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ON THE ASYMPTOTICS OF MULTIPLICITIES FOR LARGE TENSOR PRODUCT OF REPRESENTATIONS OF SIMPLE LIE ALGEBRAS

Abstract. The asymptotics of multiplicities of irreducible representations in large tensor products of finite dimensional representations of simple Lie algebras are computed for all, including non-generic, highest weights.

§1. Introduction

1.1. The study of the statistics of irreducible components in "large" representations is a natural problem in representation theory that goes back to works [5, 10, 11]. An example of such large representation is the left regular representation of the symmetric group S_N for large N where a natural probability measure is the Plancherel measure. The statistics of irreducible components in the left regular representation of S_N with respect to the Plancherel measure was exactly the focus of [5, 10, 11].

Let V be an irreducible finite dimensional representation of a finite dimensional simple Lie algebra $\mathfrak{g}$. In [1] the asymptotics of the multiplicity of an irreducible subrepresentation in $V^{\otimes N}$ was computed for when $N \to \infty$ for subrepresentations with generic highest weight(see below). These results were extended in [9] where the asymptotics of the Plancherel measure on irreducible components of $V^{\otimes N}$ was computed when $N \to \infty$. In [7] we derived the asymptotic formula for the character probability measure in this setting. This is a family of measures deforming the Plancherel measure where the deformation parameter is $t \in \mathfrak{h}_{\mathbb{R}}$. In [7] we compute the asymptotics for generic values of the deformation parameter t.

In this paper we prove the asymptotic formula for the multiplicity of an irreducible subrepresentation in $V^{\otimes N}$ when $N \to \infty$ and $\lambda/N \to \xi$ with ξ not necessary generic as in [1] and [9]. In the follow up paper [8]

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we compute the asymptotics of the character measure for all values of the deformation parameter t.

1.2. To state main results of the paper we need a few definitions.

In a finite dimensional complex simple Lie algebra $\mathfrak{g}$ choose a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ and let $\mathfrak{h} \subset \mathfrak{b}$ be the corresponding Cartan subalgebra. Fix the Killing form on $\mathfrak{g}$ with the standard normalization. This gives a bilinear form $(\cdot\,,\cdot)$ on $\mathfrak{h}$ which identifies vector spaces $\mathfrak{h}$ and $\mathfrak{h}^*$ and induces the corresponding bilinear form on $\mathfrak{h}^*$. We choose a basis of coroots $\alpha_a^\vee = \frac{2\alpha_a}{(\alpha_a,\alpha_a)}$ in $\mathfrak{h}$ (after the identification $\mathfrak{h} \simeq \mathfrak{h}^*$ via Killing form). Let $\Delta^+ \subset \mathfrak{h}^*$ be the set of positive roots. Denote simple roots by $\alpha_1, \dots, \alpha_r$, here r is the rank of $\mathfrak{g}$. Simple roots form a basis in $\mathfrak{h}^*$ and generate the root lattice $\Lambda = \oplus_a \mathbb{Z}\alpha_a$ in $\mathfrak{h}^*$. We assume that $\mathfrak{g}$ is the complexification of its compact real form $\mathfrak{k}$ and that $\mathfrak{t} \in \mathfrak{k}$ is the corresponding Cartan subalgebra. Denote by $\omega_a \in \mathfrak{h}^*$ fundamental weights of $\mathfrak{g}$, $(\omega_a, \alpha_b^\vee) = \delta_{a,b}$. We denote by $L = \oplus \mathbb{Z}\omega_a \in \mathfrak{h}^*$ the weight lattice. The Cartan matrix of the root system Δ is $C_{ab} = (\alpha_a^\vee, \alpha_b)$.

1.3. Let V_ν be an irreducible finite dimensional $\mathfrak{g}$-module with highest weight ν.

Because the Lie algebra $\mathfrak{g}$ is simple

$$V_\nu^{\otimes N} \simeq \oplus_{\lambda \in D(V_\nu, N)} V_\lambda^{\oplus m_\lambda(V_\nu,N)}. \tag{1}$$

Here $m_\lambda(V_\nu, N)$ is the multiplicity of the irreducible subrepresentation V_λ. The multiplicity $m_\lambda(V_\nu, N)$ is non-zero only if λ is dominant integral and if it is inside the Weyl group orbit through ν. This describes the domain $D(V_\nu, N)$.

Let $\chi_\nu = \mathrm{tr}_{V_\nu}(\pi_\nu(g))$ be the character of V_ν. Throughout this paper we will use notations $\mathfrak{h} \subset \mathfrak{g}$ for the Cartan subalgebra in the complex sinple Lie algebra $\mathfrak{g}$ and $\mathfrak{h}_\mathbb{R} \subset \mathfrak{g}_\mathbb{R} \subset \mathfrak{g}$ for the Cartan subalgebra of the split real form $\mathfrak{g}_\mathbb{R}$ of $\mathfrak{g}$ and $\mathfrak{t} \subset \mathfrak{k} \subset \mathfrak{g}$ for the Cartan subalgebra in the compact real form of $\mathfrak{g}$.

Define the function

$$f(x, \xi) = \ln(\chi_\nu(e^x)) - (x, \xi).$$

Here $x \in \mathfrak{h}$, $\xi \in \mathfrak{h}^* \simeq \mathfrak{h}$, we identified $\mathfrak{h}_\mathbb{R} \simeq \mathfrak{h}_{\mathbb{R}^*}$ using the Killing form $(x, \xi) \equiv \xi(x)$. In the basis of simple coroots $x = \sum_a x_a \alpha_a^\vee$, $\xi = \sum_a \xi_a \alpha_a$ and $(x, \xi) = \sum_{a,b} x_a C_{ab} \xi_b$ where $C_{ab} = (\alpha_a^\vee, \alpha_b)$ is the Cartan matrix. Recall that

ξ belongs to a wall of the principal Weyl chamber if $(\xi, \alpha) = 0$ when α belongs to Δ_0^+ and $(\xi, \alpha) > 0$ when α belongs to Δ_1^+. Here Δ_0^+ are positive roots of a Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ and Δ_1^+ are other positive roots of $\mathfrak{g}$. The Dynkin diagram of $\mathfrak{g}_0$ is a subdiagram of Dynkin diagram of $\mathfrak{g}$.

Our main result is the following theorem.

Theorem 1. *If $\xi = \lambda/N$ remain regular as $N \to \infty$ and stays strictly inside the region $D(V_\nu, N)$, the asymptotics of the multiplicity of V_λ in* (1) *has the following form:*

$$m_\lambda(V_\nu, N) = N^{-\frac{r}{2} - |\Delta_0^+|} e^{Nf(\eta,\xi) - (\eta,\rho_1)} \Delta^{(1)}(e^\eta) \frac{\kappa^{-\frac{\dim \mathfrak{g}_0}{2}}}{\sqrt{\det(f^{(2)}(\eta))}}$$
$$\times \frac{\det(B)|W_0| \prod_{\alpha \in \Delta_0^+} (\rho, \alpha)}{(2\pi)^{\frac{r}{2}} 2^{|\Delta_0^+|} \sqrt{\det(B^{\mathfrak{g}_0})}} \left(1 + O\left(\frac{1}{N}\right)\right).$$

Here $\det(B)$ *is the determinant of the symmetrized Cartan matrix of* $\mathfrak{g}$ *and* $\det(B^{\mathfrak{g}_0})$ *is the determinant of symmetrized Cartan matrix for* $\mathfrak{g}_0$*,* $|W_0|$ *is the order of the Weyl group of* $\mathfrak{g}_0$*,* $f^{(2)}(\eta)$ *is the matrix of the second derivatives of* $f(x, \xi)$ *when* $x = \eta$ *in the root basis and* $\Delta^{(1)}(e^x)$ *and* κ *are given by the formulas:*

$$\Delta^{(1)}(e^x) = \prod_{\alpha \in \Delta_1^+} (e^{\frac{(x,\alpha)}{2}} - e^{-\frac{(x,\alpha)}{2}}).$$

$$\kappa = \frac{1}{\dim \mathfrak{g}_0} \frac{\sum_\mu \mathrm{tr}_{W_\mu}(e^\eta) c_2^{\mathfrak{g}_0}(\mu) dim(V_\mu^{\mathfrak{g}_0})}{\sum_\mu \mathrm{tr}_{W_\mu}(e^\eta) \dim(V_\mu^{\mathfrak{g}_0})}.$$

Here the summation over μ corresponds to the decomposition of the $\mathfrak{g}$-module V into irreducible $\mathfrak{g}_0$ components, $V \simeq \bigoplus_\mu W_\mu \otimes V_\mu^{\mathfrak{g}_0}$.

In particular, in the extreme case $\xi = 0$ this formula yields:

$$m_0(V_\nu, N) = N^{\frac{-\dim(\mathfrak{g})}{2}} \dim(V_\nu)^N \frac{\sqrt{\det(B)}|W|}{(2\pi)^{\frac{r}{2}}} \frac{\prod_{\alpha \in \Delta^+} (\rho, \alpha)}{2^{|\Delta^+|}} \left(\frac{\dim \mathfrak{g}}{c_2(V_\nu)}\right)^{\frac{\dim \mathfrak{g}}{2}}$$
$$\times \left(1 + O\left(\frac{1}{N}\right)\right).$$

We call ξ generic if $(\xi, \alpha) > 0$. In this case $\mathfrak{g}_0 = \{0\}$ and $\mathfrak{g}_1 = \mathfrak{g}$ and we recover [1, 7, 9]:

$$m_\lambda(V_\nu, N) = N^{-\frac{r}{2}} e^{Nf(\eta,\xi)-(\eta,\rho)} \Delta(e^\eta) \frac{\det(B)}{(2\pi)^{\frac{r}{2}} \sqrt{\det(f^{(2)}(\eta))}} \times \left(1 + O\left(\frac{1}{N}\right)\right).$$

The proof of this theorem is given in Section 2.

§2. Proof of the theorem

2.1. The integral formula for the multiplicity. Let G be a complex simple connected Lie group with Lie algebra $\mathfrak{g}$ and $K \subset G$ be its compact real form. Denote by $T \subset K$ the Cartan subgroup and by $e^{i\theta} \subset T$ its elements. Here $i\theta \subset \mathbf{t} \subset \mathbf{k}$. The decomposition of $V_\nu^{\otimes N}$ into the direct sum of irreducibles gives the identity

$$\chi_\nu(g)^N = \sum_{\lambda \in D(V_\nu, N)} m_\lambda(V_\nu, N) \chi_\lambda(g),$$

Recall that characters of irreducible finite dimensional representations are orthogonal with respect to the Haar measure on K.

$$\int_K \chi_\nu(g)\overline{\chi_\lambda(g)} dg = \int_{T \subset K} \chi_\nu(e^{i\theta})\chi_\lambda(e^{-i\theta})|\Delta(e^{i\theta})|^2 d\theta_1 \dots d\theta_r$$
$$= \int_T \sum_w \sigma(w) e^{i(\theta, w(\nu+\rho))} \sum_u \sigma(u) e^{-i(\theta, u(\lambda+\rho))} d\theta_1 \dots d\theta_r$$
$$= \int_T \sum_{u,w} \sigma(w)\sigma(u) e^{i(\theta, w(\nu+\rho))-(\theta, u(\lambda+\rho))} d\theta_1 \dots d\theta_r = \delta_{\nu\lambda}|T||W| \tag{2}$$

where $|T| = (2\pi)^r$, $|W|$ is the order of the Weyl group and θ_a are the coordinates of θ in the basis corresponding to simple coroots $\alpha_a^\vee = \frac{2\alpha_a}{(\alpha_a, \alpha_a)}$. Only terms with $u = w$ give nonzero contribution and only if $\nu = \lambda$. Here we used the Weyl character formula

$$\chi_\lambda(e^{i\theta}) = \frac{\sum_w \sigma(w) e^{i(\theta, w(\lambda+\rho))}}{\Delta(e^{i\theta})},$$

where
$$\Delta(e^{i\theta}) = e^{i(\theta,\rho)} \prod_{\alpha\in\Delta^+} (1 - e^{-i(\theta,\alpha)})$$
is the Weyl denominator, and ρ is a half sum of positive roots. Here $(\theta, \lambda) \equiv \lambda(\theta) = \sum_a \theta_a \lambda_a$ and λ_a are the coordinates in the basis of fundamental weights. The torus T corresponds to $0 \leqslant \theta_a < 2\pi$, i.e. we have a natural identification $T \simeq \mathfrak{h}/2\pi\Lambda^\vee$, where $\Lambda^\vee$ is the lattice of coroots.

Due to the orthogonality of characters the multiplicity $m_\lambda(V, N)$ can be written as the following integral
$$\begin{aligned} m_\lambda(V_\nu, N) &= \int_K \chi_\nu(g)^N \overline{\chi_\lambda(g)} dg \\ &= \frac{1}{|W||T|} \int_T \chi_\nu(e^{i\theta})^N \chi_\lambda(e^{-i\theta}) |\Delta(e^{i\theta})|^2 d\theta_1 \dots d\theta_r, \end{aligned} \tag{3}$$

Using the Weyl character formula we can further simplify this formula:
$$m_\lambda(V_\nu, N) = \frac{1}{|T|} \int_T \chi_\nu(e^{i\theta})^N e^{-i(\theta,\lambda+\rho)} \Delta(e^{i\theta}) d\theta_1 \dots d\theta_r. \tag{4}$$

Lemma 1. *Let $\gamma \in \mathfrak{h}$ be such that $e^{i\alpha(\gamma)} = 1$ for all roots $\alpha \in \Delta$. Then*
$$F(\theta + \gamma) = F(\theta),$$
where
$$F(\theta) = \chi_\nu(e^{i\theta})^N e^{-i(\theta,\lambda+\rho)} \Delta(e^{i\theta}).$$
is the integrand in (4).

Proof. Because $e^{i\alpha(\gamma)} = 1$ for each root α we have:
$$\Delta(e^{i\theta+i\gamma}) = e^{i\rho(\gamma)} \Delta(e^{i\theta}).$$

Now let us show that $\chi_\nu(e^{i\theta+i\gamma}) = e^{i\nu(\gamma)} \chi_\nu(e^{i\theta})$. First, let us show that all terms in the numerator of the Weyl formula change by the same factor. If s_a is a simple reflection
$$\exp\left(i s_\alpha(\nu + \rho)(\gamma)\right) = \exp\left(i(\nu + \rho - \frac{2(\alpha, \nu + \rho)}{(\alpha, \alpha)} \alpha)(\gamma) \right),$$
but $\frac{2(\alpha,\nu+\rho)}{(\alpha,\alpha)}$ is an integer and because $e^{i\alpha(\gamma)} = 1$ we have
$$\exp\left(i s_\alpha(\nu + \rho)(\gamma)\right) = \exp\left(i(\nu + \rho)(\gamma)\right).$$

Therefore $\exp(iw(\nu+\rho)(\gamma)) = 1$ for any $w \in W$. From this and the Weyl character formula we conclude that

$$\chi_\nu(e^{i\theta+i\gamma}) = e^{i\nu(\gamma)}\chi_\nu(e^{i\theta}).$$

Thus we have

$$F(\theta+\gamma) = e^{i(\nu N-\lambda)(\gamma))}F(\theta).$$

But because λ is the highest weight in the tensor product decomposition of $V_\nu^{\otimes N}$, $N\nu-\lambda$ is in the root lattice and therefore $\exp(i(\nu N-\lambda)(\gamma)) = 1$. □

Now let Λ be the root lattice $\Lambda = \bigoplus_a \mathbb{Z}\alpha_a$ and α, and $\Lambda^\vee = \bigoplus_a \mathbb{Z}\alpha_a^\vee$ be the lattice of coroots $\alpha_a^\vee = \frac{2\alpha_a}{(\alpha_a,\alpha_a)}$. Also, denote by L the lattice of weights $L = \bigoplus_a \mathbb{Z}\omega_a$, $(\omega_a, \alpha_b^\vee) = \delta_{ab}$, and by $L^\vee$ the lattice of coweights $L^\vee = \bigoplus_a \mathbb{Z}\omega_a^\vee$, $\omega_a^\vee = \frac{(\alpha_a,\alpha_a)}{2}\omega_a$. The condition $e^{i\alpha(\gamma)} = 1$ holds for all α if and only if $e^{i\alpha_a(\gamma)} = 1$ for simple roots α_a. Writing γ in the basis of fundamental coweights $\gamma = \sum\limits_a \gamma^a\omega_a^\vee$ we have $e^{i\alpha(\gamma)} = 1$, i.e $\gamma \in \mathfrak{h}$ satisfies conditions of Lemma 1 if and only if $\gamma \in 2\pi L^\vee$. In the basis of coroots $\theta = \sum\limits_a \theta_a\alpha_a^\vee$ we have $0 \leqslant \theta_a < 2\pi$. Therefore we can naturally identify $T \simeq \mathbb{R}^r/2\pi\Lambda^\vee$. Thus vectors γ with the condition $e^{i\alpha(\gamma)} = 1$ for all α are in bijection with $L^\vee/\Lambda^\vee$. Thus we have proved the following

Lemma 2. *Vectors $\gamma \in \mathfrak{h}$ satisfying $e^{i\alpha(\gamma)} = 1$ for all roots α are in bijection with $L^\vee/\Lambda^\vee$. Since $|L^\vee/\Lambda^\vee| = \det(C_{ab})$ (where C_{ab} is the Cartan matrix of $\mathfrak{g}$) we have that many such vectors.*

2.2. The steepest descent method for generic ξ. The character function $\chi_\nu(z)$ is holomorphic for all $z \neq 0$ and therefore we can deform the integration contour.

$$m_\lambda(V_\nu, N) = (-i)^r\frac{1}{|T|}\int\limits_{C\subset G} \chi_\nu(z)^N z^{-\lambda-\rho}\Delta(z)\frac{dz_1}{z_1}\dots\frac{dz_r}{z_r}, \qquad (5)$$

where C is a deformation of the torus $T = \{|z_j| = 1, z_j = e^{i\theta_j}\}$ in the complexification G of K.

Now let us use the steepest descent method to compute the asymptotics of this integral when $N \to \infty$ and $\lambda = N\xi$, where $\xi \in \mathfrak{h}^*$ is fixed and we assume that it is generic, i.e. $(\xi, \alpha) > 0$ for all $\alpha \in \Delta_+$. For this we should deform the contour C so that it would pass through critical points in the direction of the steepest descent and we should choose critical points where the absolute value of the integrant is maximal. Here $D(V_\nu, N)$ is the set of dominant integral weights λ in the convex hull of W-orbit of V_ν. Note that for any λ in this tensor product decomposition $N\nu - \lambda \in \Lambda$.

Lemma 3. *For $x \in \mathfrak{h}_{\mathbb{R}}$ and θ in $\mathfrak{t}$ we have:*

$$|\chi_\nu(e^{x+i\theta})| \leqslant |\chi_\nu(e^x)| \tag{6}$$

The number of points for which both expressions are equal is $\det C_{ab}$.

Proof. Let $Q(\nu)$ be the set of weights of V_ν and $c(\nu,\beta)$ is the multiplicity of weight $\nu - \beta \in Q(\nu),\ \beta \in \Lambda$. The characters

$$|\chi_\nu(e^x)| = |e^{(\nu,x)}||1 + \sum_{\beta \in Q(\nu)} c(\nu;\beta) e^{-(\nu-\beta,y)}|,$$

$$|\chi_\nu(e^{x+i\theta})| = |e^{(\nu,x)}||e^{i(\nu,\theta)}||1 + \sum_{\beta \in Q(\nu)} c(\nu;\beta) e^{-(\nu-\beta,x)-i(\nu-\beta,\theta)}|.$$

The inequality is due to Cauchy Schwarz inequality. It is clear that both characters have the same modulus only if $(\nu - \beta, \theta) \in 2\pi\mathbb{Z}$. Thus the equality sign holds when $x \in \mathfrak{h}_{\mathbb{R}}$ and $\theta \in 2\pi L^\vee/\Lambda^\vee$ as in Lemmas 1, 2. $\square$

This lemma implies that asymptotically, as $N \to \infty$, the critical points giving the leading contribution to the asymptotic of the form $e^{x+i\gamma}$, where $x = \sum_a x_a \alpha_a^\vee \in \mathfrak{h}_{\mathbb{R}}$, i.e. $x_a \in \mathbb{R}$ and γ as in Lemma 1. Thus, our integral is

$$m_\lambda(V_\nu, N) = (-i)^r \frac{1}{(2\pi)^r} \int\limits_{C \subset G} e^{Nf(x,\xi)} e^{-(x,\rho)} \Delta(e^x) dx_1 \dots dx_r, \tag{7}$$

where $f(x,\xi) = \ln(\chi_\nu(e^x)) - (x,\xi)$ and C is the steepest descent path through critical point of $f(x,\xi)$.

The character function $\chi_\nu(e^x)$ is strictly convex for real x, therefore the function $f(x,\xi)$ is strictly convex. Therefore the equation

$$\frac{\partial}{\partial x_a} f(x,\xi) = \sum_b B_{ab}\xi_b \tag{8}$$

has unique real solution η.

If ξ is generic this unique solution is also generic. If ξ is not generic, i.e. if it is on a wall of the principal Weyl chamber, the real critical point η is on the same wall, in particular, it has the same stabilizer in the Weyl group. We will consider nongeneric cases in the next section.

We can either count contributions from all relevant critical points $e^{\eta+i\gamma}$, $\gamma \in 2\pi L^\vee/\Lambda^\vee$ when integrating over T or from the only relevant critical point e^η if we integrate over T/Π, where $\Pi = L^\vee/\Lambda^\vee$ Both ways we have:

$$m_\lambda(V_\nu, N) = (-i)^r e^{Nf(\eta,\xi)} e^{-(\eta,\rho)} \Delta(e^\eta) \frac{\det(B)}{(2\pi)^r}$$
$$\times \int\limits_{C_0 \subset G} e^{N\frac{1}{2}\sum\limits_{ab} y_a \frac{\partial^2 f(x,\xi)}{\partial x_z \partial x_b}|_{x=\eta} y_b} dy_1 \dots dy_r (1 + O(1/N)).$$

Here C_0 is a small interval near the real critical point in the direction of the steepest descent (which is imaginary direction), y_a are the coordinates in the root basis. The factor $\det(B)$ had appeared with the account of the Jacobian. After the change of variables $y_a = i\frac{s_a}{\sqrt{N}}$ we obtain the leading terms of the asymptotic as the Gaussian integral:

$$m_\lambda(V_\nu, N) = e^{Nf(\eta,\xi)} e^{-(\eta,\rho)} \Delta(e^\eta) \frac{\det(B) N^{-r/2}}{(2\pi)^r}$$
$$\times \int\limits_{\mathbb{R}^r} e^{-\frac{1}{2}\sum\limits_{ab} s_a \frac{\partial^2 f(x,\xi)}{\partial x_a \partial x_b}|_{x=\eta} s_b} ds_1 \dots ds_r (1 + O(1/N)).$$

Computing this Gaussian integral we obtain the asymptotics of the multiplicity from [1, 7, 9][1]

$$m_\lambda(V_\nu, N) = N^{-\frac{r}{2}} e^{Nf(\eta,\xi)-(\eta,\rho)} \Delta(e^\eta) \frac{\det(B)}{(2\pi)^{\frac{r}{2}} \sqrt{\det(f^{(2)}(\eta))}} \times \left(1 + O\left(\frac{1}{N}\right)\right). \tag{9}$$

2.3. The steepest descent method for nongeneric ξ. In this section we identify $\mathfrak{g} \simeq \mathfrak{g}^*$ using the Killing form. Recall that ξ in the principal Weyl chamber of $\mathfrak{h}$ is called generic if $(\alpha_a, \xi) > 0$ for all simple roots α_a. When ξ be non generic $(\alpha_a, \xi) = 0$ for some simple roots. These roots are simple roots for a Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$.

The root system of $\mathfrak{g}_0$ is the root subsystem of the one for $\mathfrak{g}$. In particular, we have $\Delta^+ = \Delta_0^+ \sqcup \Delta_1^+$ where Δ_0^+ are positive roots of $\mathfrak{g}_0$ and Δ_1^+ are other positive roots of $\mathfrak{g}$. We also have orthogonal decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with respect to the Killing form. Here $\mathfrak{g}_1 = \mathfrak{g}_0^\perp$.

[1]In [7] we have $\det(B)$ instead of $\det(C)$ because we used root coordinates x_a instead of the coroots as we use here.

If $\xi \in \mathfrak{h}_1$, i.e. when it is orthogonal to $\mathfrak{h}_0$, the critical point η is in $\mathfrak{h}_1$.

Lemma 4. *Let $f(y) = \ln(\chi_\nu(e^y))$, where $\{y_a\}_{a=1\dots r}$ are coordinates in Cartan subalgebra in a the root basis and η be a point on the wall of the principal Weyl chamber. Then*

$$\left[\frac{\partial^2 f(y)}{\partial y_a \partial y_b}|_{y=\eta}\right]_{a,b=1}^r = \left[\begin{array}{c|c} \kappa B_{ab}^{\mathfrak{g}_0} & 0 \\ \hline 0 & f^{(2)}(\eta) \end{array}\right],$$

where in the first diagonal block α_a and α_b are in Δ_0^+, $B_{ab}^{\mathfrak{g}_0}$ is the symmetrized Cartan matrix for $\mathfrak{g}_0$ and $f^{(2)}(y)$ is the matrix of second derivatives in y_a corresponding to $\alpha_a \in \Delta_1^+$.

$$\kappa = \frac{1}{\dim \mathfrak{g}_0} \frac{\sum\limits_\mu \mathrm{tr}_{W_\mu}(e^\eta) c_2^{\mathfrak{g}_0}(\mu) dim(V_\mu^{\mathfrak{g}_0})}{\sum\limits_\mu \mathrm{tr}_{W_\mu}(e^\eta) \dim(V_\mu^{\mathfrak{g}_0})}.$$

Here the summation over μ corresponds to the decomposition of the $\mathfrak{g}$-module V into irreducible $\mathfrak{g}_0$ components, $V \simeq \bigoplus_\mu W_\mu \otimes V_\mu^{\mathfrak{g}_0}$.

Proof. Let us compute the matrix of second derivatives of $f(y)$ when y is in the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Straightforward computation gives:

$$\frac{\partial^2 f}{\partial y_a \partial y_b}|_{y=\eta} = \frac{\partial}{\partial y_a} \frac{\frac{\partial}{\partial y_b} \mathrm{tr}_{V_\nu}(e^y)}{\mathrm{tr}_{V_\nu}(e^y)}|_{y=\eta}$$

$$= \frac{\frac{\partial^2}{\partial y_a \partial y_b}(\mathrm{tr}_{V_\nu}(e^y))\,\mathrm{tr}_{V_\nu}(e^y) - \frac{\partial}{\partial y_a}(\mathrm{tr}_{V_\nu}(e^y))\,\frac{\partial}{\partial y_b}(\mathrm{tr}_{V_\nu}(e^y))}{(\mathrm{tr}_{V_\nu}(e^y))^2}|_{y=\eta}$$

$$= \frac{\mathrm{tr}_{V_\nu}(H_a H_b e^\eta)\,\mathrm{tr}_{V_\nu}(e^\eta) - \mathrm{tr}_{V_\nu}(H_a e^\eta)\,\mathrm{tr}_{V_\nu}(H_b e^\eta)}{(\mathrm{tr}_{V_\nu}(e^\eta))^2}.$$

Here H_a is the basis of simple roots in $\mathfrak{h}$.

Now let us specialize this formula to the case when $y = \eta \in \mathfrak{h}_0 \subset \mathfrak{g}_0$. Let $\Delta_0^+ \subset \Delta^+$ be subset of positive $\mathfrak{g}_0$-roots in the set of positive $\mathfrak{g}$-roots.

- If $\alpha_a, \alpha_b \in \Delta_0^+$ we have $(\alpha_a, \eta) = 0$ and $(\alpha_b, \eta) = 0$ and we have

  $$\mathrm{tr}_{V_\nu}(H_a H_b e^\eta) = \sum_\mu \mathrm{tr}_{W_\mu}(e^\eta)\,\mathrm{tr}_{V_\mu^{\mathfrak{g}_0}}(H_a H_b).$$

  Here $H_a, H_b \in \mathfrak{h}_0 \subset \mathfrak{g}_0$ and $V_\mu^{\mathfrak{g}_0}$ is an irreducible $\mathfrak{g}_0$-module, so by Schur's lemma [7] we have:

  $$\mathrm{tr}_{V_\mu^{\mathfrak{g}_0}}(H_a H_b) = \frac{c_2^{\mathfrak{g}_0}(\mu) dim(V_\mu^{\mathfrak{g}_0})}{\dim \mathfrak{g}_0} B_{ab}^{\mathfrak{g}_0}.$$

The identities

$$\mathrm{tr}_{V_\nu}(e^\eta) = \sum_\mu \mathrm{tr}_{W_\mu^\nu}(e^\eta)\dim(V_\mu^{\mathfrak{g}_0})$$

and

$$\mathrm{tr}_{V_\nu}(H_a e^\eta) = \sum_\mu \mathrm{tr}_{W_\mu^\nu}(e^\eta)\,\mathrm{tr}(H_a) = 0$$

are clear. Thus, if $\alpha_a, \alpha_b \in \Delta_0^+$ we have

$$\frac{\partial^2 f(y)}{\partial y_a \partial y_b}|_{y=\eta} = \frac{\sum_\mu \mathrm{tr}_{W_\mu}(e^\eta) c_2^{\mathfrak{g}_0}(\mu) dim(V_\mu^{\mathfrak{g}_0})}{\dim \mathfrak{g}_0 \sum_\mu \mathrm{tr}_{W_\mu}(e^\eta)\dim(V_\mu^{\mathfrak{g}_0})} B_{ab}^{\mathfrak{g}_0}.$$

Note that in case $\eta = 0$, i.e $\mathfrak{g}_0 = \mathfrak{g}$ we have

$$\frac{\partial^2 f(y)}{\partial y_a \partial y_b}|_{y=0} = \frac{c_2(\nu)}{\dim \mathfrak{g}} B_{ab}.$$

- If only $\alpha_a \in \mathfrak{g}_0$ and α_a is not in $\mathfrak{g}_0$ we have$(\alpha_a, \eta) = 0$ and $(\alpha_b, \eta) \neq 0$ and therefore

$$\mathrm{tr}_{V_\nu}(H_a H_b e^\eta) = \sum_\mu \mathrm{tr}_{W_\mu^\nu}(H_b e^\eta)\,\mathrm{tr}_{V_\mu^{\mathfrak{g}_0}}(H_a) = 0,$$

$$\mathrm{tr}_{V_\nu}(H_a e^\eta) = 0,$$

So in this case

$$\frac{\partial^2 f(y)}{\partial y_a \partial y_b}|_{y=\eta} = 0$$ □

Now we can use this lemma to split the integration in the neighborhood $\mathfrak{t}$ of the critical point η into the integral over orthogonal components $\mathfrak{t}_0 = \mathfrak{h}_0 \cap \mathfrak{t}$ and $\mathfrak{t}_1 = \mathfrak{h}_1 \cap \mathfrak{t}$. Introducing coordinates s, t we change variables in (7) as

$$x = \eta + \frac{is}{\sqrt{\kappa N}} s + \frac{it}{\sqrt{N}},$$

where $s \in \mathfrak{h}_0,\ t \in \mathfrak{h}_1$. As $N \to \infty$ we have

$$e^{-(x,\rho)}\Delta(e^x) = \left(\frac{1}{\kappa N}\right)^{\frac{|\Delta_0^+|}{2}} e^{-(\eta,\rho_1)} \prod_{\alpha\in\Delta_1^+} (e^{\frac{(\eta,\alpha)}{2}} - e^{-\frac{(\eta,\alpha)}{2}}) \times \prod_{\alpha\in\Delta_0^+} \left(i(\alpha,s) + \frac{(\alpha,s)^2}{2\sqrt{\kappa N}} \right),$$

where $\rho_1 = \frac{1}{2}\sum_{\alpha\in\Delta_1^+} \alpha$. Note that the contribution from a monomial containing linear terms (α,s) vanishes after the integration over s because it is odd with respect to the reflection s_α. Thus, the leading term is given by the integral

$$\begin{aligned} m_\lambda(V_\nu,N) &= N^{-\frac{r}{2}-|\Delta_0^+|} e^{Nf(\eta,\xi)-(\eta,\rho_1)} \kappa^{-\frac{\dim \mathfrak{g}_0}{2}} \\ &\quad\times \prod_{\alpha\in\Delta_1^+} (e^{\frac{(\eta,\alpha)}{2}} - e^{-\frac{(\eta,\alpha)}{2}}) \frac{|\Pi|}{(2\pi)^r 2^{|\Delta_0^+|}} \\ &\quad\times \int_{\mathbb{R}^{r_0}} e^{-\frac{1}{2}(s,s)_{\mathfrak{g}_0}} \prod_{\alpha\in\Delta_0^+} (\alpha,s)^2 d^{r_0}s \int_{\mathbb{R}^{r-r_0}} e^{-\frac{1}{2}(t,f^{(2)}t)} d^{r-r_0}t \left(1 + O\left(\frac{1}{N}\right)\right) \\ &= N^{-\frac{r}{2}-|\Delta_0^+|} e^{Nf(\eta,\xi)-(\eta,\rho_1)} \prod_{\alpha\in\Delta_1^+} (e^{\frac{(\eta,\alpha)}{2}} - e^{-\frac{(\eta,\alpha)}{2}}) \frac{\kappa^{-\frac{\dim \mathfrak{g}_0}{2}}}{\sqrt{\det(f^{(2)}(\eta))}} \cdot \\ &\quad\times \frac{\det(B)|W_0| \prod_{\alpha\in\Delta_0^+}(\rho,\alpha)}{(2\pi)^{\frac{r}{2}} 2^{|\Delta_0^+|} \sqrt{\det(B^{\mathfrak{g}_0})}} \left(1 + O\left(\frac{1}{N}\right)\right). \end{aligned} \tag{10}$$

Here we used the formula

$$\int_{\mathbb{R}^r} e^{-\frac{1}{2}(s,s)} \prod_{\alpha\in\Delta^+} (\alpha,s)^2 d^r s = (2\pi)^{\frac{r}{2}} \frac{|W_0| \prod_{\alpha\in\Delta^+}(\rho,\alpha)}{\sqrt{\det(B)}},$$

where $s = \sum_{a=1}^{r} \alpha_a s_a,\ (s,s) = \sum_{a,b=1}^{r} s_a s_b (\alpha_a,\alpha_b) = \sum_{a,b=1}^{r} s_a s_b B_{ab}$, $d^r s = (\prod_{i=1}^{r} d_i) ds_1 \dots ds_r$ and d_i is the length of the root α_i, $|\Pi| = \det(C)$ is the determinant of the Cartan matrix of $\mathfrak{g}$.

In particular, in the extreme case $\xi = 0$ we have

$$m_0(V_\nu, N) = N^{\frac{-\dim(\mathfrak{g})}{2}} \dim(V_\nu)^N \frac{\sqrt{\det(B)}|W|}{(2\pi)^{\frac{r}{2}}} \frac{\prod_{\alpha\in\Delta^+}(\rho,\alpha)}{2^{|\Delta^+|}} \left(\frac{\dim \mathfrak{g}}{c_2(V_\nu)}\right)^{\frac{\dim \mathfrak{g}}{2}} \left(1 + O\left(\frac{1}{N}\right)\right).$$

For the fundamental representation of $\mathfrak{sl}_2$ we have $c_2(\nu) = \frac{3}{2}$, $|\Pi| = 2$, $|W| = 2$, $(\rho, \alpha) = 1, \det(B) = 2$ and this formula gives the multiplicity of the trivial subrepresentation in $V(\omega)^{\otimes N}$:

$$m_0(N) = N^{-\frac{3}{2}} 2^N \frac{4}{\sqrt{2\pi}} \left(1 + O\left(\frac{1}{N}\right)\right). \tag{11}$$

When $\mathfrak{g}_0 = \{0\}$, i.e. when ξ is generic, formula (10) gives (9).

2.4. Example: powers of fundamental module of $\mathfrak{sl}_2$. For $\mathfrak{g} = \mathfrak{sl}_2$ there is only one positive root α, $\alpha^\vee = \alpha$, the fundamental weight is $\omega = \frac{\alpha}{2}$. When $V = V_\omega$ is the fundamental 2-dimensional representation the character $\chi_\nu(e^{i\theta}) = \chi_\omega(e^{i\theta}) = e^{i\theta} + e^{-i\theta}$ and for the multiplicity of $V_\lambda \subset V_\omega^{\otimes N}$, $\lambda = \frac{l\alpha}{2}$ we have:

$$\begin{aligned}
m_\lambda(N) &= \int_G \chi_\nu(g)^N \overline{\chi_\lambda(g)} dg = \frac{1}{2\cdot 2\pi} \int_0^{2\pi} \chi_\omega(e^{i\theta})^N \chi_l(e^{-i\theta}) |e^{i\theta} - e^{-i\theta}|^2 d\theta \\
&= \frac{1}{4\pi} \int_0^{2\pi} \chi_\omega(e^{i\theta})^N \frac{e^{-i\theta(l+1)} - e^{i\theta(l+1)}}{e^{-i\theta} - e^{i\theta}} |e^{i\theta} - e^{-i\theta}|^2 d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \chi_\omega(e^{i\theta})^N e^{-i\theta(l+1)} (e^{i\theta} - e^{-i\theta}) d\theta \\
&= \frac{2}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \chi_\omega(e^{i\theta})^N e^{-i\theta(l+1)} (e^{i\theta} - e^{-i\theta}) d\theta.
\end{aligned}$$

As $N \to \infty$ and $\xi = \frac{l}{N}$ is fixed

$$m_\lambda(N) = \frac{2}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{N[\ln(e^{i\theta} + e^{-i\theta}) - 2i\theta\xi]} \frac{(e^{i\theta} - e^{-i\theta})}{e^{i\theta}} d\theta.$$

2.4.1. Consider first generic values of ξ, i.e. $0, \xi < \frac{1}{2}$. We can now apply the steepest descent method. It is easy to find points θ where $\ln(e^{i\theta} + e^{-i\theta}) - 2i\theta\xi$ attains its maximum, they are solutions to

$$\frac{ie^{i\theta} - ie^{-i\theta}}{e^{i\theta} + e^{-i\theta}} = 2i\xi.$$

This will give us two points η and $\eta + i\pi$

$$e^{\eta} = \pm\sqrt{\frac{\frac{1}{2} + \xi}{\frac{1}{2} - \xi}}.$$

These points are located on the real axis and outside S^1. If we integrate over the whole circle S^1 the contour $|e^{i\theta}| = 1$ should be deformed to the steepest descent contour through critical points η and $\eta + i\pi$ (Figure 1). If we integrate over the fundamental domain T/Π i.e. $\pi/2 \leqslant \theta \leqslant \pi/2$ the steepest descent contour passes through η only (Figure 2).

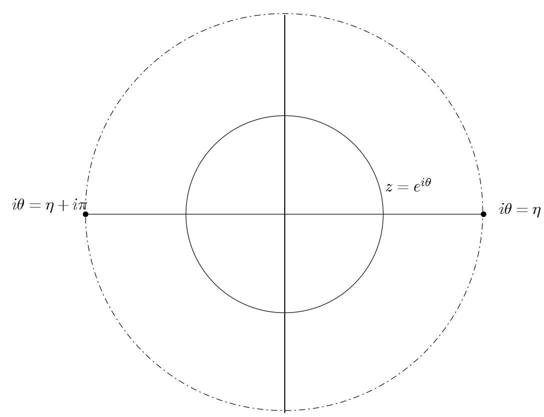

Figure 1. Deforming unit circle $z = e^{i\theta}$ into the steepest descent contour(dashed line) passing through critical points η and $\eta = i\pi$.

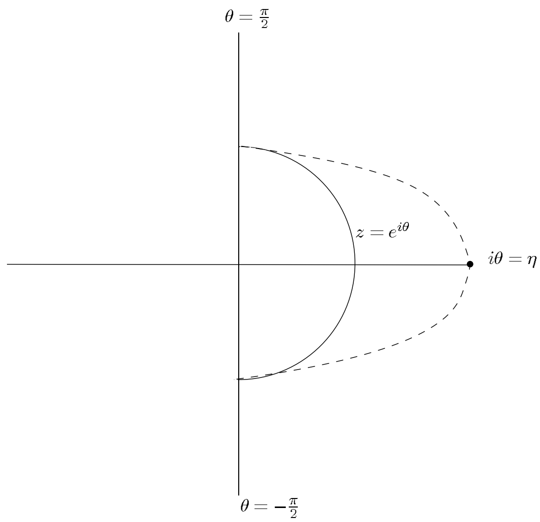

Figure 2. Deforming semicircle integration contour in the fundamental domain into the steepest descent contour(dashed line) passing through critical point η.

It is easy to compute the critical value of our function:

$$f(\eta,\xi) = \ln(e^{\eta}+e^{-\eta}) - 2\eta\xi = \ln\left(\frac{1}{\sqrt{(\frac{1}{2}+\xi)(\frac{1}{2}-\xi)}}\right) - 2\xi\ln\left(\sqrt{\frac{\frac{1}{2}+\xi}{\frac{1}{2}-\xi}}\right)$$
$$= -\left(\left(\frac{1}{2}+\xi\right)\ln\left(\frac{1}{2}+\xi\right) + \left(\frac{1}{2}-\xi\right)\ln\left(\frac{1}{2}-\xi\right)\right).$$

Near the critical point $i\theta = \eta$ we change coordinates to $i\theta = \eta + \frac{is}{\sqrt{Nf^{(2)}(\eta)}}$. This results in the asymptotic integral:

$$m_\lambda(N) = \frac{2}{2\pi}\int\limits_{-\epsilon N}^{\epsilon N} e^{Nf(\eta,\xi)-\frac{s^2}{2}}\left(1-e^{-2\eta}\right)\frac{ds}{\sqrt{Nf^{(2)}(\eta)}}.$$

Computing the Gaussian integral and taking into account that $f^{(2)}(\eta) = 1 - 4\xi^2$ and $1 - e^{2\eta} = \frac{4\xi}{1+4\xi}$ we have

$$m_\lambda(N) = N^{-\frac{1}{2}} e^{Nf(\eta,\xi)} \frac{4\xi}{\sqrt{\pi}\,(1+2\xi)^{\frac{3}{2}}\,(1-2\xi)^{\frac{1}{2}}} \left(1 + O\left(\frac{1}{N}\right)\right).$$

This expression agrees with the asymptotics of the binomial coefficients.

2.4.2. Now assume that $\xi = 0$. In this case we have two critical points $\theta = 0$ and $\theta = \pi$ on S^1 and

$$m_0(N) = \frac{2}{2\pi} \int\limits_{-\epsilon}^{\epsilon} e^{N\ln(e^{i\theta}+e^{-i\theta})} \left(1 - e^{-2i\theta}\right) d\theta.$$

Taking into account that $\ln(e^{i\theta} + e^{-i\theta}) = \ln 2 - \theta^2/2 + \dots$ and $1 - e^{-2i\theta} = 2i\theta + 2\theta^2 + \dots$ and changing variables on the steepest descent contour as $\theta = \frac{s}{\sqrt{N}}$ we obtain

$$\begin{aligned} m_0 &= \frac{N^{-\frac{3}{2}}}{\pi} 2^N \int\limits_{-\infty}^{\infty} e^{-\frac{s^2}{2}} 2s^2 ds \left(1 + O\left(\frac{1}{N}\right)\right) \\ &= N^{-\frac{3}{2}} 2^N \frac{2}{\pi} \sqrt{2\pi} \left(1 + O\left(\frac{1}{N}\right)\right) = N^{-\frac{3}{2}} 2^N \frac{4}{\sqrt{2\pi}} \left(1 + O\left(\frac{1}{N}\right)\right). \end{aligned}$$

Here we took into account that $\int\limits_{-\infty}^{\infty} e^{-\frac{s^2}{2}} s ds = 0$ and that $\int\limits_{-\infty}^{\infty} e^{-\frac{s^2}{2}} s^2 ds = \sqrt{2\pi}$. This formula agrees with (11).

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