in all dimensions $n \geqslant 2$ have obvious interpretations as such **weight polytopes**:

• **Simplices** $\alpha_n = \{3, \dots, 3\}$ with $n+1$ vertices = weight polytopes of the vector representation (A_n, ϖ_1);

• **Hyperoctahedra** $\beta_n = \{3, \dots, 3, 4\}$ with $2n$ vertices = orthoplexes = cross-polytopes = weight polytopes of the vector representation (D_l, ϖ_1);

• **Hypercubes** $\gamma_n = \{4, 3, \dots, 3\}$ with 2^n vertices = weight polytopes of the spin representation (B_l, ϖ_n).

Similarly, the exceptional **Gosset–Elte polytopes** in dimensions 6, 7 and 8 can be now interpreted as follows. Alternatively, they can be described as **Voronoi polytope**, **Delaunay polytope** or **contact polytope** of exceptional lattices, and there are many further related polytopes associated with the Weyl orbits on weights with slightly weaker regularity properties, see, in particular, [12, 22, 23, 38, 39, 54, 55].

• 2_{21} with 27 vertices – Schläfli polytope of type (E_6, ϖ_1) = Delaunay polytope for $Q(\mathrm{E}_6)$. Or, dually, 2_{12} of type (E_6, ϖ_6).

• 1_{22} with 72 vertices – the root polytope for E_6 of type (E_6, ϖ_2) = the contact polytope for $Q(\mathrm{E}_6)$.

• 3_{21} with 56 vertices = Hesse polytope of type (E_7, ϖ_7) = the contact polytope for $P(\mathrm{E}_7)$.

• 2_{31} with 126 vertices = the root polytope for E_7 of type (E_7, ϖ_1) = the contact polytope for $Q(\mathrm{E}_7)$.

• 1_{32} with 576 vertices = Voronoi polytope of $P(\mathrm{E}_7)$ of type (E_7, ϖ_2).

• 4_{21} with 240 vertices = Gosset polytope of type (E_8, ϖ_8) = the contact polytope for $Q(\mathrm{E}_8)$.

• 2_{41} with 2160 vertices = the deep hole[3] polytope for $Q(\mathrm{E}_8)$ of type (E_8, ϖ_1).

[3]The terminology for E_8 is borrowed from the book by Conway and Sloane [11]. The 240 roots of E_8 are the lattice points of norm 2. A **hole** is a point of $\mathbb{R}^8$, whose distance to $Q(\mathrm{E}_8)$ is a local maximum. The 2160 **deep holes** near the origin are *halves* of the lattice points of norm 4. The 17540 lattice points of norm 8 fall into two orbits under the action of $W(\mathrm{E}_8)$, for which 240 are twice the roots, and 17280 are 3 times the **shallow holes** near the origin.

• 1_{42} with 17280 vertices = the shallow hole polytope for $Q(\mathrm{E}_8)$ of type (E_8, ϖ_2).

As we mentioned, there are many further *extremely* interesting related examples, which are not themselves on the Gosset–Elte list, but closely related to those, like, for instance:

• Diplo-Schläfli polytope with 54 vertices = the convex hull of two dual Schläfli polytopes 2_{21} and 2_{12} = Voronoi polytope for $Q(\mathrm{E}_6)$.

• Voronoi polytope for $P(\mathrm{E}_6)$ with 720 vertices, etc.

§3. Cycle indices for the Schläfli and Hesse polytopes

Coxeter discovered that instead of **duality**, so prominent for the regular polytopes, the semiregular ones display **triality**. The exceptional polytopes come in *triples,* with the facets of each one of them corresponding to the vertices of the other two. As we already know, in dimension 6 the facets of 2_{21} correspond to the 27 vertices of 2_{12} *and* to the 72 vertices of 1_{22}.

Dually – or should one say *trially* in this case? – the facets of the root polytope of E_6 are all of them Clebsch polytopes = **5-demicubes**, but they come in two denominations, the *positive half spin* (D_5, ϖ_4) and the *negative half spin* (D_5, ϖ_5). Here $54 = 27 + 27$, the positive ones corresponding to the vertices of (E_6, ϖ_1) and the negative ones – to the vertices of the **dual** polytope (E_6, ϖ_6).

These demicubes are arranged as follows. The 5-demicube has 16 facets α_4 and 10 facets β_4. Any two adjacent 5-demicubes of the same parity intersect in α_4, whereas two adjacent demicubes of different parities intersect in β_4. In particular, (E_6, ϖ_2) has two types of α_4 faces: the positive and the negative ones.

Here we calculate the cycle indices for the action of $W(\mathrm{E}_6)$ on the vertices of the Schläfli polytope (E_6, ϖ_1) and for the action of $W(\mathrm{E}_7)$ on the vertices of the Hesse polytopes (E_7, ϖ_7).

Theorem 1. *The cycle index of the Weyl group* $W(\mathrm{E}_6)$ *in its action on the vertices of* (E_6, ϖ_1) *equals*

$$Z_{27}[x_1, \dots, x_{12}] = \frac{1}{51840}\Big(x_1^{27} + 36x_1^{15}x_2^6 + 270x_1^7x_2^{10} + 240x_1^9x_3^6$$
$$+ 585x_1^3x_2^{12} + 1440x_1^3x_2^3x_3^4x_6 + 1620x_1^5x_2x_4^5 + 2160x_1^3x_2^3x_3^4x_6$$

$$+ 560x_3^9 + 3780x_1x_2^3x_4^5 + 5184x_1^2x_5^5 + 1440x_1^3x_2^3x_6^3$$
$$+ 540x_1^3x_4^6 + 1440x_3^5x_6^2 + 5184x_2x_5^3x_{10} + 6480x_3x_6^4$$
$$\left. + 6480x_1x_2x_8^3 + 4320x_1x_4^2x_6x_{12} + 4320x_3x_{12}^2 + 5760x_9^3\right)$$

Observe the presence of rotation axes of orders 5, 8, 10 and 12, which are already possible for crystals[4] of dimensions 4 and 5, as also the appearance of a rotation axis of order 9, that first occurs in dimension 6.

This gives us the following number of essentially different colourings of the *vertices* of (E_6, ϖ_1) into n colours, for $n = 2, \dots, 10$:

$n = 2$	5550
3	155284437
4	350661193456
5	144058220931500
6	19758585250013658
7	1267988749077947862
8	46647074029346916224
9	1121791681317791814588
10	19290818437992445765750

However, since all elements of $W(\mathrm{E}_6)$ are *real* (= conjugate to their inverses), its action on the *vertices* of (E_6, ϖ_6) is exactly the same. This means that to calculate the cycle index of the action of $W(\mathrm{E}_6)$ on the *facets* of (E_6, ϖ_2), one only has to replace the variables in the above formula by their squares.

This gives us the following number of essentially different colourings of the *facets* of (E_6, ϖ_2) into n colours, for $n = 2, \dots, 10$:

$n = 2$	350661193456
3	1121791681317791814588
4	6260016398154707016138243072
5	10708171183809427472144240697 18750
6	2020703227080796075439136632727 3490800

[4]The root polytope of type E_6 folds to icosidodecahedron in dimension 3, which inherits part of these symmetries, but that's not crystallographic.

7	83296963322289671227853489024090078184319
8	112770188124732915609319727357553863872675840
9	65227467475959432188119059515773419707456153763
10	19290123457484567953333815587426819457226555750000

We were shocked to see that there are 350 661 193 456 essentially different colourings of the facets of the root polytope of type E_6 in 2 colours and already 1 121 791 681 317 791 814 588 such colourings in 3 colours. Even for a professional mathematician it is hard to develop the gut feeling of what *polynomial* size really means.

Theorem 2. *The cycle index of the Weyl group* $W(E_7)$ *in its action on the vertices of* (E_7, ϖ_7) *equals*

$$\begin{aligned}
Z_{56}[x_1,\dots,x_{30}] = \frac{1}{2903040}\Big(& x_1^{56} + 63x_1^{32}x_2^{12} + 945x_1^{16}x_2^{20} \\
&+ 4095x_1^8x_2^{24} + 672x_1^{20}x_3^{12} + 15680x_1^2x_3^{18} + 3780x_1^8x_4^{12} \\
&+ 48384x_1^6x_5^{10} + 161280x_1^2x_9^6 + 7560x_1^{12}x_2^2x_4^{10} + 52920x_1^4x_2^6x_4^{10} \\
&+ 10080x_1^8x_2^6x_6^6 + 90720x_1^4x_2^2x_8^6 + 40320x_1^2x_3^{10}x_6^4 + 181440x_1^2x_3^2x_6^8 \\
&+ 120960x_1^2x_3^2x_{12}^4 + 10080x_1^8x_2^6x_3^8x_6^2 + 30240x_1^4x_2^8x_3^4x_6^4 \\
&+ 145152x_1^2x_2^2x_5^6x_{10}^2 + 60480x_1^4x_4^4x_6^2x_{12}^2 + 5104x_2^{28} + 60480x_2^8x_4^{10} \\
&+ 26460x_2^4x_4^{12} + 51072x_2^{10}x_6^6 + 237440x_2x_6^9 + 90720x_2^4x_8^6 \\
&+ 193536x_2^3x_{10}^5 + 161280x_2x_{18}^3 + 120960x_2x_6x_{12}^4 + 60480x_2^2x_3^4x_4^4x_{12}^2 \\
&+ 120960x_2^2x_4^4x_6^2x_{12}^2 + 96768x_3^2x_5^4x_{15}^2 + 181440x_4^2x_8^6 \\
&+ 96768x_6x_{10}^2x_{30} + 207360x_7^8 + 207360x_{14}^4\Big)
\end{aligned}$$

This gives us the following number of essentially different colourings of the *vertices* of (E_7, ϖ_7) into n colours, for $n = 2, \dots, 10$:

$n = 2$	25233248480
3	180297145195775729262
4	1788578993276493332697527680

5	478043409291242022183613257247000
6	12990235192362148376418849830315701152
7	72884843049179495495383764040536401913143
8	128880215050332093241657695180553256160753664
9	94346872608175085448244095692323952023743549339
10	34446649032152502530270596015780468799225246452000

It is nothing special that such things can be easily done nowadays. What seems to be a bit special, is that the above calculation for the Schläfli polytope 2_{21} was essentially done by us *manually* within a couple of evenings[5].

The calculation for the Hesse polytope 3_{21} with 56 vertices, which by triality gives the colourings of the 56 facets of type 2_{21} of the root polytope 2_{31}, required somewhat more effort, but was still quite manageable. However, at that point we decided that to calculate by hand the cycle index on the 576 simplicial facets would be a bit too much of a good thing.

Similarly, performing such calculations by hand for the 1_{23} – not to say for 2_{14} and 1_{24} – would require much more leisure, and should be rather relegated to a computer. For the polytope 4_{12} it was indeed implemented by David Madore, and can be found at his home page [30]. Computer realisations for other small cases are straightforward[6].

§4. Our toolkit

In the above calculations we assumed the following background information and visual aids.

[5]Samuel Wagstaff: "Multiply $2071723 \times 5363222357$ by hand. Feel the joy."

[6]Well, because we calculate in the smallest representations of the Weyl groups $W(\mathrm{E}_l)$, of dimensions 6, 7 and 8, respectively. If one is interested in the explicit construction of the irreducible constituents of the corresponding permutation representation considered as a linear representation, it becomes a computational challenge on a completely different scale. To give some idea, John Stembridge [44] has computed *explicit* matrices of all irreducible complex representations of the exceptional Weyl groups. Just the construction of the matrix for the fundamental reflection w_8 in the largest irreducible complex representation R_{7168} of $W(\mathrm{E}_8)$ of dimension 7168 involved solution of 14597 [quadratic] Coxeter relations in 593 variables + vanishing of 2 matrix entries + one clone equation to distinguish it from another representation of $W(\mathrm{E}_8)$ having the same restriction to $W(\mathrm{E}_7)$. Too many, for a general purpose CAS.

• Most of the non-trivial calculations with root systems pertain to the cases $\Phi = \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$. As in our previous works that relied on massive computations in root systems, such as [25, 47–49, 51–53] we use the **hyperbolic realisation** of these systems in the $(l+1)$-dimensional Minkowski space [31]. This realisation is *considerably* more adapted to the large-scale calculations, than the usual realisations in Euclidean space.

• Classification of all **subsystems of root systems**, including the maximal ones, was obtained by Borel–de Siebenthal and Dynkin in 1948–1952, many further details are produced in [8, 19, 25, 40, 51]

• Most of our actual computations depend on an explicit knowledge of the **conjugacy classes of the Weyl groups**. An *ad hoc* description of the conjugacy classes of the exceptional Weyl groups was given by Sutherland Frame in 1951–1967. Roger Carter [7,8] proposed a conceptual explanation. Roughly the situation can be described as follows. Most – but by no means all! – of the conjugacy classes of the Weyl group $W(\Phi)$ are represented by the class $C(\Delta)$ of **Coxeter elements** corresponding to a subsystem $\Delta \leqslant \Phi$.

Other conjugacy classes come from **Carter graphs**, which are essentially bipartite Dynkin diagrams with cycles. The vertices of a Carter diagram C are partitioned into two sets $C = C_1 \sqcup C_2$, both consisting of pairwise orthogonal roots, and the remaining conjugacy classes are represented as products of two involutions $w = w_1 w_2$, where w_i, $i = 1, 2$, is the product of all root reflections w_α, $\alpha \in C_i$. These guys are called **semi-Coxeter elements**.

A further remarkable conceptual advance was made by Rafael Stekolshchik [43], who has modified Carter's list by observing that Carter graphs with long cycles are equivalent to Carter graphs containing only cycles of length 4. In other words, the graphs with cycles of length 6 in the original Carter's list (in our case one for D_6, one for E_7 and two for E_8), can be reduced to other forms, more suitable for actual computations.

In fact, all Carter diagrams, both in the original form and Stekolshchik form, can be readily accounted for by the **enhanced Dynkin diagrams** [19], which look as follows.

• The 8 vertex graph consisting of three squares with common edge, for E_6.

• The 11 vertex graph, consisting of the 4 vertices and the 6 edge midpoints of a tetrahedron + its centre joined to the vertices, for E_7.

- The 4×4 rectangular net on a torus, for E_8.

The details of the identification of all Dynkin diagrams of root subsystems and all Carter diagrams within an enhanced Dynkin diagram are described in our previous paper [37].

This considerably simplifies all calculations since the action of the fundamental reflections is clear from the weight diagram so that to construct representatives of all conjugacy classes we now need to record the action of only 2, 5 or 8 further root reflections, respectively.

- Our *major* device are **weight diagrams**, which are a standard tool in the representation theory of Lie algebras and algebraic groups, see [41, 46–49] for the details and many further references. There are two other ways to render exceptional polytopes as 2D pictures.

Weight graphs, like the usual **publicity photo** of E_8, as reproduced in hundreds of places [1,2,5,14,20,28,30,45]. These pictures are beautiful, but completely unsuitable for actual computations. The orthogonal projections to smaller dimensions, usually, 2D, 3D or 4D, which try to keep vertices distinct and faithfully depict all edges become a *complete mess*. Already for the root polytope of type E_8 with 240 vertices there are as many as 6720 edges, which makes the corresponding picture completely unfit for human calculations.

The **McMullen projections** [33] are *terribly* much handier, but they give a very schematic picture, where some of the vertices represent actual vertices, whereas some other represent higher dimensional faces, sometimes the whole facet! As a result, you should be able to use several of those in conjunction, in the same calculation. To visualise the whole symmetry of a multidimensional object with these pictures requires some *serious* mental exercise.

We chose the middle way. The pictures we use to visualise the polytopes are related to **Schreier graphs** depicting the cosets of the Weyl group modulo a parabolic subgroup. They are much more schematic and shorthand than the usual Coxeter like projections, and at the same time much more faithful and informative than McMullen diagrams. One such picture serves as a genuine shorthand reproduction of the whole multidimensional object. With *moderate* practice, all properties of this multidimensional object can be read off from such a picture purely combinatorially.

Roughly, the difference is as follows. All polytopes we consider can be scaled so that all of their vertices are *integral weights* = lattice points of $P(\Phi)$. We depict all vertices of the polytope, but [as a first approximation] only draw the edges that correspond to the *fundamental* roots, marking them accordingly.

The corresponding **weight graph** is obtained when you draw the edges corresponding to all positive roots, instead of drawing just the ones that correspond to the fundamental ones. The missing edges can be easily restored as those *paths* in these graphs, for which the multiplicities of marks coincide with the coefficients in the linear expansion of a given root with respect to the fundamental ones.

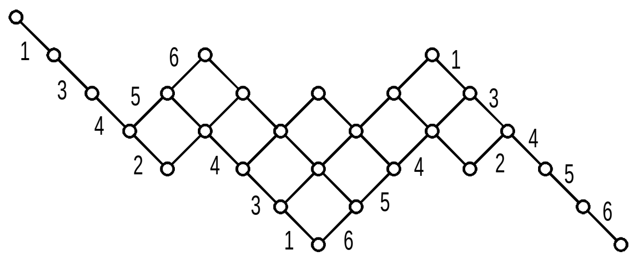

Figure 1. Weight diagram (E_6, ϖ_1)

For the three *microweight polytopes* – the Clebsch one, the Schläfli one and the Hesse one – all of their vertices are *extremal* weights of the corresponding representation. Moreover, the action of a root reflection consists in subtracting/adding the corresponding root. Thus, in these cases we get a *genuine* picture that fully captures all properties of the corresponding polytope. These are precisely the (E_6, ϖ_1) and (E_7, ϖ_7), reproduced in dozens of texts, including [41, 46–48, 52, 53].

§5. Sample calculations

Here we give some idea how we visualise exceptional polytopes with these means, and how Theorems 1 and 2 were proven (see [35] for all details).

5.1. Structure of exceptional polytopes. We start with repeating with our methods all results on the structure, number and adjacency of faces of the above polytopes. With our tools, such a description becomes immediate.

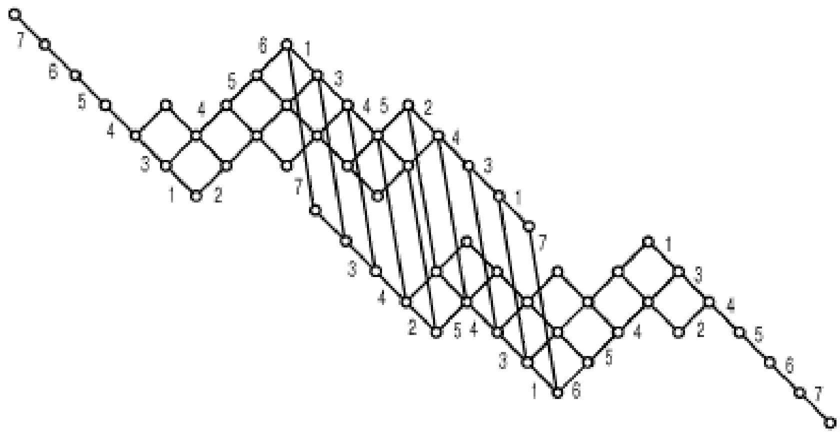

Figure 2. Weight diagram (E_7, ϖ_7)

For instance, look at Figure 1. Since the polytope is uniform, the highest weight $\varpi_1 =$ the left-most node of the diagram, is incident to faces of all types, which thus correspond to parabolic root subsystems containing α_1.

Since from ϖ_1 there are unique descending paths of lengths 1, 2 and 3, and the roots susbystems they generate have types A_1, A_2 and A_3, respectively. This means that there are only one type of the faces of dimensions 2 and 3 each, and they are triangles and tetrahedra.

Their number can be easily computed as well. Since the Weyl group $W(\mathrm{E}_6)$ acts transitively on roots, the number of edges equals $6 \cdot |\mathrm{E}_6^+|$. Alternatively, $W(\mathrm{E}_6)$ acts transitively on vertices, and there are 16 vertices at distance 1 from a given one in the weight graph. Thus, the number of edges equals $27 \cdot 16/2 = 36 \cdot 6 = 216$.

Obviously, in dimension 4 something funny happens. Namely, there are two *different* ways to embed $\mathrm{A}_3 = \langle \alpha_1, \alpha_3, \alpha_4 \rangle$ into A_4. One is to proceed with α_2, and this cannot be further embedded into A_5, and another one is to proceed with α_5, which can then be embedded into A_5 by further adjoining α_6. Both ways produce faces of type A_4, which are 4-simplices, but they form two distinct orbits.

Finally, there are two types of facets. There are 5-simplices α_5, 72 of them, that correspond to the roots of E_6, and there are 5-hyperoctahedra β_5, that correspond to the 27 pairs of non-comparable weights.

This accounts for the distinction between two types of 4-dimensional faces. Indeed, α_5 has 6 facets, which gives $72 \cdot 6 = 432$, whereas β_5 has 32 facets, which gives $27 \cdot 32 = 864$. This means that 432 of the 4-faces are common faces of an α_5 and a β_5, whereas the 216 remaining ones are shared by two β_5. Clearly, they form two distinct Weyl orbits.

Of course, the case of (E_6, ϖ_1) is by far the simplest one. Nevertheless, for all other cases the types of faces, their incidence numbers, etc. can be easily recuperated within half an hour by such similar means, perhaps with some little help of the tables of root subsystems, orders of the Weyl group, and the like. For that one even does not need the whole Schreier graph $W(\Phi)/W(\Delta)$ or, respectively, the whole weight diagram (Φ, ϖ_i), just the neighbourhood of the highest weight.

5.2. Weyl orbits. In each $\Phi = \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$ we fix a subset of roots Π^* that contains a given fundamental system Π and forms an **enhanced Dynkin diagram** of the corresponding type. For instance, in E_6 we adjoin to the usual fundamental root system Π the maximal root of E_6 itself, and the maximal root of D_4 spanned by $\alpha_2, \alpha_3, \alpha_4, \alpha_5$. This is done in a compatible way so that the calculations for E_6 can be then reused for E_7 and so on, see [37] for details.

Next, we fix representatives in all conjugacy classes as **Coxeter elements** or **semi-Coxeter elements**, with respect to the roots in the above Π^*. In [37] we observed this can be done. Such a choice is not unique, to simplify calculations in each case we minimise the number of occuring non-fundamental roots. The orders of the conjugacy classes themselves are taken from Carter's tables [7, 8].

For each representative we compute the sizes of its orbits on weights in the corresponding weight diagram. The action of the fundamental reflections w_i is obvious from the picture itself. They just interchange the end nodes of the edges marked by i. For other nodes, one has to find paths, whose sequence of marks coincides with the expansion of the root as a linear combination of the fundamental roots. But, as we observed above, there are only 2 further roots apart from the fundamental ones in E_6 and only 5 of them in E_7.

The most difficult and interesting case are the cuspidal classes that do not come from any smaller rank subsystem. In $W(\mathrm{E}_6)$ there are 5 such classes – the Coxeter classes of E_6 itself, $\mathrm{A}_5 + \mathrm{A}_1$ and $3\mathrm{A}_2$, and the semi-Coxeter classes $\mathrm{E}_6(a_1)$ and $\mathrm{E}_6(a_2)$. According to [37] we take

$x = w_1w_4w_6w_2w_3w_\gamma$, where $\gamma = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$, as a representative of $\mathrm{E}_6(a_1)$ and $y = w_1w_4w_6w_3w_5w_\gamma$ as a representative of $\mathrm{E}_6(a_2)$.

Tracing the action of x on Figure 1, we see that it has 3 orbits of size 9, which gives the most amazing last summand $5760x_9^3$ of Z_{27}. Similarly, tracing the action of y we see that it has one orbit of size 3 and 4 orbits of size 6, which contributes $720x_3x_6^4$ to the last summand $6480x_3x_6^4$ in the penultimate row. The other two types of elements with the same cycle type are the Coxeter elements of $\mathrm{A}_5 + A_1$, which comes with the coefficient 1440, and of A_5, which comes with the coefficient 4320. All other classes can be accounted for in the same leisurely style.

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