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MORE ON THE CONVERGENCE OF GAUSSIAN CONVEX HULLS

Abstract. A "law of large numbers" for consecutive convex hulls for weakly dependent Gaussian sequences $\{X_n\}$, having the same marginal distribution, is extended to the case when the sequence $\{X_n\}$ has a weak limit. Let $\mathbb{B}$ be a separable Banach space with a conjugate space $\mathbb{B}^*$. Let $\{X_n\}$ be a centered $\mathbb{B}$-valued Gaussian sequence satisfying two conditions: 1) $X_n \Rightarrow X$ and 2) For every $x^* \in \mathbb{B}^*$

$$\lim_{n,m,\,|n-m|\to\infty} E\langle X_n, x^*\rangle\langle X_m, x^*\rangle = 0.$$

Then with probability 1 the normalized convex hulls

$$W_n = \frac{1}{(2\ln n)^{1/2}} \operatorname{conv}\{ X_1, \ldots, X_n \}$$

converge in Hausdorff distance to the concentration ellipsoid of a limit Gaussian $\mathbb{B}$-valued random element X. In addition, some related questions are discussed.

§1. Introduction and formulation of results

Let $\mathbb{B}$ be a separable Banach space with a norm $||\cdot||$ and let $\mathbb{B}^*$ and $\langle \cdot, \cdot \rangle$ denote its conjugate space and the corresponding inner product, respectively. For $A \in \mathbb{B}$ the notation $\operatorname{conv}\{A\}$ is used for the closed convex hull of A. If X is a $\mathbb{B}$-valued centered Gaussian random element with a distribution $\mathcal{P}$ then by H we denote its reproducing kernel Hilbert space and $\mathcal{E}$ will stand for the closed unit ball in H, see, e.g., [8, p. 207]. The set $\mathcal{E}$ is also called *concentration ellipsoid* of X.

Finally, we introduce the separable complete metric space $\mathcal{K}_{\mathbb{B}}$ of all nonempty compact subsets of a Banach space $\mathbb{B}$ equipped with the Hausdorff distance $d_{\mathbb{B}}$:

$$d_{\mathbb{B}}(A,B) = \max\{\inf\{ \epsilon \mid A \subset B^{\epsilon}\},\ \ \inf\{ \epsilon \mid B \ \subset A^{\epsilon}\}\},$$

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where A^ϵ is the open ϵ-neighborhood of A. Convergence of compact sets in $\mathbb{B}$ always will be considered in this metric.

It is well known that the set $\mathcal{E}$ belongs to $\mathcal{K}_{\mathbb{B}}$ (see, e.g. the previous reference).

Investigation of the asymptotic behavior of convex hulls

$$W_n = \mathrm{conv}\{ X_1, \dots, X_n\}$$

of multivariate Gaussian random variables is an important part of Extreme Value Theory and has various applications, see for example, [9] and reference list, containing 160 items, in it. In 1988 Goodman [7] proved a fundamental result that the normalized set $\{ X_1, \dots, X_n\}$ of independent and identically distributed B-valued centered Gaussian random elements with a distribution $\mathcal{P}$ is approaching the concentration ellipsoid of $\mathcal{P}$ as n grows to infinity. From this result one can immediately derive that a.s.

$$\frac{1}{b(n)} W_n \overset{\mathcal{K}_{\mathbb{B}}}{\to} \mathcal{E}, \quad \text{as } n \to \infty, \tag{1}$$

where $b(t) = \sqrt{2\ln(t)}$, $t > 1$. Moreover, the rate of convergence in this relation is of the order $o(b(n)^{-1})$.

Later the convergence of the type (1) was proved first for stationary d-dimensional weakly dependent Gaussian sequences in [2], and then the similar result was proved for $\mathbb{B}$-valued Gaussian random fields on $\mathbb{R}^m$ or $\mathbb{Z}^m$ in [3]. Although at the introduction of the paper [3] it was said that only the case $m > 1$ is considered, inspection of the proof of the main result – Theorem 1.1 in [3] – shows that the result holds for $m = 1$, too. In particular, this result states that if a $\mathbb{B}$-valued centered Gaussian sequence $\{X_k, k \in \mathbb{N}\}$ has the same marginal distribution $\mathcal{P}$ and satisfies the following condition:

$$E \langle X_n, x^*\rangle \langle X_m, x^*\rangle \to 0, \quad \text{for all } x^* \in \mathbb{B}^* \ \text{ as } n, m, |n-m| \to \infty, \tag{2}$$

then (1) holds.

In the paper we show that the condition of equality of marginal distributions can be essentially relaxed substituting it by the weak convergence of the sequence $\{X_n\}$, and for weak convergence we use the sign $\Rightarrow$.

Theorem 1. *Suppose that a centered Gaussian sequence of* $\mathbb{B}$*-valued random elements* $\{X_k, k \in \mathbb{N}\}$ *satisfies* (2) *and the following condition:*

$$X_n \Rightarrow X. \tag{3}$$

Then a.s.

$$\frac{1}{b(n)} W_n \overset{\mathcal{K}_{\mathbb{B}}}{\to} \mathcal{E}, \quad \text{as } n \to \infty, \tag{4}$$

where $\mathcal{E}$ is concentration ellipsoid of X.

Since the proof of this theorem will be carried in two steps, and in the first step we consider the case $\mathbb{B} = \mathbb{R}$, we look more closely what is the meaning of the result in this particular case. Let $N(0, \sigma^2)$ stand for a Gaussian random variable with mean zero and variance σ^2, and $\{X_k\},\ k \in \mathbb{Z}_+$, be a sequence of $N(0, \sigma_k^2)$ random variables. Without loss of generality we can assume that X in (3) is $N(0,1)$. We have

$$W_n = [\min\{X_1, X_2, \ldots, X_n\}, \max\{X_1, X_2, \ldots, X_n\}] \tag{5}$$

and $\mathcal{E} = [-1, 1]$. If the covariance function $\rho(m,n) := EX_m X_n \to 0$ as $n, m, |n-m| \to \infty$, then we have the relation (4). It is clear, that if the dependence between elements X_k of the sequence is stronger, the sequence of their convex hulls is more concentrated. One can consider the extreme case, when $X_k \equiv X_0$ for all $k \geqslant 1$, then $W_n = \{X_0\}$ is one point and $\lim_{n\to\infty} (g(n))^{-1} W_n = \{0\}$ for any sequence $g(n) \to \infty$. The following example gives us additional information in this question.

Remark 2. Let us consider the sequence of i.i.d. $N(0,1)$ random variables $\{\xi_j\},\ j \geqslant 1$, and let $S_k = k^{-1/2} \sum_{j=1}^{k} \xi_j$. Taking $X_k = S_k$, we are in the setting of Theorem 1, but the condition (2) is not satisfied, since, if $n = m + k, k > 0$, then

$$\rho(m,n) = \frac{ES_m S_{m+k}}{(m(m+k))^{1/2}} = \frac{m}{(m(m+k))^{1/2}} = \left(1 + \frac{k}{m}\right)^{-1/2}. \tag{6}$$

Thus, in order to get $\rho(m,n) \to 0$, it is not sufficient to require $m \to \infty,\ k \to \infty$, but stronger condition is required $m \to \infty,\ k/m \to \infty$. On the other hand, S_k is a sum of i.i.d. random variables, therefore, denoting $c(t) = (2 \ln\ln t)^{1/2},\ t > e$, the classical LIL gives us that the cluster set for the sequence $\{X_n / c(n)\}$ is $[-1, 1]$, while for the sequence $\{X_n / b(n)\}$ with probability one limit is zero. We shall prove that for this example we have the following result:

Proposition 3. *With probability one*

$$\frac{1}{c(n)} W_n \overset{\mathcal{K}_{\mathbb{R}}}{\to} [-1, 1], \quad \textit{as } n \to \infty, \tag{7}$$

where W_n *is from* (5).

In connection with this example it is possible to formulate the following problem.

Suppose that a sequence $\{X_n\}$ has standard normal marginal distributions and covariance function $\rho(m,n)$. For which functions $g(n)$ and under what conditions for covariance function ρ we can get the relation (7) with function g instead of c?

This Proposition and Theorem 1 give us two examples of such functions g. What other normalizing functions are possible in relation (7)?

Let us make three final remarks.

Remark 4. As in [3], having (4), we can get some information on asymptotic behavior of $Ef(W_n)$ for some functions defined on $\mathcal{K}_{\mathbb{B}}$. In case $\mathbb{B} = \mathbb{R}^m$ typical examples of such functions are diameter, volume or surface measure.

Remark 5. From the proof of Theorem 1 we can extract some information about the rate of convergence in (4). Namely, in the proof we have the equality

$$\mathbb{P}\left\{d_{\mathbb{B}}(X, b(n)\mathcal{E}) > \varepsilon\right\} = \mathbb{P}\{\exp\frac{1}{2}\psi_\varepsilon \geqslant n\},$$

and since these probabilities are monotonically non-increasing and

$$\sum_n \mathbb{P}\{\exp\psi_\varepsilon/2 \geqslant n\} = E\{\exp\psi_\varepsilon/2\} < \infty$$

we get

$$\mathbb{P}\left\{d\left(\frac{X_n}{b_n}, \mathcal{E}\right) > \varepsilon\right\} = o(n^{-1}).$$

Let us note that this result cannot be compared with the result from [7], where it is proved that with probability 1

$$d_{\mathbb{B}}\left(\frac{X_n}{b_n}, \mathcal{E}\right) = o(b_n^{-1}).$$

Remark 6. In Theorem 1 and in previous results for Gaussian sequences limit set of convex hulls was ellipsoid of some Gaussian measure. If we dismiss the condition of weak convergence of Gaussian sequence $\{X_k\}$, the limit set may exist, but not necessarily will be an ellipsoid. For example, the following statement holds.

Proposition 7. *Let $\mathbb{B}$ be a separable Banach space. Let $V \subset \mathbb{B}$ be a central symmetric polytope, $V = \operatorname{conv}\{a_k, -a_k,\, k = 1, \dots, m\}$. Then there exists a sequence of independent Gaussian vectors $\{X_k\}$ such that a.s.*

$$\frac{1}{b(n)} W_n \ \to\ V.$$

§2. Proofs

Proof of Theorem 1. As it was mentioned above, the proof will be carried in two steps, and in the first step we consider the case $\mathbb{B} = \mathbb{R}$.

I. Without loss of generality we can suppose that X has a standard Gaussian distribution; then $\mathcal{E} = [-1, 1]$.

The condition (2) transforms now in

$$EX_nX_m \to 0 \quad \text{as} \quad n, m, |n - m| \to \infty.$$

From weak convergence of X_n to X it follows that $\sigma_n^2 := EX_n^2 \to 1$ as $n \to \infty$.

For r.v. $Y_n = X_n/\sigma_n$ the conditions (3) and (2) are fulfilled. Since Y_n are identically distributed, setting $U_n = \operatorname{conv}\{Y_1, \dots, Y_n\}$, by Theorem 1.1 from [3]

$$\frac{1}{b(n)} U_n \quad \overset{\mathcal{K}_1}{\longrightarrow} \quad [-1, 1], \ \ a.s. \quad \text{as } n \to \infty. \tag{8}$$

Let us show that a.s.

$$\Delta_n := d_{\mathbb{R}^1}\left(\frac{1}{b(n)} W_n,\ \frac{1}{b(n)} U_n\right) \to 0. \tag{9}$$

We have

$$\Delta_n \leqslant \frac{1}{b(n)} \max_{k \leqslant n}\{|X_k - Y_k|\} = \frac{1}{b(n)} \max_{k \leqslant n}\{|Y_k||1 - \sigma_k|\}. \tag{10}$$

Let $Z_n = \max\limits_{k \leqslant n}\{|Y_k|\}$; $\quad M_n = \max\limits_{k \leqslant n}\{|1 - \sigma_k|\}$. For $\varepsilon > 0$ find n_0 such that $\sup\limits_{k > n_0} |1 - \sigma_k| < \varepsilon$. Then for $n \geqslant n_0$ by (10)

$$\Delta_n \leqslant \frac{1}{b(n)} \max_{n_0 \leqslant k \leqslant n}\{|Y_k|\}\varepsilon + \frac{M_{n_0} Z_{n_0}}{b(n)}.$$

It follows from (8) that $\limsup_n \Delta_n \leqslant \varepsilon$. Hence we get (9) and (4) is proved for $\mathbb{B} = \mathbb{R}$.

II. General case. We shall show that with the probability 1 the sequence $\{b(n)^{-1} W_n\}$ is relatively compact in $\mathcal{K}_{\mathbb{B}}$. Due to Lemmas 2.2 and 2.3 from [3] it is sufficient to prove that there exists a compact set K such that for every $\varepsilon > 0$ with probability 1 for all sufficiently large n we have the following inclusion $\{b(n)^{-1}W_n\} \subset K^{\varepsilon}$. We take $K = \mathcal{E}$. Since the space $\mathbb{B}$ is fixed, instead of $d_{\mathbb{B}}$ we shall write simply d. It is clear that this inclusion will follow from the following relation: for every $\varepsilon > 0$ a.s.

$$\limsup_n d(X_n, (1+\varepsilon)b(n)\mathcal{E}) = 0. \tag{11}$$

By Skorokhod representation theorem we can suppose that $X_n \to X$ a.s.

Let $\sigma^2 = E\|X\|^2$, $\sigma^2_{\max} = \sup_n E\|X_n\|^2$. It follows from Fernique's theorem about integrability of exponential moments (see [6]) that for every γ, $0 < \gamma < (2\sigma^2)^{-1}$,

$$E\exp\{\gamma\|X\|^2\} \, < \, \infty.$$

Moreover, from the proof of Fernique's theorem one can deduce that for every γ, $0 < \gamma < (2\sigma^2_{\max})^{-1}$,

$$\limsup_n E\exp\{\gamma\|X_n\|^2\} \, < \, \infty.$$

It means, in particular, that for every γ, $0 < \gamma < (2\sigma^2_{\max})^{-1}$, the family $\exp\{\gamma\|X_n\|^2\}$ is uniformly integrable. Therefore

$$\delta_n^2 := E\|X_n - X\|^2 \;\to\; 0 \;\text{ and }\; E\exp\{\gamma\|X_n - X\|^2\} \;\to\; 1.$$

For $\varepsilon > 0$ let n_ε be such that $\sigma^2_\varepsilon := \sup_{n>n_\varepsilon} \delta_n^2 \, < \varepsilon^2$.

We have

$$\mathbb{P}\{d(X_n, (1+\varepsilon)b(n)\mathcal{E}) \, > \, \varepsilon\} \, = \, A_n + B_n,$$

where

$$A_n = \mathbb{P}\{d(X_n, (1+\varepsilon)b(n)\mathcal{E}) \, > \, \varepsilon, \; \|X_n - X\| < \varepsilon b(n)\},$$
$$B_n = \mathbb{P}\{d(X_n, (1+\varepsilon)b(n)\mathcal{E}) \, > \, \varepsilon, \; \|X_n - X\| \geqslant \varepsilon b(n)\}.$$

Evidently

$$A_n \leqslant \mathbb{P}\{d(X, b(n)\mathcal{E}) > \varepsilon\}.$$

Now we formulate Talagrand's lemma [11] as it is formulated in [7], see Lemma 3.1 therein.

Lemma 8. *Let X be a $\mathbb{B}$-valued centered Gaussian random element with a concentration ellipsoid $\mathcal{E}$. Then for any $\varepsilon > 0$ there is a random variable ψ_ε such that*

$$E\{\exp\{\psi_\varepsilon/2\}\} < \infty$$

and for all $\lambda > 0$

$$\mathbb{P}\{d(X, \lambda\mathcal{E}) \leqslant \varepsilon\} = \mathbb{P}\{\psi_\varepsilon < \lambda^2\}.$$

We apply this lemma taking $\lambda = b(n)$ and obtain

$$A_n \leqslant \mathbb{P}\{d(X, b(n)\mathcal{E}) > \varepsilon\} = \mathbb{P}\{\psi_\varepsilon \geqslant 2\ln n\} = \mathbb{P}\{\exp\{\psi_\varepsilon/2\} \geqslant n\}$$

Hence $\sum_n A_n < \infty$.

For $0 < \gamma < (2\sigma_\varepsilon^2)^{-1}$ we have $E\exp\{\gamma\|X_n - X\|^2\} < \infty$ for each $n > n_\varepsilon$, therefore, denoting

$$L_\varepsilon(a) = \sup_{n > n_\varepsilon} \{E\exp\{a\|X_n - X\|^2\}\},$$

for $a \in \left(\frac{1}{2\varepsilon^2}, \frac{1}{2\sigma_\varepsilon^2}\right)$ and $n \geqslant n_\varepsilon$ we apply once more Fernique's theorem and get

$$B_n \leqslant \mathbb{P}\{\|X_n - X\| > \varepsilon b(n)\} \leqslant \frac{E\exp\{a\|X_n - X\|^2\}}{n^{2a\varepsilon^2}} \leqslant \frac{L_\varepsilon(a)}{n^{2a\varepsilon^2}}.$$

Since $2a\varepsilon^2 > 1$, this estimate gives the convergence of the series $\sum_n B_n$.

Therefore we see that for every $\varepsilon > 0$

$$\sum_n \mathbb{P}\{d(X_n, (1+\varepsilon)b(n)\mathcal{E}) \ > \ \varepsilon\} < \infty.$$

Then the Borel–Cantelli lemma gives us (11), which shows that for every $\delta > 0$ with probability 1 for all sufficiently large n

$$\frac{1}{b(n)} W_n \subset \mathcal{E}^\delta.$$

This proves the relative compactness of $\{b(n)^{-1}W_n\}$.

It follows from Lemma 2.7 [3] that now it is sufficient to prove the convergence for every $\theta \in S_1^*(0) := \{x^* \in \mathbb{B}^* : ||x^*|| = 1\}$

$$M_n(\theta) \overset{\text{a.s.}}{\to} M_{\mathcal{E}}(\theta), \quad n \to \infty, \tag{12}$$

where $M_n, M_{\mathcal{E}}$ are support functions for $b(n)^{-1}W_n$ and $\mathcal{E}$, respectively. We recall that a function $M_A(\theta)$, defined by the relation

$$M_A(\theta) := \sup_{x \in A} \langle x, \theta\rangle, \quad A \in \mathcal{K}_{\mathbb{B}},\ \theta \in S_1^*(0),$$

is called the support function of a set $A \in \mathcal{K}_{\mathbb{B}}$. Since

$$M_n(\theta) = \frac{1}{b(n)} \max_{k \leqslant n} \{\langle X_k, \theta\rangle\},$$

the convergence (12) follows from the first part of the proof. □

Proof of Proposition 3. Let us denote

$$c_1 := \liminf_n \frac{V(n)}{c(n)}.$$

Let $l \geqslant 2$ be a fixed integer, then we have $c(l^m) \sim b(m)$, as $m \to \infty$. $V(n)$ is non-decreasing, therefore, for $n \in [l^k, l^{k+1}]$, we have

$$\frac{V(n)}{c(n)} \geqslant \frac{c(l^k)}{c(n)} \frac{V(l^k)}{c(l^k)}.$$

Since

$$\frac{c(l^k)}{c(n)} \geqslant \frac{c(l^k)}{c(l^{k+1})} \to 1 \quad \text{as } n \to \infty,$$

we get

$$c_1 \geqslant \liminf_m \left\{ \frac{V(l^m)}{c(l^m)} \right\} \geqslant \liminf_m \left\{ \frac{\max\{X_l, X_{l^2}, \dots, X_{l^m}\}}{c(l^m)} \right\}.$$

In our example we have (6), therefore $r := \sup_{i \neq j} E X_{l^i} X_{l^j} = l^{-1/2}$. Due to Lemma 2.5 from [3] we get

$$\liminf_m \left\{ \frac{X_l, X_{l^2}, \dots, X_{l^m}}{c(l^m)} \right\} \geqslant \sqrt{1-r} = \sqrt{1 - l^{-1/2}}.$$

This quantity can be made close to 1 if we choose l sufficiently large, therefore with probability one we have

$$c_1 \geqslant 1. \tag{13}$$

In order to get the estimate from above for

$$c_2 := \limsup_n \left\{ \frac{V(n)}{c(n)} \right\}$$

we shall need the following lemmas.

Lemma 9. *Suppose that a sequence of random variables $\{Y_k\}$ satisfies the following condition: for all $\gamma < (2\sigma^2)^{-1}$, $\sigma > 0$,*

$$\sup_n E \exp\{\gamma Y_n^2\} < \infty. \tag{14}$$

Then

$$\limsup_n \left\{ \frac{\max_{k \leqslant n}\{Y_k\}}{b(n)} \right\} \leqslant \sigma.$$

The proof of this lemma coincides with the proof of Lemma 1 in [1], despite of the fact that in this paper the variables $\{Y_k\}$ was i.i.d. It turns out that independence is not used at all and condition of identical distributions of Y_k can be replaced by condition (14).

Lemma 10 ([10, Theorem 2.2])**.** *Let* $\{\xi_k\}$, $k \geqslant 1$, *be independent symmetric random variables,* $S_n = \sum_{k=1}^{n} \xi_k$. *Then for every* $x \geqslant 0$

$$\mathbb{P}\left\{\max_{1\leqslant k\leqslant n} |S_k| \geqslant x\right\} \leqslant 2\,\mathbb{P}\{|S_n| \geqslant x\}.$$

Lemma 11. *Let* X *and* Y *be two non-negative random variables with distribution functions* F *and* G*. If for some* $a \geqslant 1$ *and for all* $x \geqslant 0$ *we have* $1 - F(x) \leqslant a(1 - G(x))$*. Then for every* $\gamma > 0$

$$E\exp\{\gamma X^2\} \leqslant aE\exp\{\gamma Y^2\}.$$

Elementary proof of this statement follows from equalities

$$E\exp\{\gamma X^2\} = \sum_{k=0}^{\infty} \gamma^k (k!)^{-1} E(X^{2k}),\ E(X^{2k}) = \int_0^\infty (1 - F(x))d(x^{2k}).$$

Let us fix a non-integer (we have in mind that we shall choose a close to 1) number $a > 1$ and let us denote

$$\Delta_j = \{i \in N : a^j \leqslant i \leqslant a^{j+1}\},\quad Y_j = \max_{i\in\Delta_j} X_i.$$

As in the case of lower bound we can prove that

$$c_2 \leqslant \limsup_m \left\{\frac{V(\lfloor a^m \rfloor)}{c(\lfloor a^m \rfloor)}\right\}.$$

Note that

$$\Delta_j = \{i \in N : \lfloor a^j \rfloor + 1 \leqslant i \leqslant \lfloor a^{j+1} \rfloor\}$$

and $V(\lfloor a^m \rfloor) = \max_{0\leqslant j\leqslant m-1} Y_j$, therefore

$$\begin{aligned}|Y_j| &\leqslant \max_{i\in\Delta_j}\{|X_i|\} = \max_{i\in\Delta_j}\left\{\left|\frac{S_i}{\sqrt{i}}\right|\right\}\\ &\leqslant \frac{1}{\sqrt{\lfloor a^j \rfloor + 1}} \max_{i\in\Delta_j}\{|S_i|\} \leqslant \frac{1}{\sqrt{a^j}} \max_{i\in[\lfloor a^j \rfloor+1, \lfloor a^{j+1} \rfloor]}\{|S_i|\}.\end{aligned}$$

Applying Lemma 10, we have

$$\mathbb{P}\{\max_{i\in\Delta_j} |S_i| \geqslant x\} \leqslant 2\mathbb{P}\{|S_{\lfloor a^{j+1} \rfloor}| \geqslant x\},$$

whence

$$\mathbb{P}\{|Y_j| \geqslant x\} \leqslant 2\mathbb{P}\left\{\left|\frac{S_{\lfloor a^{j+1}\rfloor}}{\sqrt{a^j}}\right| \geqslant x\right\}.$$

Let $\xi(j,a) = (a^j)^{-1} S_{\lfloor a^{j+1}\rfloor}$. It is easy to see that $\xi(j,a)$ has distribution $N(0, \sigma^2(j,a))$ with

$$\sigma^2(j,a) = \frac{\lfloor a^{j+1}\rfloor}{a^j} \to a \text{ as } j \to \infty, \quad \text{and} \quad \sigma^2(j,a) \leqslant a. \tag{15}$$

Applying Lemma 11 with $\gamma < 1/2a$, we get

$$E\exp\{\gamma Y_j^2\} \leqslant 2E\exp\{\gamma\xi(j,a)^2\}.$$

Due to (15) we have $\sup_j E\exp\{\gamma\xi(j,a)^2\} := C(a) < \infty$, therefore, using Lemma 9 with $\sigma^2 = a$ and recalling that $c(\lfloor a^m\rfloor) \sim b(m)$, as $m \to \infty$, we get that with probability 1

$$c_2 \leqslant \limsup_m \frac{1}{b(m)} V(\lfloor a^m\rfloor) \leqslant \sqrt{a}.$$

Since the last estimate holds for any $a > 1$, we get that with probability 1

$$c_2 \leqslant 1. \tag{16}$$

Estimates (13) and (16) prove (7).

Proof of Proposition 7. Let $\mathbb{N} = \cup_{k=1}^m T_k$, where the sets T_k, $k = 1, \dots, m$, are disjoint and have positive densities p_k. Let $\{X_n\}$ be a sequence of independent random vectors such that for each k and $j \in T_k$, X_j has Gaussian distribution concentrated on the line $\{ta_k, t \in R^1\}$ with zero mean and variance $\sigma_k^2 = \| a_k \|$. We denote $W_n^{(k)} = \text{conv}\{X_j\ j \leqslant n,\ j \in T_k\}$. Since for any $p > 0$,

$$\lim_n \frac{b(np)}{b(n)} = 1,$$

Theorem 1 implies that a.s. for any $k = 1, \dots, m$,

$$\frac{1}{b(n)} W_n^{(k)} \ \to\ \text{conv}\{-a_k, a_k\}.$$

Clearly, we have $W_n = \text{conv}\{W_n^{(1)}, \dots, W_n^{(m)}\}$, therefore a.s.

$$\frac{1}{b(n)} W_n \ \to\ \text{conv}\{\cup_{k=1}^m \text{conv}\{-a_k, a_k\}\} \ =\ V. \qquad \square$$

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