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ON THE PREDICTION ERROR FOR SINGULAR STATIONARY PROCESSES AND TRANSFINITE DIAMETERS OF RELATED SETS

Abstract. The paper is devoted to the prediction problem for discrete-time singular stationary processes with spectral density f and related topics in the case where f vanishes on a set of positive Lebesgue measure. We first discuss the Fekete theorem and its extension due to Robinson on the transfinite diameters of related sets, and prove an extension of Robinson's theorem. For some special sets the transfinite diameters are calculated explicitly by using Robinson's theorem. The obtained results are applied to describe the asymptotic behavior of the prediction error. Then we discuss the Davisson theorem concerning upper bound for the prediction error, and prove its extension. As an application, we obtain estimates for the minimal eigenvalue of a Toeplitz matrix associated with spectral density f.

Dedicated to our teacher Academician Il'dar Abdullovich Ibragimov on the occasion of his 90th birthday.

§1. Introduction

One of the fundamental result of geometric complex analysis is the classical theorem by Fekete and Szegő, stating that for any closed bounded set F in the complex plane $\mathbb{C}$ the transfinite diameter, the Chebyshev constant and the capacity of F *coincide*, although they are defined from very different points of view. Namely, the transfinite diameter of the set F characterizes the asymptotic size of F, the Chebyshev constant of F characterizes the minimal uniform deviation of a monic polynomial on F, and the capacity of F describes the asymptotic behavior of the Green function at infinity.

Key words and phrases: Prediction error, singular stationary process, transfinite diameter, Robinson's theorem, Davisson's theorem, eigenvalues of truncated Toeplitz matrices.

It is worth to note that in only very few cases can the transfinite diameter (and hence, the capacity and the Chebyshev constant) be exactly calculated.

In 1930, Fekete [9] proved that if F is a bounded closed set in the complex w-plane $\mathbb{C}_w$ and F^* is the preimage of F in the z-plane $\mathbb{C}_z$ under the mapping $w = p(z)$, where $p(z) = z^n + \dots$ is an arbitrary monic polynomial of degree n ($n \in \mathbb{N} := \{1, 2, \dots\}$), then the transfinite diameters $\tau(F)$ and $\tau(F^*)$ of the sets F and F^* are related by the formula: $\tau(F^*) = \sqrt[n]{\tau(F)}$.

In 1969, Robinson [17] has extended Fekete's result to the case where the mapping is carried out by a rational function $w = \varphi(z)$ instead of a polynomial, and proved that if F is a bounded closed set of the complex plane $\mathbb{C}_w$ lying on the unit circle $\mathbb{T}$ and symmetric with respect to real axis, and if F^x is the projection of F onto the real axis, then $\tau(F) = \sqrt{2\tau(F^x)}$.

The notion of the transfinite diameter plays an important role in the prediction theory of second-order stationary processes. This issue goes back to the classical paper of Rosenblatt [18], where he proved that if the spectral density f of a discrete-time stationary process $X(t)$ is positive and continuous on a segment of length 2α and is zero elsewhere, then the best linear one-step ahead prediction error $\sigma_n(f)$ of $X(0)$ based on the finite past of length n of the process $X(t)$ approaches zero exponentially as $n \to \infty$. More precisely, using the technique of orthogonal polynomials and Szegő's results, Rosenblatt proved in [18] that the following asymptotic relation holds:

$$\lim_{n\to\infty} \sqrt[n]{\sigma_n(f)} = \sin(\alpha/2).$$

Later in Babayan [1, 2] and Babayan et al. [5] Rosenblatt's result was extended to the case of several segments, without having to stipulate continuity of the spectral density $f(\lambda)$. Using constructive methods, Davisson [7] obtained an upper bound (rather than an asymptote) for the prediction error $\sigma_n(f)$.

In this paper we prove extensions of the above quoted Robinson's and Davisson's theorems. Then using Robinson's theorem and some properties of the transfinite diameter, we explicitly calculate the transfinite diameter for some special sets. For these special cases, we obtain asymptotic formulas and upper bounds for the prediction error $\sigma_n(f)$, extending the above mentioned Rosenblatt's relation and Davisson's bound. Finally, as an application, we obtain estimates for the minimal eigenvalue of a Toeplitz matrix generated by the spectral density f.

§2. Preliminaries

In this section we introduce some metric characteristics of bounded closed sets in the plane, such as, the transfinite diameter, the Chebyshev constant and the capacity, and briefly discuss some properties of these characteristics. For the definitions and results stated below we refer the reader to the following references: Fekete [9], Goluzin [12], Chapter 7, and Tsuji [22], Chapter III (see also Babayan et al. [5]).

Transfinite diameter. Let F be an infinite bounded closed (compact) set in the complex plane $\mathbb{C}$. Given any natural number $n \geqslant 2$, choose n points $z_1, \ldots, z_n \in F$ so as to maximize the product of the distances between them. Then the geometric mean of these distances, denoted by $d_n(F)$, is called the nth *transfinite diameter* of the set F. Note that $d_2(F)$ is the diameter of F. Fekete [8] (see also Goluzin [12, p. 294]) proved that the sequence $d_n(F)$ is non-increasing and does not exceed the diameter $d_2(F)$ of F, implying that $d_n(F)$ has a finite limit as $n \to \infty$. This limit, denoted by $d_\infty(F)$, is called the *transfinite diameter* of the set F. Thus,

$$d_\infty(F) := \lim_{n\to\infty} d_n(F). \tag{2.1}$$

If F is empty or consists of a finite number of points, then we put $d_\infty(F) = 0$.

Chebyshev constant. Let F be as before, we put $m_n(F) := \inf \max_{z\in F} |q_n(z)|$, where the infimum is taken over all monic polynomials $q_n(z)$ from the class:

$$\mathcal{Q}_n := \left\{q_n \,:\, q_n(z) = z^n + c_1 z^{n-1} + \cdots + c_n\right\}. \tag{2.2}$$

Then there exists a unique monic polynomial $T_n(z) := T_n(z, F)$ form the class $\mathcal{Q}_n$, called the *Chebyshev polynomial* of F of order n, such that $m_n(F) = \max_{z\in F} |T_n(z, F)|$. Fekete [9] proved that $\lim_{n\to\infty} (m_n(F))^{1/n}$ exists. This limit, denoted by $\tau(F)$, is called the *Chebyshev constant* for the set F. Thus,

$$\tau(F) := \lim_{n\to\infty} (m_n(F))^{1/n}. \tag{2.3}$$

Along with the polynomial $T_n(z)$ we also consider the Chebyshev auxiliary polynomial $t_n(z) := t_n(z, F)$, which deviates least from zero on the set F in the uniform metric, among all the monic polynomials, having roots only on F. We set $\mu_n(F) = \max_{z\in F} |t_n(z, F)|$. Then, as it is known (see

Goluzin [12, Section 7.1, p. 295]), we have

$$\lim_{n\to\infty}(\mu_n(F))^{1/n} = \tau(F). \tag{2.4}$$

Capacity. Let F be as above, and let D_F denote the complementary domain to F, containing the point $z=\infty$. If the boundary $\Gamma := \partial D_F$ of the domain D_F consists of a finite number of rectifiable Jordan curves, then for the domain D_F can be constructed a Green function $G_F(z,\infty) := G_{D_F}(z,\infty)$ with a pole at infinity. This function is harmonic everywhere in D_F, except at the point $z=\infty$, is continuous including the boundary Γ and vanishes on Γ. It is known that in a vicinity of the point $z=\infty$ the function $G_F(z,\infty)$ admits the representation (see, e.g., Goluzin [12, pp. 309–310]):

$$G_F(z,\infty) = \ln|z| + \gamma + O(z^{-1}) \quad \text{as} \quad z\to\infty. \tag{2.5}$$

The number γ in (2.5) is called the *Robin's constant* of the domain D_F, and the number

$$C(F) := e^{-\gamma} \tag{2.6}$$

is called the *capacity* (or the *logarithmic capacity*) of the set F.

Now we are in position to state the above mentioned fundamental result of geometric complex analysis, due to M. Fekete and G. Szegő (see, e.g., Goluzin [12, p. 197], or Tsuji [22, p. 73]).

Proposition 1 (Fekete - Szegő's theorem)**.** *For any compact set $F\subset\mathbb{C}$, the transfinite diameter $d_\infty(F)$ defined by* (2.1)*, the Chebyshev constant $\tau(F)$ defined by* (2.3)*, and the capacity $C(F)$ defined by* (2.6) *coincide, that is,*

$$d_\infty(F) = C(F) = \tau(F). \tag{2.7}$$

It what follows, we will use the term "transfinite diameter" and the notation $\tau(F)$ for (2.7).

In the next proposition we list some properties of the transfinite diameter (and hence, of the capacity and the Chebyshev constant), which will be used later (see, e.g., Tsuji [22, pp. 56, 84] and Babayan et al. [5]).

Proposition 2. *The transfinite diameter possesses the following properties.*

(a) *The transfinite diameter is monotone, that is, for any closed sets F_1 and F_2 with $F_1\subset F_2$, we have $\tau(F_1)\leqslant\tau(F_2)$.*

(b) *If a set F_1 is obtained from a compact set $F \subset \mathbb{C}$ by a linear transformation, that is, $F_1 := aF + b = \{az + b : z \in F\}$, then $\tau(F_1) = |a|\tau(F)$. In particular, the transfinite diameter $\tau(F)$ is invariant with respect to parallel translation and rotation of F.*

(c) *The transfinite diameter of an arbitrary circle of radius R is equal to its radius R. In particular, the transfinite diameter of the unit circle $\mathbb{T}$ is equal to 1.*

(d) *The transfinite diameter of an arc Γ_α of a circle of radius R with central angle α is equal to $R\sin(\alpha/4)$. In particular, for the unit circle $\mathbb{T}$, we have $\tau(\Gamma_\alpha) = \sin(\alpha/4)$.*

(e) *The transfinite diameter of an arbitrary line segment F is equal to one-fourth its length, that is, if $F := [a, b]$, then $\tau(F) = \tau([a, b]) = (b - a)/4$.*

§3. Transfinite diameters of related sets

In this section we state the theorems of Fekete and Robinson on the transfinite diameters of related sets, calculate transfinite diameters of some special sets, and prove an extension of Robinson's theorem.

3.1. Fekete's and Robinson's theorems. The following classical theorem about the relationship between transfinite diameters of a compact set and its preimage under a mapping given by a polynomial was proved by Fekete [9] (see also Goluzin [12, pp. 299–300]).

Theorem A (Fekete [9])**.** Let F be a bounded closed set in the complex w-plane $\mathbb{C}_w$, and let $p(z) := p_n(z) = z^n + c_1z^{n-1} \cdots + c_n$ be an arbitrary monic polynomial of degree n. Let F^* be the preimage of F in the z-plane $\mathbb{C}_z$ under the mapping $w = p(z)$, that is, F^* is the set of all points $z \in \mathbb{C}_z$ such that $w := p(z) \in F$. Then

$$\tau(F^*) = [\tau(F)]^{1/n}, \tag{3.1}$$

where $\tau(F)$ and $\tau(F^*)$ stand for the transfinite diameters of the sets F and F^*, respectively.

In 1969, Robinson [17], developing the idea of Fekete's proof of Theorem A, extends Theorem A to the case where the mapping is carried out by a rational function instead of a polynomial. More precisely, in Robinson [17] the following theorem was proved.

Theorem B (Robinson [17])**.** Let $p(z) := p_n(z) = z^n + a_1 z^{n-1} \cdots + a_n$ and $q(z) := q_k(z)$ be arbitrary relatively prime polynomials of degrees n and k, respectively, with $k < n$. Let F be a bounded closed set in the complex w-plane $\mathbb{C}_w$, and let F^* be the preimage of F in the z-plane $\mathbb{C}_z$ under the mapping $w = \varphi(z) := p(z)/q(z)$. Assume that $|q(z)| = 1$ for all $z \in F^*$. Then

$$\tau(F^*) = [\tau(F)]^{1/n}. \tag{3.2}$$

Remark 1. It is clear that the condition $|q(z)| = 1$ for all $z \in F^*$ in Theorem B can be replaced by the condition $|q(z)| = C$ for all $z \in F^*$ with an arbitrary positive constant C, and, in this case, in view of Proposition 2(b), the relation (3.2) becomes

$$\tau(F^*) = [C\tau(F)]^{1/n}. \tag{3.3}$$

Observe that in the special case where $q(z) \equiv 1$, Theorem B reduces to the Fekete theorem (Theorem A). A special interest represents the other special case where $p(z) = z^2 + 1$ and $q(z) = 2z$. In this case, the mapping given by the rational function

$$\varphi(z) := \frac{p(z)}{q(z)} = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

projects the subsets of the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ onto the real axis $\mathbb{R}$, and, in view of Remark 1, Theorem B reads as follows.

Theorem C (Robinson [17])**.** Let F be a bounded closed subset of the complex plane $\mathbb{C}$ lying on the unit circle $\mathbb{T}$ and symmetric with respect to real axis, and let F^x be the projection of F onto the real axis. Then

$$\tau(F) = [2\tau(F^x)]^{1/2}. \tag{3.4}$$

3.2. Calculation of transfinite diameters of some sets. Examples. In this section we give examples of calculation of transfinite diameters of some specific subsets of the unit circle, using formula (3.4) and the properties of the transfinite diameter listed in Proposition 2.

We will use the following notation: given $0 < \beta < 2\pi$ and $z_0 = e^{i\theta_0}$, $\theta_0 \in (-\pi, \pi]$, we denote by $\Gamma_\beta(\theta_0)$ an arc of the unit circle of length β which is symmetric with respect to the point $z_0 = e^{i\theta_0}$, that is,

$$\Gamma_\beta(\theta_0) := \{e^{i\theta} : |\theta - \theta_0| \leqslant \beta/2\} = \{e^{i\theta} : \theta \in [\theta_0 - \beta/2, \theta_0 + \beta/2]\}. \tag{3.5}$$

Example 1. Let $\Gamma_{2\alpha} := \Gamma_{2\alpha}(0)$. Then the projection $\Gamma_{2\alpha}^x$ of $\Gamma_{2\alpha}$ onto the real axis is the segment $[\cos\alpha, 1]$ (see Fig. 1a)), and by Proposition 2(e) for the transfinite diameter $\tau(\Gamma_{2\alpha}^x)$ we have

$$\tau(\Gamma_{2\alpha}^x) = \frac{1-\cos\alpha}{4} = \frac{\sin^2(\alpha/2)}{2}.$$

Hence, according to formula (3.4), we obtain

$$\tau(\Gamma_{2\alpha}) = [2\tau(\Gamma_{2\alpha}^x)]^{1/2} = \left[2\frac{\sin^2(\alpha/2)}{2}\right]^{1/2} = \sin(\alpha/2). \qquad (3.6)$$

Taking into account that the transfinite diameter is invariant with respect to rotation (see Proposition 2(b)), from (3.6) for any $\theta_0 \in (-\pi, \pi]$ we have

$$\tau(\Gamma_{2\alpha}(\theta_0)) = \sin(\alpha/2). \qquad (3.7)$$

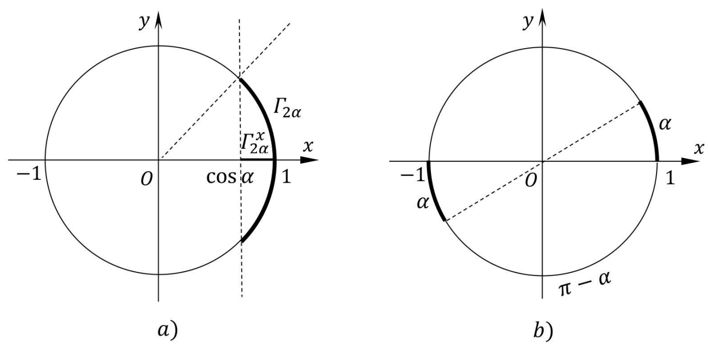

Figure 1. a) The sets $\Gamma_{2\alpha}$ and $\Gamma_{2\alpha}^x$. b) The set $\Gamma(k,\alpha)$ with $k=2$.

Remark 2. Notice that the expression $\sin(\alpha/2)$ in (3.6) was first obtained by Szegő [21], where he calculated it as the Chebyshev constant of the arc $\Gamma_{2\alpha}(\pi/2)$, then it was deduced by Rosenblatt [18], as the capacity of $\Gamma_{2\alpha}(\pi/2)$.

Example 2. Let $\Gamma_{2\alpha}(\alpha)$ be an arc of length 2α, defined by (3.5): $\Gamma_{2\alpha}(\alpha) = \{e^{i\theta} : \theta \in [0, 2\alpha]\}$, and let $\Gamma(2,\alpha)$ be the preimage of the arc $\Gamma_{2\alpha}(\alpha)$ under the mapping $p(z) = z^2$. It can be shown that the set $\Gamma(2,\alpha)$ is the union of two closed arcs of equal lengths α, symmetrically located with respect to the center of the unit circle (see Fig. 1b):

$$\Gamma(2,\alpha) = \{e^{i\omega} : \omega \in [-\pi, -\pi+\alpha] \cup [0,\alpha]\}. \tag{3.8}$$

Then, by the Fekete theorem (Theorem A) and formula (3.7), for the transfinite diameter $\tau(\Gamma(2,\alpha))$ we have

$$\tau(\Gamma(2,\alpha)) = [\tau(\Gamma_{2\alpha}(\alpha))]^{1/2} = (\sin(\alpha/2))^{1/2}. \tag{3.9}$$

The above result can easily be extended to the case of k ($k > 2$) arcs. Let $\Gamma(k,\alpha)$ be the union of k ($k \in \mathbb{N}$, $k \geqslant 2$) closed arcs of equal lengths α, which are symmetrically located on the unit circle (the arcs are assumed to be equidistant). It can be shown that the set $\Gamma(k,\alpha)$ is the preimage (to within rotation) under the mapping $p(z) = z^k$ of the arc $\Gamma_{k\alpha}(k\alpha/2)$ of length $k\alpha$ defined by (3.5). Therefore, by (3.1) and the invariance property of the transfinite diameter with respect to rotation (see Proposition 2(b)), for the transfinite diameter $\tau(\Gamma(k,\alpha))$, we have

$$\tau(\Gamma(k,\alpha)) = (\sin(k\alpha/4))^{1/k}. \tag{3.10}$$

Example 3. Let $\alpha > 0$, $\delta \geqslant 0$ and $\alpha + \delta \leqslant \pi$. Consider the set

$$\Gamma_{\alpha,\delta}(\theta_0) := \Gamma_{\alpha+\delta}(\theta_0) \setminus \Gamma_\delta(\theta_0) \tag{3.11}$$

consisting of the union of two arcs of the unit circle of lengths α, the distance of which (over the circle) is equal to 2δ. Define (see Fig. 2a)):

$$\Gamma_{\alpha,\delta} := \Gamma_{\alpha,\delta}(0) = \{e^{i\theta} : \theta \in [-(\delta+\alpha), -\delta] \cup [\delta, \delta+\alpha]\}. \tag{3.12}$$

Then the projection $\Gamma^x_{\alpha,\delta}$ of $\Gamma_{\alpha,\delta}$ onto the real axis is the segment $\Gamma^x_{\alpha,\delta} = [\cos(\alpha+\delta), \cos\delta]$, and by Proposition 2(e) for the transfinite diameter $\tau(\Gamma^x_{\alpha,\delta})$ we have

$$\tau(\Gamma^x_{\alpha,\delta}) = \frac{\cos\delta - \cos(\alpha+\delta)}{4} = \frac{\sin(\alpha/2)\sin(\alpha/2+\delta)}{2}.$$

Hence, according to formula (3.4), for the transfinite diameter $\tau(\Gamma_{\alpha,\delta})$, we obtain

$$\tau(\Gamma_{\alpha,\delta}) = [2\tau(\Gamma^x_{\alpha,\delta})]^{1/2} = (\sin(\alpha/2)\sin(\alpha/2+\delta))^{1/2}. \tag{3.13}$$

In view of Proposition 2(b), from (3.13) for any $\theta_0 \in (-\pi, \pi]$ we have

$$\tau(\Gamma_{\alpha,\delta}(\theta_0)) = (\sin(\alpha/2)\sin(\alpha/2+\delta))^{1/2}. \tag{3.14}$$

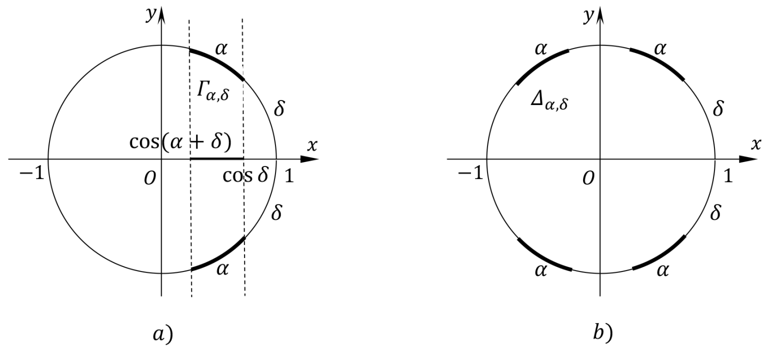

Figure 2. a) The set $\Gamma_{\alpha,\delta}$. b) The set $\Delta_{\alpha,\delta}$.

Observe that for $\delta = 0$ we have $\Gamma_{\alpha,\delta}(\theta_0) = \Gamma_{2\alpha}(\theta_0)$ (see (3.5) and (3.11)), and formula (3.14) becomes (3.7).

Example 4. Let the arc $\Gamma_{\alpha,\delta}$ be as in Example 3 (see (3.12)) with α, δ satisfying $\alpha + \delta \leqslant \pi/2$, that is, $\Gamma_{\alpha,\delta}$ is a subset of the right semicircle $\mathbb{T}$. Denote by $\Gamma'_{\alpha,\delta}$ the symmetric to $\Gamma_{\alpha,\delta}$ set with respect to y-axis, that is,

$$\Gamma'_{\alpha,\delta} := \{e^{i\theta} : \theta \in [-\pi + \delta, -\pi + (\delta + \alpha)] \cup [\pi - (\delta + \alpha), \pi - \delta]\}.$$

Define $\Delta_{\alpha,\delta} := \Gamma_{\alpha,\delta} \cup \Gamma'_{\alpha,\delta}$, and observe that the set $\Delta_{\alpha,\delta}$ consists of four arcs of equal lengths α, which are symmetrically located with respect to both axes (see Fig. 2b)). The set $\Delta_{\alpha,\delta}$ is the preimage (to within rotation) of the set $\Gamma_{2\alpha,2\delta}$ under the mapping $p(z) = z^2$. Hence, according to formulas (3.1) and (3.13), for the transfinite diameter $\tau(\Delta_{\alpha,\delta})$, we obtain

$$\tau(\Delta_{\alpha,\delta}) = (\tau(\Gamma_{2\alpha,2\delta}))^{1/2} = (\sin\alpha \sin(\alpha + 2\delta))^{1/4}. \tag{3.15}$$

Denote by $\Delta_{\alpha,\delta}(\theta_0)$ the image of the set $\Delta_{\alpha,\delta}$ under mapping $q(z) = e^{i\theta_0} z$, that is, under the rotation by the angle θ_0 around the origin. Then, in view of Proposition 2(b), from (3.15) for any $\theta_0 \in (-\pi, \pi]$ we have

$$\tau(\Delta_{\alpha,\delta}(\theta_0)) = (\sin\alpha \sin(\alpha + 2\delta))^{1/4}. \tag{3.16}$$

3.3. An extension of Robinson's theorem. Returning to the Robinson theorem (Theorem B), observe that the condition $|q(z)| = C$ for all

$z \in F^*$ (see Remark 1) is too restrictive, and it essentially reduces the range of applicability of the theorem into the following two cases:

(a) $q(z) \equiv 1$, and Theorem B reduces to the Fekete theorem (Theorem A);

(b) $p(z) = z^2 + 1$ and $q(z) = 2z$, and, in this case, the rational function $\varphi(z) = (z^2+1)/(2z)$ projects the subsets of the unit circle $\mathbb{T}$ onto the real axis $\mathbb{R}$.

Therefore, the question of extending Robinson's theorem to the case where the condition $|q(z)| = C$ is replaced by a weaker condition becomes topical. The next theorem provides such an extension.

Theorem 1. *Let the polynomials $p(z)$, $q(z)$, the sets F, F^*, and the mapping $w = \varphi(z) := p(z)/q(z)$ be as in Theorem B, and let $m := \min_{z\in F^*} |q(z)|$ and $M := \max_{z\in F^*} |q(z)|$. Then the following inequalities hold:*

$$[m\tau(F)]^{1/n} \leqslant \tau(F^*) \leqslant [M\tau(F)]^{1/n}. \tag{3.17}$$

Proof. Observe first that the polynomials $p(z)$ and $q(z)$ have no common zeros because by assumption the fraction $\varphi(z) = p(z)/q(z)$ is noncancellable. Also, the polynomial $q(z)$ has no roots on the set F^*, because otherwise the function $\varphi(z)$ would have a pole on the set F^*, making the set F unbounded. Therefore, $\varphi(z)$ is continuous on F^* and F^* is a closed set. Next, since $\varphi(z) \sim z^{n-k}$ as $z \to \infty$ and $n > k$ from the boundedness of F follows boundedness of F^*. Thus, F^* also is a bounded closed set, and hence $\min_{z\in F^*} |q(z)| = m$ and $\max_{z\in F^*} |q(z)| = M$ are attained, and $m > 0$.

We first prove the second inequality in (3.17). To this end, for an arbitrary $\nu \in \mathbb{N}$ by $T_\nu(w) = w^\nu + c_1 w^{\nu-1} + \ldots$ and $T^*_\nu(z) = z^\nu + \ldots$ we denote the Chebyshev polynomials for sets F and F^*, respectively, and set $m_\nu := m_\nu(F) = \max_{w\in F} |T_\nu(w)|$ and $m^*_\nu := m_\nu(F^*) = \max_{z\in F^*} |T^*_\nu(z)|$. According to the definition of the preimage $\varphi^{-1}(F) = F^*$, we have the equivalence relation: $z \in F^* \Leftrightarrow w = \varphi(z) \in F$. Therefore for the polynomial $s_{n\nu}(z)$ of degree $n\nu$:

$$s_{n\nu}(z) := q^\nu(z) T_\nu(\varphi(z)) = p^\nu(z) + c_1 p^{\nu-1}(z) q(z) + \ldots \tag{3.18}$$

we have

$$\begin{aligned}\max_{z\in F^*} |s_{n\nu}(z)| &= \max_{z\in F^*} |q^\nu(z) T_\nu(\varphi(z))| \leqslant M^\nu \max_{z\in F^*} |T_\nu(\varphi(z))| \\ &= M^\nu \max_{w\in F} |T_\nu(w)| = M^\nu m_\nu.\end{aligned}$$

From (3.18) and the condition $n > k$, we conclude that the leading term of the polynomial $s_{n\nu}(z)$ is equal to 1, that is, $s_{n\nu}(z) \in \mathcal{Q}_n$. Therefore

$$m^*_{n\nu} = \max_{z\in F^*} |T^*_{n\nu}(z)| \leqslant \max_{z\in F^*} |s_{n\nu}(z))| \leqslant M^\nu m_\nu,$$

implying that

$$(m^*_{n\nu})^{1/(n\nu)} \leqslant \left(M\,(m_\nu)^{1/\nu}\right)^{1/n}. \tag{3.19}$$

Passing to the limit in (3.19) as $\nu \to \infty$ and using (2.3), we obtain the second inequality in (3.17).

To prove the first inequality in (3.17), we choose an arbitrary point $w \in F$ and consider the parametric monic polynomial $p_w(z) := p(z) - wq(z)$, the roots of which we denote by $z_i = z_i(w)$ $(i = 1, \ldots, n)$:

$$p_w(z) = p(z) - wq(z) = (\varphi(z) - w)q(z) = \prod_{i=1}^{n}(z - z_i). \tag{3.20}$$

Observe that $q(z_i) \neq 0$ $(i = 1, \ldots, n)$ because otherwise by (3.20) we would have $p(z_i) = 0$ for some $i = 1, \ldots, n$, which contradicts the assumptions that $p(z)$ and $q(z)$ are relatively prime polynomials. Therefore setting $z = z_i$ in (3.20) we get

$$\varphi(z_i) = \frac{p(z_i)}{q(z_i)} = w \in F, \quad i = 1, \ldots, n,$$

implying that all the roots z_i belong to the set F^*.

Let $t^*_\nu(z) := t^*_\nu(z, F^*)$ be the Chebyshev auxiliary polynomial for the set F^* and let $\mu^*_\nu := \mu_\nu(F^*) = \max_{z\in F^*} |t^*_\nu(z)|$. Denote by z^*_j $(j = 1, \ldots, \nu)$ the roots of the polynomial $t^*_\nu(z)$:

$$t^*_\nu(z) = \prod_{j=1}^{\nu}(z - z^*_j). \tag{3.21}$$

Setting $z = z_i$ $(i = 1, \ldots, n)$ in (3.21) and multiplying all the obtained equalities, we get

$$\prod_{i=1}^{n} t^*_\nu(z_i) = \prod_{i=1}^{n}\prod_{j=1}^{\nu}(z_i - z^*_j) = \prod_{j=1}^{\nu}\prod_{i=1}^{n}(z_i - z^*_j). \tag{3.22}$$

Similarly, setting $z = z_j^*$ $(j = 1, \dots, \nu)$ in (3.20) and multiplying all the obtained equalities, we get

$$\prod_{j=1}^{\nu} p_w(z_j^*) = \prod_{j=1}^{\nu} (p(z_j^*) - wq(z_j^*)) = \prod_{j=1}^{\nu}\prod_{i=1}^{n} (z_j^* - z_i)$$

or, equivalently,

$$\prod_{j=1}^{\nu} \left(\varphi(z_j^*) - w\right) = \prod_{j=1}^{\nu} \frac{1}{q(z_j^*)} \prod_{i=1}^{n} (z_j^* - z_i) \tag{3.23}$$

Hence, taking into account that $z_i, z_j^* \in F^*$, from (3.22) and (3.23), for all $w \in F$, we obtain

$$\left|\prod_{j=1}^{\nu} \left(w - \varphi(z_j^*)\right)\right| \leqslant \frac{1}{m^\nu} \prod_{j=1}^{\nu}\prod_{i=1}^{n} |z_i - z_j^*| = \frac{1}{m^\nu} \prod_{i=1}^{n} |t_\nu^*(z_i)| \leqslant \frac{(\mu_\nu^*)^n}{m^\nu}. \tag{3.24}$$

Now observe that the product under the modulus on the left-hand side of (3.24) is a polynomial $r_\nu(w)$ of degree ν of the variable w with leading term equal to 1. Therefore, in view of the definition of the polynomial $T_\nu(w)$, we have

$$m_\nu = m_\nu(F) = \max_{w\in F} |T_\nu(w)| \leqslant \max_{w\in F} |r_\nu(w))| \leqslant \frac{(\mu_\nu^*)^n}{m^\nu},$$

implying that

$$\left(m\,(m_\nu)^{1/\nu}\right)^{1/n} \leqslant (\mu_\nu^*)^{1/\nu}. \tag{3.25}$$

Passing to the limit in (3.25) as $\nu \to \infty$ and using (2.3) and (2.4), we obtain the first inequality in (3.17). This completes the proof of Theorem 1. □

Remark 3. If the condition $|q(z)| = C$ is satisfied for all $z \in F^*$, then we have $m = M = C$, and Theorem 1 reduces to Robinson's Theorem B (see Remark 1).

Remark 4. Theorem 1 can easily be extended to more general case where $p(z) := p_n(z)$ is an arbitrary (not necessarily monic) polynomial of degree n: $p(z) = az^n + a_1 z^{n-1} \cdots + a_n$, $a \neq 0$. Indeed, in this case, canceling the fraction $\varphi(z) := p(z)/q(z)$ by a, we get $\varphi(z) := p_1(z)/q_1(z)$, where now $p_1(z) = p(z)/a = z^n +$ lower order terms, is a monic polynomial. Also, we have $\min_{z\in F^*}|q_1(z)| = m/|a|$ and $\max\limits_{z\in F^*}|q_1(z)| = M/|a|$, where m and M are as in Theorem 1. Hence we can apply the inequality (3.17) to obtain

$$\left[\frac{m}{|a|}\tau(F)\right]^{1/n} \leqslant \tau(F^*) \leqslant \left[\frac{M}{|a|}\tau(F)\right]^{1/n}. \tag{3.26}$$

§4. Asymptotic behavior and estimation of the prediction error for singular stationary processes

4.1. The prediction problem. Let $X(t)$, $t \in \mathbb{Z} := \{0, \pm1, \dots\}$, be a second-order stationary stochastic sequence possessing a spectral density function $f(\lambda)$, $\lambda \in \Lambda := [-\pi, \pi]$. The *"finite" linear prediction problem* is as follows. Suppose we observe a finite realization of the process $X(t)$: $\{X(t),\ -n \leqslant t \leqslant -1\}$, $\ n \in \mathbb{N}$. We want to predict the random variable $X(0)$, which is the unobserved one-step ahead value of the process $X(t)$, using the *linear predictor* $Y = \sum\limits_{k=1}^{n} c_k X(-k)$. The coefficients c_k, $k = 1, 2, \dots, n$, are chosen so as to minimize *the mean-squared error*: $\mathbb{E}\left|\mathrm{X}(0) - \mathrm{Y}\right|^2$, where $\mathbb{E}[\cdot]$ stands for the expectation operator. If such minimizing constants $\widehat{c}_k := \widehat{c}_{k,n}$ can be found, then the random variable $\widehat{X}_n(0) := \sum\limits_{k=1}^{n} \widehat{c}_k X(-k)$ is called *the best linear one-step ahead predictor* of the random variable $X(0)$ based on the observed finite past: $X(-n), \dots, X(-1)$. The minimum mean-squared error:

$$\sigma_n^2(f) := \mathbb{E}\left|X(0) - \widehat{X}_n(0)\right|^2 > 0$$

is called *the best linear one-step ahead prediction error* of $X(0)$ based on the finite past of length n of the process $X(t)$.

One of the main problem in prediction theory of second-order stationary processes is to describe the asymptotic behavior of the prediction error $\sigma_n^2(f)$ as $n \to \infty$. This behavior depends on the regularity (deterministic or nondeterministic) of the observed process $X(t)$.

Observe that $\sigma_{n+1}^2(f) \leqslant \sigma_n^2(f)$, $n \in \mathbb{N}$, and hence the limit of $\sigma_n^2(f)$ as $n \to \infty$ exists. Denote by $\sigma^2(f) := \sigma_\infty^2(f)$ the prediction error by the entire infinite past: $\{X(t),\, t \leqslant -1\}$.

From the prediction point of view it is natural to distinguish the class of processes for which we have *error-free prediction* by the entire infinite past, that is, $\sigma^2(f) = 0$. Such processes are called *singular* or *deterministic*. Processes for which $\sigma^2(f) > 0$ are called *nondeterministic*.

The well-known *Kolmogorov–Szegő theorem* gives a spectral characterization of deterministic and nondeterministic processes, and states that the following limiting relation hold (see, e.g., Grenander and Szegő [13, p. 44]):

$$\lim_{n\to\infty} \sigma_n^2(f) = \sigma^2(f) = 2\pi G(f), \tag{4.1}$$

where $G(f)$ is the *geometric mean* of $f(\lambda)$, namely

$$G(f) = \begin{cases} \exp\left\{\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \ln f(\lambda)\, d\lambda\right\} & \text{if} \quad \ln f \in L^1(\Lambda) \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

The condition $\ln f \in L^1(\Lambda)$ in (4.2) is equivalent to the *Szegő condition*:

$$\int\limits_{-\pi}^{\pi} \ln f(\lambda)\, d\lambda > -\infty \tag{4.3}$$

(this equivalence follows because $\ln f(\lambda) \leqslant f(\lambda)$ and $f(\lambda) \in L^1(\Lambda)$). The Szegő condition (4.3) is also called the *non-determinism condition*.

Observe that the Szegő condition is related to the character of the singularities (zeroes and poles) of the spectral density f, and does not depend on the differential properties of f (see, e.g., Babayan et al. [5]).

Define the *relative prediction error* $\delta_n(f) := \sigma_n^2(f) - \sigma^2(f)$, and observe that $\delta_n(f)$ is nonnegative and tends to zero as $n \to \infty$. But what about the speed of convergence of $\delta_n(f)$ to zero as $n \to \infty$? The prediction problem we are interested in is to describe the rate of decrease of $\delta_n(f)$ to zero as $n \to \infty$, depending on the regularity nature (deterministic or nondeterministic) of the observed process $X(t)$.

The prediction problem stated above goes back to classical works of A. Kolmogorov, G. Szegő and N. Wiener. It was then considered by many authors for different classes of nondeterministic processes (see, e.g., the survey papers Bingham [6] and Ginovyan [11], and references therein).

We focus in this paper on singular processes, that is, when $\sigma^2(f) = 0$, and hence $\delta_n(f) = \sigma_n^2(f)$. This case is not only of theoretical interest, but is also important from the point of view of applications. For example, as pointed out by Rosenblatt [18] (see also Pierson [15]), situations of this type arise in Neumann's theoretical model of storm-generated ocean waves. Such models are also of interest in meteorology (see, e.g., Fortus [10]).

Only few works are devoted to the study of the speed of convergence of $\sigma_n^2(f)$ to zero as $n \to \infty$, that is, the asymptotic behavior of the prediction error for deterministic processes. One needs to go back to the classical work of M. Rosenblatt [18]. Using the technique of orthogonal polynomials on the unit circle and Szegő's results, M. Rosenblatt investigated the asymptotic behavior of the prediction error $\sigma_n^2(f)$ for discrete-time deterministic processes in the following two cases:

(a) the spectral density $f(\lambda)$ is continuous and positive on a segment of $[-\pi, \pi]$ and is zero elsewhere,

(b) the spectral density $f(\lambda)$ has a very high order of contact with zero at points $\lambda = 0, \pm\pi$, and is strictly positive otherwise.

Later the problems (a) and (b) were studied by Babayan [1,2], Babayan and Ginovyan [3,4], Babayan et al. [5], (see also Davisson [7] and Fortus [10]), where some generalizations and extensions of Rosenblatt's results have been obtained.

In this paper we discuss the case (a).

4.2. Asymptotic behavior of the prediction error. Taking into account that the formula $z = e^{i\lambda}$ establishes a bijection between the interval $(-\pi, \pi]$ and the unit circle $\mathbb{T}$, the spectral density f can also be considered as a function defined on $\mathbb{T}$.

For the case (a) above, that is, when the spectral density $f(\lambda)$ is continuous and positive on a segment of $[-\pi, \pi]$ and is zero elsewhere, M. Rosenblatt proved in [18] that the prediction error $\sigma_n^2(f)$ decreases to zero exponentially as $n \to \infty$. More precisely, M. Rosenblatt proved the following theorem.

Theorem D (Rosenblatt [18])**.** Let the spectral density $f(\lambda)$ of a discrete-time stationary process $X(t)$ be positive and continuous on the segment $[\pi/2 - \alpha, \pi/2 + \alpha]$, $0 < \alpha < \pi$, and zero elsewhere. Then the prediction error $\sigma_n^2(f)$ approaches zero exponentially as $n \to \infty$. More precisely, the following asymptotic relation holds:

$$\sigma_n^2(f) \simeq \left(\sin(\alpha/2)\right)^{2n+1} \quad \text{as} \quad n \to \infty. \tag{4.4}$$

Notice that the relation (4.4) implies that

$$\lim_{n\to\infty} \sqrt[n]{\sigma_n(f)} = \sin(\alpha/2). \tag{4.5}$$

In what follows, by E_f we denote the spectrum of the process $X(t)$, that is, $E_f := \{e^{i\lambda} : f(\lambda) > 0\}$. Thus, the closure $\bar{E}_f$ of E_f is the support of the spectral density f.

The next result, which was proved in Babayan et al. [5] (see also Babayan [1,2]) extends Rosenblatt's theorem (Theorem D). More precisely, the theorem that follows extends the asymptotic relation (4.5) to the case of several arcs, without having to stipulate continuity of the spectral density f.

Theorem E (Babayan et al. [5])**.** Let the support $\bar{E}_f$ of the spectral density f of the process $X(t)$ consist of a finite number of closed arcs of the unit circle $\mathbb{T}$, and let $f > 0$ a.e. on $\bar{E}_f$. Then the sequence $\sqrt[n]{\sigma_n(f)}$ converges, and

$$\lim_{n\to\infty} \sqrt[n]{\sigma_n(f)} = \tau_f, \tag{4.6}$$

where $\tau_f := \tau(\bar{E}_f)$ is the transfinite diameter of $\bar{E}_f$.

Remark 5. Theorem E shows that the question of exponential decay of the prediction error $\sigma_n(f)$ as $n \to \infty$ in fact does not depend on the form of the spectral density $f(\lambda)$ and is determined solely by the value of the transfinite diameter of the support $\bar{E}_f$ of the spectral density f.

Remark 6. In Theorem D we have $\bar{E}_f := \{e^{i\lambda} : \lambda \in [\pi/2 - \alpha, \pi/2 + \alpha]\}$, which represents a closed arc of length 2α, and, according to Proposition 2(d), we have $\tau(\bar{E}_f) = \sin(\alpha/2)$. Thus, the asymptotic relation (4.5) is a special case of (4.6).

Now we apply Theorem E to obtain the asymptotic behavior of the prediction error $\sigma_n(f)$ in the cases where the spectrum of a stationary process $X(t)$ is as in Examples 1–4. Specifically, putting together Theorem E and Examples 1–4, we obtain the following result.

Theorem 2. *Let $\bar{E}_f$ be the support of the spectral density f of a stationary process $X(t)$, and let $f > 0$ a.e. on $\bar{E}_f$. Then for the prediction error $\sigma_n(f)$ the following assertions hold.*

(a) *If $\bar{E}_f = \Gamma_{2\alpha}(\theta_0)$, where $\Gamma_{2\alpha}(\theta_0)$ is as in Example 1, then* $\lim_{n\to\infty} \sqrt[n]{\sigma_n(f)} = \sin(\alpha/2)$.

(b) *If $\bar{E}_f = \Gamma(k, \alpha)$, where $\Gamma(k, \alpha)$ is as in Example 2, then* $\lim_{n\to\infty} \sqrt[n]{\sigma_n(f)} = (\sin(k\alpha/4))^{1/k}$.

(c) *If $\bar{E}_f = \Gamma_{\alpha,\delta}(\theta_0)$, where $\Gamma_{\alpha,\delta}(\theta_0)$ is as in Example 3, then* $\lim_{n\to\infty} \sqrt[n]{\sigma_n(f)} = (\sin(\alpha/2)\sin(\alpha/2 + \delta))^{1/2}$.

(d) *If $\bar{E}_f = \Delta_{\alpha,\delta}(\theta_0)$, where $\Delta_{\alpha,\delta}(\theta_0)$, is as in Example 4, then* $\lim_{n\to\infty} \sqrt[n]{\sigma_n(f)} = (\sin\alpha \sin(\alpha + 2\delta))^{1/4}$.

Remark 7. The assertion (a) is a slight extension of Rosenblatt's relation (4.5). The assertion (c) is an extension of assertion (a), which reduces to assertion (a) if $\delta = 0$.

4.3. Davisson's theorem and its extension. Using constructive methods, Davisson [7] obtained an upper bound (rather than an asymptote) for the prediction error $\sigma_n^2(f)$ without imposing continuity requirement on the spectral density $f(\lambda)$. Specifically, in Davisson [7] the following result was proved:

Theorem F (Davisson [7])**.** Let the spectral density $f(\lambda)$, $\lambda \in [-\pi, \pi]$ of the process $X(t)$ be identically zero on a closed interval of length $2\pi - 2\alpha$, $0 < \alpha < \pi$. Then for the prediction error $\sigma_n^2(f)$ the following inequality holds:

$$\sigma_n^2(f) \leqslant 4c\,(\sin(\alpha/2))^{2n-2}, \quad \text{where} \quad c = \int_{-\pi}^{\pi} f(\lambda)\,d\lambda. \tag{4.7}$$

Here we extend Davisson's theorem to the case where the spectrum of the process $X(t)$ consists of a union of two equal arcs.

Let $\alpha > 0$, $\delta \geqslant 0$ and $\alpha + \delta \leqslant \pi$, and let $\Gamma_{\alpha,\delta}$ be the set defined by (3.12). Recall that $\Gamma_{\alpha,\delta}$ is the union of two arcs of the unit circle of lengths α, the distance between which (over the circle) is equal to 2δ (see Example 3 and Figure 2a)).

Theorem 3. *Let the spectral density $f(\lambda)$, $\lambda \in [-\pi, \pi]$ of the process $X(t)$ be nonnegative on the set $\Gamma_{\alpha,\delta}$ $(\alpha > 0, \delta \geqslant 0, \alpha + \delta \leqslant \pi)$ and vanishes outside $\Gamma_{\alpha,\delta}$. Then for the prediction error $\sigma_n^2(f)$ the following inequality holds:*

$$\sigma_n^2(f) \leqslant 4c\,(\sin(\alpha/2))^{n-1}\,(\sin(\alpha/2 + \delta))^{n-1}, \tag{4.8}$$

where c is as in Theorem F.

Proof. According to the definition of the prediction error $\sigma_n^2(f)$ (see, e.g., Babayan et al. [5, formulas (3.1), (3.2), (3.11)]), we have

$$\sigma_n^2(f) = \min_{\{q_n \in \mathcal{Q}_n\}} \int_{-\pi}^{\pi} |q_n(e^{i\lambda})|^2 f(\lambda)\,d\lambda = \int_{-\pi}^{\pi} |p_n(e^{i\lambda})|^2 f(\lambda)\,d\lambda, \tag{4.9}$$

where $p_n(z) := p_n(z, f)$ is the optimal polynomial for spectral density $f(\lambda)$ in the class $\mathcal{Q}_n$ given by (2.2). Thus, for any polynomial $q_n \in \mathcal{Q}_n$ we have

$$\begin{aligned}\sigma_n^2(f) &\leqslant \int_{-\pi}^{\pi} |q_n(e^{i\lambda})|^2 f(\lambda)\, d\lambda = \int_{\delta\leqslant|\lambda|\leqslant\delta+\alpha} |q_n(e^{i\lambda})|^2 f(\lambda)\, d\lambda \\ &\leqslant \max_{\delta\leqslant|\lambda|\leqslant\delta+\alpha} |q_n(e^{i\lambda})|^2 \int_{-\pi}^{\pi} |f(\lambda)\, d\lambda = c \max_{\delta\leqslant|\lambda|\leqslant\delta+\alpha} |q_n(e^{i\lambda})|^2.\end{aligned} \tag{4.10}$$

To obtain the upper bound for $\sigma_n^2(f)$, we construct a special sequence of polynomials $q_n(z)$, $n \in \mathbb{N}$ as follows. If n is even $n = 2m$, $m \in \mathbb{N}$, then we choose m values λ_k with $\delta \leqslant \lambda_k \leqslant \delta + \alpha$, $k = 1, 2, \ldots, m$, and the corresponding points $z_k = e^{i\lambda_k}$ on the unit circle, and difine the polynomials $q_n \in \mathcal{Q}_n$ with zeros z_k and $\bar{z}_k$:

$$q_n(z) = q_{2m}(z) := \prod_{k=1}^{m} (z - z_k)(z - \bar{z}_k), \quad z = e^{i\lambda}. \tag{4.11}$$

If n is odd $n = 2m + 1$, $m \in \mathbb{N}$, then we set

$$q_n(z) = q_{2m+1}(z) := z q_{2m}(z) = \prod_{k=1}^{m} z(z - z_k)(z - \bar{z}_k), \quad z = e^{i\lambda}. \tag{4.12}$$

Thus, for $z = e^{i\lambda}$ we have

$$|q_{2m+1}(z)| = |q_{2m}(z)|. \tag{4.13}$$

The next step is to choose the zeros z_k in (4.11) and (4.12) so as to minimize the factor $\max_{\delta\leqslant|\lambda|\leqslant\delta+\alpha} |q_n(e^{i\lambda})|^2$ in (4.10) among all polynomials $q_n \in \mathcal{Q}_n$. To do this, observe first that in view of equality $|z| = |z_k| = 1$ $(k = 1, 2, \ldots, m)$, we can write

$$\begin{aligned}&|(z - z_k)(z - \bar{z}_k)|^2 = |(e^{i\lambda} - e^{i\lambda_k})(e^{i\lambda} - e^{-i\lambda_k})|^2 \\ &= |(e^{2i\lambda} - e^{i\lambda}(e^{i\lambda_k} + e^{-i\lambda_k}) + 1|^2 \\ &= |e^{2i\lambda} + 1 - 2e^{i\lambda}\cos\lambda_k)|^2 = |e^{i\lambda}(e^{i\lambda} + e^{-i\lambda} - 2\cos\lambda_k)|^2 \\ &= |2\cos\lambda - 2\cos\lambda_k|^2 = 4(\cos\lambda - \cos\lambda_k)^2.\end{aligned} \tag{4.14}$$

In view of relations (4.11)–(4.14), the inequality (4.10) becomes

$$\sigma_n^2(f) \leqslant 4^m c \cdot \max_{\delta\leqslant|\lambda|\leqslant\delta+\alpha} \prod_{k=1}^{m} (\cos\lambda - \cos\lambda_k)^2. \tag{4.15}$$

Denote $a := \cos(\delta+\alpha)$, $b := \cos\delta$ $(a < b)$, and consider the mapping $y = y(x)$ given by formula

$$y = \frac{b-a}{2}x + \frac{b+a}{2}. \tag{4.16}$$

Observe that this mapping is bijection and maps $[-1,1]$ to $[a,b]$: $y(-1) = a$ and $y(1) = b$.

Next, since $\delta \leqslant \lambda \leqslant \delta+\alpha$ and $\delta \leqslant \lambda_k \leqslant \delta+\alpha$, we have $\cos\lambda \in [a,b]$ and $\cos\lambda_k \in [a,b]$. Therefore, there are $x \in [-1,1]$ and $x_k \in [-1,1]$ such that $y(x) = \cos\lambda$ and $y(x_k) = \cos\lambda_k$, $k = 1,2,\dots,m$. So, in view of (4.16), we have

$$\begin{aligned} &\frac{b-a}{2}x + \frac{b+a}{2} = \cos\lambda, \\ &\frac{b-a}{2}x_k + \frac{b+a}{2} = \cos\lambda_k, \quad k = 1,2,\dots,m. \end{aligned} \tag{4.17}$$

In view of relations in (4.17), we have

$$\begin{aligned} \cos\lambda - \cos\lambda_k &= \frac{b-a}{2}(x-x_k) = \frac{\cos\delta - \cos(\delta+\alpha)}{2}(x-x_k) \\ &= \sin(\alpha/2)\sin(\alpha/2+\delta)(x-x_k). \end{aligned} \tag{4.18}$$

Substituting (4.18) into (4.15) we obtain

$$\sigma_n^2(f) \leqslant 4^m c \cdot \sin^{2m}(\alpha/2)\sin^{2m}(\alpha/2+\delta) \max_{x\in[-1,1]} \prod_{k=1}^{m}(x-x_k)^2. \tag{4.19}$$

Consider the sequence of Chebyshev polynomials $T_m(x)$ which have least deviation from the zero in the segment $[-1,1]$ in the uniform metric among all polynomials $r_m(x)$ of degree at most m:

$$\max_{x\in[-1,1]} |T_m(x)| \leqslant \max_{x\in[-1,1]} |r_m(x)|.$$

It is known that the polynomials $T_m(x)$ are given by formula (see, e.g., Goluzin [12, p. 298]):

$$T_m(x) = \cos(m \arccos x), \quad x \in [-1,1], \tag{4.20}$$

from which it follows that

$$\max_{x\in[-1,1]} |T_m(x)| = 1. \tag{4.21}$$

Besides, Chebyshev polynomials $T_m(x)$ satisfy the recurrence relations:

$$T_0(x) \equiv 1, \quad T_1(x) = x, \quad T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x), \tag{4.22}$$

from which it follows that the leading coefficient of $T_m(x)$ is equal to 2^{m-1}. Therefore, the normed Chebyshev polynomial $\widehat{T}_m(x) = 2^{1-m}T_m(x)$ has the least maximum modulus on the segment $[-1,1]$ in the uniform metric among all monic polynomials of degree m, and this maximum is equal to 2^{1-m}. Thus, setting $\prod_{k=1}^{m}(x-x_k) \equiv \widehat{T}_m(x)$ in (4.19) we obtain

$$\begin{aligned} \sigma_n^2(f) &\leqslant 4^m c \cdot \sin^{2m}(\alpha/2)\sin^{2m}(\alpha/2+\delta) \max_{x\in[-1,1]} |\widehat{T}_m(x)|^2 \\ &= 4^m \cdot 4^{1-m} c \cdot \sin^{2m}(\alpha/2)\sin^{2m}(\alpha/2+\delta) \max_{x\in[-1,1]} |T_m(x)|^2, \end{aligned} \tag{4.23}$$

or, in view of (4.21),

$$\sigma_n^2(f) \leqslant 4c \cdot \sin^{2m}(\alpha/2)\sin^{2m}(\alpha/2+\delta). \tag{4.24}$$

From the relations (4.11) and (4.12) we have

$$m = \begin{cases} n/2 & \text{for } n \text{ even}, \\ (n-1)/2 & \text{for } n \text{ odd}. \end{cases} \tag{4.25}$$

Thus, $m \geqslant (n-1)/2$, or $2m \geqslant n-1$, and from (4.24) we obtain the desired inequality (4.8):

$$\sigma_n^2(f) \leqslant 4c \cdot \sin^{2m}(\alpha/2)\sin^{2m}(\alpha/2+\delta) \leqslant 4c \cdot \sin^{n-1}(\alpha/2)\sin^{n-1}(\alpha/2+\delta).$$

This completes the proof of the theorem. □

Remark 8. The procedure of definition of the sequence of polynomials $q_n(z)$ is as follows. For given n we first specify m by formula (4.25) and the roots x_k of the Chebyshev polynomial $T_m(x)$ by formula:

$$x_k = \cos\frac{2k+1}{2m}\pi, \quad k = 0, 1, \dots, m-1.$$

Then, we use x_k and the second relation in (4.17) to specify λ_k and $z_k = e^{i\lambda_k}$. Finally, we define the polynomials $q_n(z)$ by formulas (4.11) and (4.12).

Remark 9. For $\delta = 0$ the set $\Gamma_{\alpha,\delta}$ defined by (3.12) is an arc of length 2α, and, in this case, the inequality (4.8) becomes Davisson's inequality (4.7).

§5. Estimates for the minimal eigenvalue of truncated Toeplitz matrices

The problem of asymptotic behavior of the extreme eigenvalues of truncated (finite sections) Toeplitz matrices goes back to the classical work by Kac, Murdoch and Szegö [14] (see also Grenander and Szegő [13, p. 72]), where the asymptotic behavior of extreme eigenvalues was studied for truncated Toeplitz matrices generated by continuous and continuously differentiable functions (symbols). Since then the problem for different classes of symbols, which are not (necessarily) continuous nor differentiable, was studied by many authors (see, e.g., Pourahmadi [16], Serra [19, 20] and references therein).

In this section we analyze the relationship between the the minimal eigenvalue of a truncated Toeplitz matrix and the finite prediction error for a stationary process, by showing how it is possible to obtain information about the minimal eigenvalue from that of the prediction error.

We use the notation. Let $f(\lambda)$ be a real-valued Lebesgue integrable function defined on $\Lambda := [-\pi, \pi]$, $T_n(f) := ||r_{k-j}||_{j,k=0,1,\dots,n}$ be the truncated Toeplitz matrix generated by the Fourier coefficients of f, and let $\lambda_{1,n}(f) \leqslant \lambda_{2,n}(f) \leqslant \dots \lambda_{n+1,n}(f)$ be the eigenvalues of $T_n(f)$. We denote by $m_f := \text{ess inf} f$ and $M_f := \text{ess sup} f$ the essential minimum and the essential maximum of f, respectively. In the following we consider the case where $f(\lambda)$ is a spectral density, that is, $f(\lambda) \geqslant 0$. Also, without loos of generality, we assume that $m_f := \text{ess inf} f = 0$.

The next proposition provides a relationship between the minimal eigenvalue $\lambda_{1,n}(f)$ of a truncated Toeplitz matrix $T_n(f)$ generated by spectral density f and the prediction error $\sigma_n^2(f)$ (see Pourahmadi [16] and Serra [19]).

Proposition 3. *Let f, $\lambda_{1,n}(f)$ and $\sigma_n^2(f)$ be as above. Then for any $n \in \mathbb{N}$ the following inequalities hold:*

$$\lambda_{1,n}(f) \leqslant \sigma_n^2(f) \leqslant M_f \frac{\lambda_{1,n}(f)}{\lambda_{1,n-1}(f)}. \tag{5.1}$$

The first inequality in (5.1) was proved in Pourahmadi [16], while the proof of the second inequality in (5.1) can be found in Serra [19].

Now we can apply Theorem 2 and Proposition 3 to obtain asymptotic estimates for the minimal eigenvalue $\lambda_{1,n}(f)$ of the matrix $T_n(f)$ in the cases where the support $\bar{E}_f$ of the generating function f is as in Examples 1–4. In the next theorem we state the corresponding result when $\bar{E}_f$ is as in Examples 1 and 3, similar estimates can be stated in the cases where $\bar{E}_f$ is as in Examples 2 and 4.

Theorem 4. *Let f, $\bar{E}_f$ and $\lambda_{1,n}(f)$ be as above. Then the following asymptotic estimates hold.*

(a) *If* $\bar{E}_f = \Gamma_{2\alpha}(\theta_0)$*, where* $\Gamma_{2\alpha}(\theta_0)$ *is as in Example* 1*, then*

$$\lambda_{1,n}(f) = O\left(\sin^{2n}(\alpha/2)\right) \quad \text{as} \quad n \to \infty. \tag{5.2}$$

(b) *If* $\bar{E}_f = \Gamma_{\alpha,\delta}(\theta_0)$*, where* $\Gamma_{\alpha,\delta}(\theta_0)$ *is as in Example* 3*, then*

$$\lambda_{1,n}(f) = O\left((\sin(\alpha/2)\sin(\alpha/2+\delta))^n\right) \quad \text{as} \quad n \to \infty. \tag{5.3}$$

Using Davisson's theorem (Theorem F), its extension (Theorem 3) and Proposition 3 we obtain exact upper bounds for the minimal eigenvalue $\lambda_{1,n}(f)$ rather than the asymptotic estimates (5.2) and (5.3). Specifically, we have the following result.

Theorem 5. *Let* f*,* $\bar{E}_f$ *and* $\lambda_{1,n}(f)$ *be as above. Then the following estimates hold.*

(a) *If* $\bar{E}_f = \Gamma_{2\alpha}(\theta_0)$*, where* $\Gamma_{2\alpha}(\theta_0)$ *is as in Example* 1*, then*

$$\lambda_{1,n}(f) \leqslant 4c\left(\sin(\alpha/2)\right)^{2n-2}. \tag{5.4}$$

(b) *If* $\bar{E}_f = \Gamma_{\alpha,\delta}(\theta_0)$*, where* $\Gamma_{\alpha,\delta}(\theta_0)$ *is as in Example* 3*, then*

$$\lambda_{1,n}(f) \leqslant 4c\left(\sin(\alpha/2)\right)^{n-1}\left(\sin(\alpha/2+\delta)\right)^{n-1}, \tag{5.5}$$

where the constant c *is as in Theorem F.*

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