Записки научных
семинаров ПОМИ
Том 509, 2021 г.

D. P. Solovyev

TOWARDS COUNTING PATHS IN LATTICE PATH MODELS WITH FILTER RESTRICTIONS AND LONG STEPS

Abstract. In this paper we introduce the notion of congruence for regions in lattice path models. This turns out to be useful for deriving path counting formula for the auxiliary lattice path model in the presence of long steps, source and target points of which are situated near the filter restrictions. This problem was motivated by the fact, that weighted numbers of paths in such model mimic multiplicities in tensor power decomposition of $U_q(sl_2)$-module $T(1)^{\otimes N}$ at roots of unity. We expand on combinatorial properties of such model and introduce the punchline of a proof for explicit path counting formula.

Introduction

The problem of tensor power decomposition can be considered from the combinatorial perspective as a problem of counting lattice paths in Weyl chambers [1–4]. In this paper we introduce preliminary results on counting paths on Bratteli diagram, reproducing decomposition of tensor powers of the fundamental module of the quantum group $U_q(sl_2)$ with divided powers, where q is a root of unity ([5–7]) into indecomposable modules. Combinatorial treatment of this problem gives rise to some interesting structures on lattice path models, such as filter restrictions of type 1, first introduced in [8], and long steps, which are introduced in the present paper.

In [8] Postnova and the author considered lattice path model motivated by the problem of finding explicit formulas for multiplicities of indecomposable modules in decomposition of tensor power of fundamental module $T(1)$ of the small quantum group $u_q(sl_2)$. This model is also known as the auxiliary lattice path model [7]. It consists of the left wall restriction at $x = 0$ and filter restrictions located periodically at $x = nl - 1$ for $n \in \mathbb{N}$. For $n = 1$ the filter restriction is of type 1, for the rest of values of n filter restrictions are of type 2. Applying periodicity conditions

Key words and phrases: lattice path models, quantum groups, representation theory.

This research is supported by the Russian Science Foundation under grant No. 21-11-00141.

$(M+2l,N)=(M,N)$, $M,N\geqslant l-1$ to the Bratteli diagram of this model allows one to obtain another lattice path model, recursion for weighted numbers of paths of which coincides with recursion for the multiplicities of indecomposable $u_q(sl_2)$-modules in decomposition of $T(1)^{\otimes N}$. Counting weighted numbers of paths descending from $(0,0)$ to (M,N) on this folded Bratteli diagram allows one to obtain desired formulas for multiplicities, where M stands for the highest weight of the module, multiplicity of which is in question, and N stands for the tensor power of $T(1)$. This has been done in [7].

Turns out that the auxiliary lattice path model can be modified in a different way, giving interesting results for representation theory of $U_q(sl_2)$, the quantized universal enveloping algebra of sl_2 with divided powers, when $q=\exp(\frac{\pi i}{l})$ and l is odd. Instead of applying periodicity conditions to the auxiliary lattice path model, as in the case of $u_q(sl_2)$, for $U_q(sl_2)$ we consider all filters to be of the 1-st type and also allow additional steps from $x=nl-2$ to $x=(n-2)l-1$, where $n\geqslant 3$. It was shown [7] that recursion for weighted numbers of paths for such model coincides with recursion for the multiplicities of indecomposable $U_q(sl_2)$-modules in decomposition of $T(1)^{\otimes N}$. Counting weighted numbers of paths descending from $(0,0)$ to (M,N) on the Bratteli diagram of the lattice path model obtained by this modification gives formulas for multiplicities in decomposition of $T(1)^{\otimes N}$, where $T(1)$ is the fundamental representation of $U_q(sl_2)$.

The main goal of this paper is to give a more in-depth combinatorial treatment of the auxiliary lattice path model in the presence of long steps. We focus on combinatorial properties of long steps, boundaries and congruence of regions in lattice path models. Latter turn out to be useful for deriving formulas for weighted numbers of paths. For any considered region, weighted numbers of paths at boundary points uniquely define such for the rest of the region by means of recursion. So, for congruent regions in different lattice path models, regions, where, roughly speaking, recursion is similar, it is sufficient to prove identities only for boundary points of such regions. This, in turn, allows us to provide a preliminary proof for explicit formulas for multiplicities of indecomposable $U_q(sl_2)$-modules in decomposition of $T(1)^{\otimes N}$ for q roots of unity, which relies purely on combinatorics of lattice path models.

This paper is organized as follows. In Section 1 we introduce the necessary notation. In Section 2 we give background on the auxiliary lattice path model. In Section 3 we explore combinatorial properties of long steps in periodically filtered lattice path models and consider the auxiliary lattice path model in the presence of long steps. In Section 4 we give a punchline for the proof of formulas for the weighted numbers of descending paths, relevant to considered model.

§1. Notations

As in [8], in this paper we will be using notation following [9]. For our purposes of counting multiplicities in tensor power decomposition of $U_q(\mathfrak{sl}_2)$-module $L(1)$ throughout this paper we will be considering the lattice

$$\mathcal{L} = \{(n,m)|n+m = 0 \bmod 2\} \subset \mathbb{Z}^2,$$

and the set of steps $\mathbb{S} = \mathbb{S}_L \cup \mathbb{S}_R$, where

$$\mathbb{S}_R = \{(x,y) \to (x+1, y+1)\},\ \mathbb{S}_L = \{(x,y) \to (x-1, y+1)\}.$$

A *lattice path* $\mathcal{P}$ in $\mathcal{L}$ is a sequence $\mathcal{P} = (P_0, P_1, \ldots, P_m)$ of points $P_i = (x_i, y_i)$ in $\mathcal{L}$ with starting point P_0 and the endpoint P_m. The pairs $P_0 \to P_1, P_1 \to P_2 \ldots P_{m-1} \to P_m$ are called steps of $\mathcal{P}$. Given starting point A and endpoint B, a set $\mathbb{S}$ of steps and a set of restrictions $\mathcal{C}$ we write

$$L(A \to B; \mathbb{S} \mid \mathcal{C})$$

for the set of all lattice paths from A to B that have steps from $\mathbb{S}$ and obey the restrictions from $\mathcal{C}$. We will denote the number of paths in this set as

$$|L(A \to B; \mathbb{S} \mid \mathcal{C})|.$$

The set of restrictions $\mathcal{C}$ in lattice path models considered throughout this paper will mostly contain wall restrictions and filter restrictions.

To each step from (x,y) to $(\widetilde{x}, \widetilde{y})$ we will assign the weight function $\omega : \mathbb{S} \longrightarrow \mathbb{R}_{>0}$ and use notation $(x,y) \xrightarrow{\omega} (\widetilde{x}, \widetilde{y})$ to denote that the step from (x,y) to $(\widetilde{x}, \widetilde{y})$ has the weight ω. By default, all unrestricted steps from $\mathbb{S}$ will have weight 1 and will denoted by an arrow with no number at the top. The *weight* of a path $\mathcal{P}$ is defined as the product

$$\omega(\mathcal{P}) = \prod_{i=0}^{m-1} \omega(P_i \to P_{i+1}).$$

For the set $L(A \to B; \mathbb{S} \mid \mathcal{C})$ we define the *weighted number of paths* as

$$Z(L(A \to B; \mathbb{S} \mid \mathcal{C})) = \sum_{\mathcal{P}} \omega(\mathcal{P}),$$

where the sum is taken over all paths $\mathcal{P} \in L(A \to B; \mathbb{S} \mid \mathcal{C})$.

§2. The auxiliary lattice path model

In this section we will briefly revise notions and results obtained in [8]. It will be convenient for us to omit mentioning $\mathbb{S}$ in $L(A \to B; \mathbb{S} \mid \mathcal{C})$. All paths considered below will be involving steps from set $\mathbb{S}$ unless stated otherwise.

2.1. Unrestricted paths, wall restrictions, filter restrictions. Let $L(A \to B)$ be the set of unrestricted paths from A to B on lattice $\mathcal{L}$ with the steps $\mathbb{S}$.

Lemma 2.1. *For set of unrestricted paths with steps $\mathbb{S}$ we have*

$$|L((0,0) \to (M,N))| = \binom{N}{\frac{N-M}{2}}.$$

Definition 2.2. *For lattice paths that start at $(0,0)$ we will say that $\mathcal{W}_d^L$ with $d \leqslant 0$ is a left wall restriction (relative to $x = 0$) if at points (d,y) paths are allowed to take steps of type $\mathbb{S}_R$ only*

$$\mathcal{W}_d^L = \{(d,y) \to (d+1, y+1) \ \ only\}.$$

Lemma 2.3. *The number of paths from $(0,0)$ to (M,N) with the set of steps $\mathbb{S}$ and one wall restriction $\mathcal{W}_a^L$ can be expressed via the number of unrestricted paths as*

$$|L((0,0) \to (M,N) \mid \mathcal{W}_a^L)| = \binom{N}{\frac{N-M}{2}} - \binom{N}{\frac{N-M}{2} + a - 1}, \quad for \ \ M \geqslant a,$$

We will be involved with left walls located at $x = 0$.

Definition 2.4. *For $n \in \mathbb{N}$, we will say that there is a filter $\mathcal{F}_d^n$ of type n, located at $x = d$ if at $x = d,\ d+1$ only the following steps are allowed:*

$$\mathcal{F}_d^n = \{(d,y) \xrightarrow{n} (d+1, y+1),\ (d+1, y+1) \to (d+2, y+2),$$
$$(d+1, y+1) \xrightarrow{2} (d, y+2)\}.$$

The index above the arrow is the weight of the step.

Lemma 2.5. *The number of lattice paths from $(0,0)$ to (M,N) with steps from $\mathbb{S}$ and filter restriction $\mathcal{F}_d^n$ with $x = d > 0$ and $n \in \mathbb{N}$ is*

$$Z(L_N((0,0) \to (M,N) \mid \mathcal{F}_d^n)) = \binom{N}{\frac{N-M}{2}} - \binom{N}{\frac{N-M}{2} + d}, \ for\ M < d,$$

$$Z(L_N((0,0) \to (M,N) \mid \mathcal{F}_d^n)) = n\binom{N}{\frac{N-M}{2}}, \ for\ M > d.$$

Proof. The proof is the same as for Lemma 4.8 and Lemma 4.9 in [8]. ☐

2.2. Counting paths in the auxiliary lattice path model. In [8] we considered lattice path model for the set of paths on $\mathcal{L}$ descending from $(0,0)$ to (M,N) with steps $\mathbb{S}$ in the presence of restrictions $\mathcal{W}_0^L$, $\mathcal{F}_{l-1}^1$, $\mathcal{F}_{nl-1}^2$, $n \in \mathbb{N}$, $n \geqslant 2$. Such set is denoted as

$$L_N((0,0) \to (M,N); \mathbb{S} \mid \mathcal{W}_0^L, \mathcal{F}_{l-1}^1, \{\mathcal{F}_{nl-1}^2\}_{n=2}^{\infty})$$

and such model is called the auxiliary lattice path model [7]. The main theorem of [8] gives explicit formula for weighted numbers of paths in the auxiliary lattice path model.

We will be interested in paths descending from $(0,0)$ to (M,N) with steps $\mathbb{S}$ in the presence of restrictions $\mathcal{W}_0^L$, $\mathcal{F}_{nl-1}^1$, $n \in \mathbb{N}$, instead of filters of type 2.

Definition 2.6. *We will denote by multiplicity function in the j-th strip $M^j_{(M,N)}$ the weighted number of paths in set*

$$L_N((0,0) \to (M,N); \mathbb{S} \mid \mathcal{W}_0^L, \{\mathcal{F}_{nl-1}^1\}, n \in \mathbb{N})$$

with the endpoint (M,N) that lies within $(j-1)l-1 \leqslant M \leqslant jl-2$

$$M^j_{(M,N)} = Z(L_N((0,0) \to (M,N); \mathbb{S} \mid \mathcal{W}_0^L, \{\mathcal{F}_{nl-1}^1\}, n \in \mathbb{N})),$$

where $M \geqslant 0$ and $j = \left[\frac{M+1}{l} + 1\right]$.

Now consider the version of the main theorem in [8], corresponding to this model.

Theorem 2.7. [8] *The multiplicity function in the j-th strip is given by*

$$M^j_{(M,N)} = \sum_{k=0}^{\left[\frac{N-(j-1)l+1}{4l}\right]} P_j(k) F^{(N)}_{M+4kl} + \sum_{k=0}^{\left[\frac{N-jl}{4l}\right]} P_j(k) F^{(N)}_{M-4kl-2jl}$$
$$- \sum_{k=0}^{\left[\frac{N-(j+1)l+1}{4l}\right]} Q_j(k) F^{(N)}_{M+2l+4kl} - \sum_{k=0}^{\left[\frac{N-jl-2l}{4l}\right]} Q_j(k) F^{(N)}_{M-4kl-2(j+1)l},$$

where

$$P_j(k) = \sum_{i=0}^{\left[\frac{j}{2}\right]} \binom{j-2}{2i} \binom{k-i+j-2}{j-2},$$

$$Q_j(k) = \sum_{i=0}^{\left[\frac{j}{2}\right]} \binom{j-2}{2i+1} \binom{k-i+j-2}{j-2},$$

$$F_M^{(N)} = \binom{N}{\frac{N-M}{2}} - \binom{N}{\frac{N-M}{2}-1}.$$

Proof. The proof is the same as the proof of the main theorem in [8], except that instead of Lemma 4.9 from [8], for the slightly tweaked model one should use Lemma 2.5. □

From now on, mentioning the auxiliary lattice path model we will mean its slightly tweaked version. This model will be further modified in subsequent sections, so that recursion for the weighted numbers of paths on the resultant Bratteli diagram will recreate recursion for multiplicities of indecomposable $U_q(sl_2)$-modules in decomposition of $T(1)^{\otimes N}$.

§3. Boundary points, congruence and long steps

3.1. Boundary points and congruent regions. We will argue, that multiplicities on the boundary of any considered region uniquely define multiplicities in the rest of the region. For proving identities between multiplicities in two congruent regions, it is sufficient to prove such identity for their boundary points.

Definition 3.1. *Consider lattice path model, defined by set of steps $\mathbb{S}$ and set of restrictions $\mathcal{C}$ on lattice $\mathcal{L}$. Subset $\mathcal{L}_0 \subset \mathcal{L}$ with steps $\mathbb{S}$ and restrictions $\mathcal{C}$ is called a region of such lattice path model.*

Definition 3.2. *Consider $\mathcal{L}_0 \subset \mathcal{L}$ a region of the lattice path model defined by steps $\mathbb{S}$ and restrictions $\mathcal{C}$. Point $B \in \mathcal{L}_0$ is called a boundary point of $\mathcal{L}_0$ if there exists $B' \in \mathcal{L}$, $B' \notin \mathcal{L}_0$ such that step $B' \to B$ is allowed in $\mathcal{L}$ by set of steps $\mathbb{S}$ and restrictions $\mathcal{C}$. Union of all such points is a boundary of $\mathcal{L}_0$ and is denoted by $\partial\mathcal{L}_0$.*

Note, that boundary points are defined with respect to some lattice path model under consideration. For brevity, we assume that this lattice path model is known from the context and mentioning it will be mostly omitted.

Lemma 3.3. *Weighted numbers of paths* $Z(L_N((0,0) \to (M,N);\dots)$ *for* $(M,N) \in \mathcal{L}_0$ *are uniquely defined by weighted numbers of paths for its boundary points* $\partial\mathcal{L}_0$.

Proof. It is easy to see, that weighted numbers of descending paths for each point of the region $\mathcal{L}_0$ can be expressed in terms of weighted numbers of paths at either points of the region $\mathcal{L}_0$, or at boundary points $\partial\mathcal{L}_0$ by means of recursion. Full proof will be given in the consequent paper. □

Definition 3.4. *Consider two lattice path models with steps* $\mathbb{S}_1$, $\mathbb{S}_2$ *and restrictions* $\mathcal{C}_1$, $\mathcal{C}_2$ *defined on lattice* $\mathcal{L}$. *Subset* $\mathcal{L}_1 \subset \mathcal{L}$ *is a region in the lattice path model defined by* $\mathbb{S}_1, \mathcal{C}_1$. *Subset* $\mathcal{L}_2 \subset \mathcal{L}$ *is a region in the lattice path model defined by* $\mathbb{S}_2, \mathcal{C}_2$. *Regions* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are congruent if there exists a translation* T *in* $\mathcal{L}$ *such that*

- $T\mathcal{L}_1 = \mathcal{L}_2$ *as sets of points in* $\mathcal{L}$
- *Translation* T *is a bijection between steps in* $\mathcal{L}_1$ *and* $\mathcal{L}_2$, *meaning that there is a one-to-one correspondence between steps with source and target points related by* T, *with preservation of weights.*

To put it simply, if we forget about lattice path models outside $\mathcal{L}_1$ and $\mathcal{L}_2$, these two regions will be indistinguishable. Due to translations in $\mathcal{L}$ being invertible, it is easy to see that congruence defines an equivalence relation. Now we formulate the main theorem of this subsection.

Theorem 3.5. *Consider two lattice path models with steps* $\mathbb{S}_1$, $\mathbb{S}_2$ *and restrictions* $\mathcal{C}_1$, $\mathcal{C}_2$ *defined on lattice* $\mathcal{L}$. *Region* $\mathcal{L}_1$ *of the lattice path model defined by* $\mathbb{S}_1$, $\mathcal{C}_1$ *is congruent to region* $\mathcal{L}_2$ *of the lattice path model defined by* $\mathbb{S}_2$, $\mathcal{C}_2$, *where* $T\mathcal{L}_1 = \mathcal{L}_2$. *If equality*

$$Z(L_N((0,0) \to (M,N); \mathbb{S}_1 \mid \mathcal{C}_1)) = Z(L_N((0,0) \to T(M,N); \mathbb{S}_2 \mid \mathcal{C}_2)) \tag{1}$$

holds for all $(M,N) \in \partial\mathcal{L}_1 \cup T^{-1}(\partial\mathcal{L}_2)$, *then it holds for all* $(M,N) \in \mathcal{L}_1$.

Note, that if $(M,N) \in \partial\mathcal{L}_1$ it doesn't necessarily follows that $T(M,N) \in \partial\mathcal{L}_2$, due to $\mathcal{C}_1$ and $\mathcal{C}_2$ being different. So, it is natural to ask formula (1) to hold for $\partial\mathcal{L}_1 \cup T^{-1}(\partial\mathcal{L}_2)$.

Proof. The proof follows from Lemma 3.3 and linearity of recursion. □

Corollary 3.6. *Consider lattice path models with steps $\mathbb{S}_1$, $\mathbb{S}_2$, $\mathbb{S}_3$ and restrictions $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ defined on lattice $\mathcal{L}$. Region $\mathcal{L}_1$ is congruent to $\mathcal{L}_2$ and $\mathcal{L}_3$, where $T_1(M_1, N_1) = (M_2, N_2)$, $T_2(M_1, N_1) = (M_3, N_3)$ for $(M_1, N_1) \in \mathcal{L}_1$. If equality*

$$Z(L_N((0,0) \to (M,N); \mathbb{S}_1 \mid \mathcal{C}_1)) = Z(L_N((0,0) \to T_1(M,N); \mathbb{S}_2 \mid \mathcal{C}_2)) \\ + Z(L_N((0,0) \to T_2(M,N); \mathbb{S}_3 \mid \mathcal{C}_3))$$

holds for all $(M,N) \in \partial\mathcal{L}_1 \cup T_1^{-1}(\partial\mathcal{L}_2) \cup T_2^{-1}(\partial\mathcal{L}_3)$, then it holds for all $(M,N) \in \mathcal{L}_1$.

The morale of this subsection is that for two congruent regions weighted numbers of paths are defined by values of such at the boundary of the considered regions. For proving identities, it is sufficient to establish equality for weighted numbers of paths at boundary points, while equality for the rest of the region will follow due to the congruence.

3.2. Long steps in lattice path models with filter restrictions. Long step is a step $(x,y) \xrightarrow{w} (x', y+1)$ in $\mathcal{L}$ such that $|x - x'| > 1$. We denote the sequence of long steps as

$$\mathbb{S}[M_1, M_2] = \{(M_1, M_1 + 2m) \to (M_2, M_1 + 1 + 2m)\}_{m=0}^{\infty},$$

where $x = M_1$ is the source point for the sequence and $x = M_2$ is the target point, $|M_1 - M_2| > 1$. For the purposes of this paper we will be mainly interested in sequences

$$\mathbb{S}(k) \equiv \mathbb{S}[l(k+2) - 2, lk - 1] = \{(l(k+2) - 2, lk - 2 + 2m) \\ \to (lk - 1, lk - 2 + 1 + 2m)\}_{m=0}^{\infty},$$

where $k \in \mathbb{N}$ and $\mathcal{C}$ consists of $\mathcal{F}^1_{lk-1}$ and $\mathcal{F}^1_{l(k+2)-1}$. We will need such sequences of long steps for modification of the auxiliary lattice path model, relevant to representation theory of $U_q(sl_2)$ at roots of unity.

Lemma 3.7. *Fix $j, k \in \mathbb{N}$, $j \leqslant k$. Let*

$$Z_{(M,N)} \equiv Z(L_N((0,0) \to (M,N)); \mathbb{S} \mid \mathcal{W}_0^L, \{\mathcal{F}^1_{nl-1}\}_{n=j}^{\infty})$$

be the weighted number of lattice paths from $(0,0)$ to (M,N) with filter restrictions $\{\mathcal{F}^1_{nl-1}\}_{n=j}^{\infty}$ and set of unrestricted elementary steps $\mathbb{S}$. Let

$$Z'_{(M,N)} \equiv Z(L_N((0,0) \to (M,N)); \mathbb{S} \cup \mathbb{S}(k) \mid \mathcal{W}_0^L, \{\mathcal{F}^1_{nl-1}\}_{n=j}^{\infty})$$

be the weighted number of lattice paths from $(0,0)$ *to* (M,N) *with the same restrictions, with steps* $\mathbb{S}\cup\mathbb{S}(k)$. *Then for* $lk-1\leqslant M\leqslant l(k+2)-2$ *we have*

$$Z'_{(M,N)}=Z_{(M,N)},\quad \text{if}\quad N\leqslant M+2l-2, \tag{2}$$

$$Z'_{(M,N)}=Z_{(M,N)}+Z_{(M+2l,N)},\quad \text{if}\quad M+2l\leqslant N\leqslant l(k+4)-2. \tag{3}$$

Proof. Proof of this theorem is technical and will be given in the consequent paper. □

Note, that formula (3) is true for greater values of N, as making region II into a parallelogram-like region will not add new boundary points. Consider the auxiliary lattice path model in the presence of sequence of steps $\mathbb{S}(k)$.

Definition 3.8. *We will denote by multiplicity function in the* j*-th strip* $\widetilde{M}^j_{(M,N)}$ *the weighted number of paths in set*

$$L_N((0,0)\to(M,N);\mathbb{S}\cup\widetilde{\mathbb{S}}\mid\mathcal{W}_0^L,\{\mathcal{F}^1_{nl-1}\},n\in\mathbb{N})$$

with the endpoint (M,N) *that lies within* $(j-1)l-1\leqslant M\leqslant jl-2$

$$\widetilde{M}^j_{(M,N)}=Z(L_N((0,0)\to(M,N);\mathbb{S}\cup\widetilde{\mathbb{S}}\mid\mathcal{W}_0^L,\{\mathcal{F}^1_{nl-1}\},n\in\mathbb{N})),$$

where $\widetilde{\mathbb{S}}$ *is a set of some additional steps and* $M\geqslant 0$ *and* $j=\left[\frac{M+1}{l}+1\right]$.

In this subsection $\widetilde{\mathbb{S}}=\mathbb{S}(k)$ if not stated the otherwise.

Lemma 3.9. *For fixed* $k\in\mathbb{N}$

$$\widetilde{M}^{k+1}_{(M,N)}=\sum_{j=0}^{[\frac{N-lk+1}{2l}]}M^{k+1+2j}_{(M+2jl,N)}, \tag{4}$$

$$\widetilde{M}^{k+3}_{(M,N)}=\sum_{j=0}^{[\frac{N-l(k+2)+1}{2l}]}M^{k+3+2j}_{(M+2jl,N)}, \tag{5}$$

where $\widetilde{M}^j_{(M,N)}$ *is the multiplicity function for* j*-th strip in the auxiliary model with steps* $\mathbb{S}\cup\widetilde{\mathbb{S}}$, $M^j_{(M,N)}$ *is the multiplicity function for* j*-th strip in the auxiliary model with steps* $\mathbb{S}$.

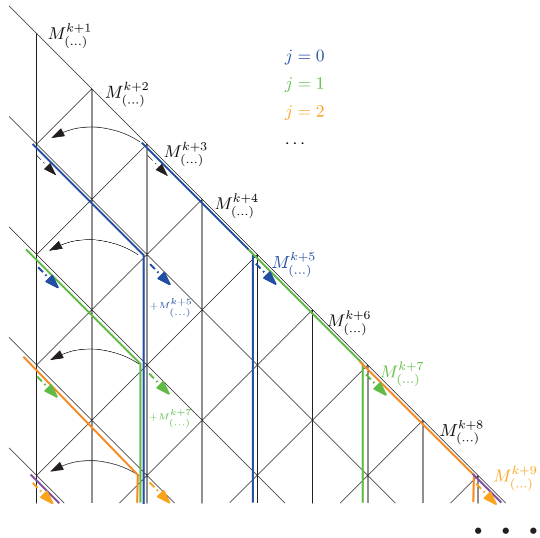

Fig. 1. Color emphasizes the number of iteration in the induction. Paths induced at the boundary of $(k+1)$-th strip during $(j-1)$-th iteration descend in the region, highlighted with color, corresponding to j-th iteration. Colored lines outline regions congruent to each other. Dashed colored arrows denote weighted numbers of induced paths, inflicted to $(k+3)$-th strip once they have descended, and their equivalents in strips of the auxiliary lattice path model.

Proof. The proof of this lemma heavily relies on applying Lemma 3.7, as shown in Figure 1, and induction principle. This figure also shows, how long steps act on descended paths corresponding to dashed colored arrows, inducing paths at boundary points of $(k+1)$-th strip, highlighted with the dashed arrow of the same color. These induced paths, in turn, will be descending in region, highlighted with color corresponding to the next, $(j+1)$-th iteration. Proving that long steps induce paths at boundary points of $(k+1)$-th strip following this scenario amounts to proving the inductive step for formula (4). Full proof will be given in the consequent paper. $\square$

Corollary 3.10. *For fixed $k \in \mathbb{N}$ and $m \geqslant k$*

$$\widetilde{M}^{m+1}_{(M,N)} = \sum_{j=0}^{[\frac{N-lm+1}{2l}]} M^{m+1+2j}_{(M+2jl,N)},$$

Proof. The proof relies on the same procedure as was used in proof of the Lemma 3.9. $\square$

3.3. Modifying auxiliary lattice path model. The auxiliary lattice path model was considered in [8]. In [7] periodicity conditions $(M+2l, N) = (M, N)$ for $M, N \geqslant l-1$ were applied, the resultant lattice path model was denoted by $\mathcal{L}_u$ and corresponding set of steps in the folded Bratteli diagram as $\mathbb{S}_u$. Note, that periodicity conditions can be considered as applying right wall restriction $\mathcal{W}^R_{3l-2}$ and sequence of long steps $\mathbb{S}(3)$. Consider the following theorem.

Theorem 3.11. [7] *The multiplicity of the titling $u_q(sl_2)$-module $T(k)$ in the decomposition of $T(1)^{\otimes N}$ is equal to the weighted number of lattice paths in $\mathcal{L}_u$ connecting $(0,0)$ and (k,N) with weights given by multiplicities of elementary steps $\mathbb{S}_u$.*

This theorem is due to the fact that recursion for multiplicities of $u_q(sl_2)$-modules in tensor product decomposition of $T(1)^{\otimes N}$ coincides with the recursion for the weighted numbers of paths. The explicit formula for the latter was obtained in [7], using weighted numbers of paths for the auxiliary lattice path model. Now consider the auxiliary lattice path model with filter restrictions of type 1, in the presence of steps

$$\mathbb{S}_U \equiv \mathbb{S} \cup \Big(\bigcup_{k=1}^{\infty} \mathbb{S}(k) \Big),$$

and, following [7], denote it as $\mathcal{L}_U$. The arrangement of steps for points of $\mathcal{L}_U$ is depicted in Figure 2. Consider the following theorem.

Theorem 3.12. [7] *The multiplicity of the tilting $U_q(sl_2)$-module $T(k)$ in the decomposition of $T(1)^{\otimes N}$ is equal to the weighted number of lattice paths on $\mathcal{L}_U$ connecting $(0,0)$ and (k,N) with weights given by multiplicities of elementary steps $\mathbb{S}_U$.*

This theorem is also due to the fact that recursion for multiplicities of $U_q(sl_2)$-modules in tensor product decomposition of $T(1)^{\otimes N}$ coincides with the recursion for the weighted numbers of paths in $\mathcal{L}_U$. The explicit

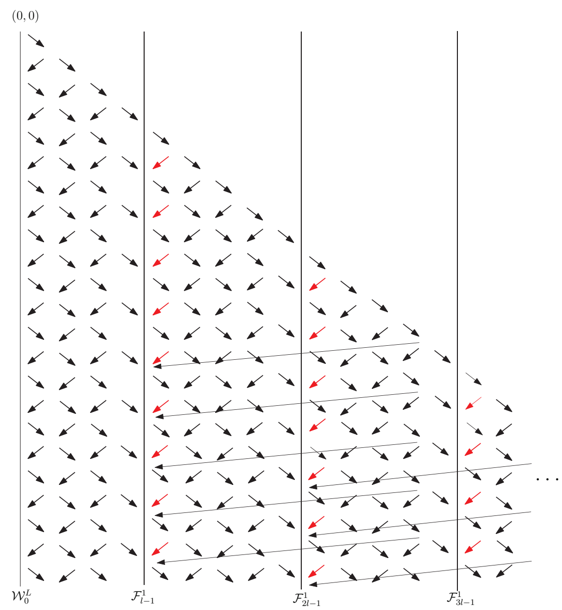

Fig. 2. Arrangement of steps for points of the lattice path model $\mathcal{L}_U$. Here we depict the case, where $l = 5$.

formula for the former is known. The main goal of the following section is to hint at the same formula by combinatorial means, mainly counting lattice paths in this modification of the auxiliary lattice path model.

§4. Towards counting paths in the auxiliary lattice path model in the presence of long steps

Consider lattice path model $\mathcal{L}_U$. which is the lattice path model in the presence of long steps. From now on, following Definition 3.8, we denote by multiplicity function in the j-th strip $\widetilde{M}^j_{(M,N)}$ the weighted number of paths in set

$$L_N((0,0) \to (M,N); \mathbb{S}_U \mid \mathcal{W}_0^L, \{\mathcal{F}^1_{nl-1}\}, n \in \mathbb{N})$$

with the endpoint (M, N) that lies within $(j-1)l - 1 \leqslant M \leqslant jl - 2$

$$\widetilde{M}^j_{(M,N)} = Z(L_N((0,0) \to (M,N); \mathbb{S} \cup \widetilde{\mathbb{S}} \mid \mathcal{W}_0^L, \{\mathcal{F}^1_{nl-1}\}, n \in \mathbb{N})),$$

where
$$\mathbb{S}_U = \mathbb{S} \cup \widetilde{\mathbb{S}} = \mathbb{S} \cup \Big(\bigcup_{k=1}^{\infty} \mathbb{S}(k) \Big).$$
and $M \geqslant 0$ and $j = \left[\frac{M+1}{l} + 1\right]$. The main goal of this section is to derive the explicit formula for $\widetilde{M}^j_{(M,N)}$.

Lemma 4.1. *For the lattice path model $\mathcal{L}_U$*
$$\widetilde{M}^1_{(M,N)} = M^1_{(M,N)}, \tag{6}$$
and for $k \in \mathbb{N}$,
$$\widetilde{M}^{k+1}_{(M,N)} = \sum_{j=0}^{[\frac{N-lk+1}{2l}]} F^{(k-1+2j)}_{k-1} M^{k+1+2j}_{(M+2jl,N)}, \tag{7}$$

Proof. The idea of the proof relies on applying long steps from strip to strip using results of the Corollary 3.10. In a similar fashion to the one, depicted in Figure 1, this leads to a recursion on coefficients near terms corresponding to induced descending paths. This recursion is satisfied by Catalan numbers, which proves formula (7). □

Theorem 4.2. *For $k \in \mathbb{N} \cup \{0\}$ we have*
$$\widetilde{M}^{k+1}_{(M,N)} = F^{(N)}_M + \sum_{j=1}^{[\frac{N-lk+1}{2l}+\frac{1}{2}]} F^{(N)}_{-2lk+M-2jl} + \sum_{j=1}^{[\frac{N-lk+1}{2l}]} F^{(N)}_{M+2jl} \tag{8}$$
where $lk - 1 \leqslant M \leqslant l(k+1) - 2$.

Proof. This proof relies on induction method and Lemma 4.1. Remarkably, if we resum coefficients near terms $F^{(\dots)}_{\dots}$ in formula (7) explicitly, using formulas for strips from Theorem 2.7, by means of Wilf–Zeilberger pairs we prove formula (8). Full proof will be given in a consequent paper. □

§5. Conclusions

In this paper we considered lattice path model $\mathcal{L}_U$, which is the auxiliary lattice path model in the presence of long steps. Weighted numbers of paths in this model recreate multiplicities of $U_q(sl_2)$-modules in tensor product decomposition of $T(1)^{\otimes N}$, where $U_q(sl_2)$ is a quantum deformation of universal enveloping algebra of sl_2 with divided powers and q is a root of unity. Explicit formulas for multiplicities of all tilting modules in tensor product decomposition were hinted at by purely combinatorial means in the main theorem of this paper 4.2. Turns out that the auxiliary lattice model defined in [8] is of great use for counting multiplicities of modules of differently defined quantum deformations of $U(sl_2)$ at q root of unity. For instance, in [7] we applied periodicity conditions to the auxiliary lattice path model to obtain folded Bratteli diagram, weighted numbers of paths for which recreate multiplicities of modules in tensor product decomposition of $T(1)^{\otimes N}$, where $T(1)$ is a tilting module of the small quantum group $u_q(sl_2)$. In this paper, we modified the auxiliary lattice path model by applying long steps to obtain multiplicities for the case of $U_q(sl_2)$ with divided powers in a similar fashion. Model defined in [8] required analysis of combinatorial properties of filters, which we heavily relied on. In this paper we introduced long steps and explored their combinatorial properties. In order to derive formulas for weighted numbers of paths in this setting, we also defined boundary points and congruence of regions in lattice path models. Philosophy of congruence is fairly easy to understand. Two different lattice path models can be locally indistinguishable due to coinciding recursions for weighted numbers of paths in these regions. Weighted numbers of paths at boundary points of the considered region uniquely define weighted numbers of paths for the rest of the region by recursion. So, instead of proving identities for the whole region it is sufficient to prove such only for boundary points of the region. At boundary points an identity can be represented as a linear combination of weighted numbers of paths from different lattice path models and one needs to take into account boundary points of congruent regions with respect to all these models.

We are grateful to O. Postnova, N. Reshetikhin, F. Petrov for fruitful discussions.

References

1. D. Grabiner, P. Magyar, *Random walks in Weyl chambers and the decomposition of tensor powers*, — J. Algebraic Combin., **2(3)** (1993), 239–260.
2. D. Grabiner, *Random walk in an alcove of an affine Weyl group, and non-colliding random walks on an interval*, — J. Combin. Theory, Series A, **97(2)** (2002), 285–306.
3. T. Tate, S. Zelditch, *Lattice path combinatorics and asymptotics of multiplicities of weights in tensor powers*, — J. Funct. Anal., **217(2)** (2004), 402–447.
4. O. Postnova, N. Reshetikhin, *On multiplicities of ireducibles in large tensor product of representations of simple Lie algebras*, `arXiv:1812.11236`.

5. L. Faddeev, N. Reshetikhin, L. Takhtajan, *Quantization of Lie groups and Lie algebras*, — Algebraic analysis (1988), 129–139.
6. H. Andersen, *Tensor products of quantized tilting modules*, — Comm. Math. Phys., **149** (1992), 149–159.
7. A. Lachowska, O. Postnova, N. Reshetikhin, D. Solovyev, *Tensor powers of vector representation of $U_q(sl_2)$ at even roots of unity*, in preparation.
8. O. Postnova, D. Solovyev, *Counting filter restricted paths in $\mathbb{Z}^2$ lattice*, `arXiv:2107.09774`
9. C. Krattenthaler, *Lattice path combinatorics chapter in Handbook of Enumerative combinatorics* edited by Miklos Bona, Chapman and Hall (2015).
10. P. Littelmann, *Paths and root operators in representation theory*, — Annals Math., **142 (3)** (1995), 499–525.
11. O. Bratteli, *Inductive limits of finite dimensional C*-algebras*, — Transactions of the American Math. Soci., **171** (1972), 195–234.
12. D. Zeilberger, *The Method of Creative Telescoping*, — J. Symb. Comput., **11** (1991), 195–204.
13. Herbert Wilf, Doron Zeilberger, *Rational Functions Certify Combinatorial Identities* — J. Amer. Math. Soc., **3** (1990), 147–158.
14. P. Paule, M. Schorn, *A Mathematica Version of Zeilberger's Algorithm for Proving Binomial Coefficient Identities*, J. Symb. Comput., **11** (1994), 1-000.

Department of Physics,
St.Petersburg State University,
Ulyanovkaya str.1,
St. Peterburg, Russia
E-mail: dimsol42@gmail.com

Поступило 22 ноября 2021 г.