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**MELLIN–BARNES INTEGRALS RELATED TO THE LIE
ALGEBRA $u(N)$**

ABSTRACT. We present an alternative proof of Gustafson’s generalization of the second Barnes’ lemma.

§1. INTRODUCTION

In refs. [1,2] R. A. Gustafson generalized the first and the second Barnes’ lemmas to the case of the Lie algebra $u(n)$. Namely, he calculated the following multidimensional Mellin - Barnes (MB) integrals in closed form:

$$\int_{-i\infty}^{i\infty} \dots \int_{-i\infty}^{i\infty} \frac{\prod_{k=1}^{N+1} \prod_{j=1}^N \Gamma(\alpha_k - z_j) \Gamma(\beta_k + z_j)}{\prod_{1 \leq k < j \leq N} \Gamma(z_k - z_j) \Gamma(z_j - z_k)} \prod_{k=1}^N \frac{dz_k}{2\pi i} = \frac{N! \prod_{k,j=1}^{N+1} \Gamma(\alpha_k + \beta_j)}{\Gamma\left(\sum_{k=1}^{N+1} (\alpha_k + \beta_k)\right)} \quad (1.1)$$

and

$$\int_{-i\infty}^{i\infty} \dots \int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^N \prod_{k=1}^{N+2} \Gamma(\alpha_k - z_j) \prod_{m=1}^{N+1} \Gamma(\beta_m + z_j)}{\prod_{m=1}^N \Gamma(\gamma - z_m) \prod_{1 \leq k < j \leq N} \Gamma(z_k - z_j) \Gamma(z_j - z_k)} \prod_{k=1}^N \frac{dz_k}{2\pi i} = \frac{N! \prod_{k=1}^{N+2} \prod_{j=1}^{N+1} \Gamma(\alpha_k + \beta_j)}{\prod_{k=1}^{N+2} \Gamma(\gamma - \alpha_k)}. \quad (1.2)$$

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Here $\gamma = \sum_{k=1}^{N+2} \alpha_k + \sum_{k=1}^{N+1} \beta_k$ and it is assumed that the integration contours separate sequences of poles going to the right, $\alpha_n + k$, and to the left, $-\beta_m - k$, $n = 1, \dots, N + 1(N + 2)$, $m = 1, \dots, N + 1$, which are due to the gamma-functions in the numerators.

The integral (1.2) depends on $2n+3$ external parameters and implies the relation in (1.1). Indeed, sending $\alpha_{N+2} \rightarrow \infty$ in (1.2) one recovers (1.1). The integral (1.2) was calculated in ref. [2] making use of the residues theorem and evaluating the corresponding sums with the help of Milne's $U(n)$ generalization of the Gauss summation theorem [3]. The proof of the MB integral (1.2) given in [1] is more involved and follows a different route. It relies on the integral (1.1) and uses induction on N to show that the expressions on the l.h.s. and the r.h.s. of (1.2) coincide for special values of the external parameters. The general case then follows from Carlson's theorem. In the present paper we present an alternative non-inductive derivation of the integral (1.2).

Classical MB integrals can be extended to the q -beta and elliptic integrals, see refs. [1, 2, 4-6], and to the MB integrals which involve the gamma-functions over the field of complex numbers [7]. In the latter case it was shown [8] that studying properties of the counterpart of the integral (1.1) as a function of external parameters one can recover the corresponding analogue of the second integral (1.2). We want to apply the same approach, with some modifications, to the classical MB integrals with Euler's gamma functions. However, the integral (1.1) is not a good starting point for such an analysis so we first consider a different MB integral with more suitable analytic properties.

§2. AUXILIARY INTEGRAL

Let

$$\mathcal{R}_{\pm}(a, \alpha, \beta) = \int_{-i\infty}^{i\infty} \dots \int_{-i\infty}^{i\infty} e^{\pm i\pi \sum z_k} \times \frac{\prod_{j=1}^N \left(\prod_{k=1}^{N+1} \Gamma(\alpha_k - z_j) \prod_{m=1}^N \Gamma(z_j + \beta_m) \right)}{\prod_{k=1}^N \Gamma(a - z_k) \prod_{k < j} \Gamma(z_k - z_j) \Gamma(z_j - z_k)} \prod_{k=1}^N \frac{dz_k}{2\pi i}. \tag{2.1}$$

These integrals are close cousins of the integral (1.1): the only difference is that we moved one gamma function from the numerator to the denominator, $\Gamma(z_k + \beta_{N+1}) \rightarrow 1/\Gamma(a - z_k)$ and added the exponential factor, $e^{\pm i\pi \sum z_k}$. This factor compensates some sign factors arising during an evaluation of the integral by the residues theorem and changes its analytic properties as a function of the external parameters, $\{a, \alpha_k, \beta_j\}$. Indeed, while the integral (1.1) converges (at large z_k) for all values of the external parameters, the convergence of the integrals (2.1) is controlled by the parameter

$$\nu = a - \sum_{k=1}^{N+1} \alpha_k - \sum_{k=1}^N \beta_k. \quad (2.2)$$

The integrals \mathcal{R}_{\pm} converge only if $\text{Re}(\nu) > 0$.

Assuming that this condition is fulfilled, the integrals (2.1) can be calculated according to the strategy used in ref. [2] for the integral (1.1). Namely, one closes the integration contours in the left-half plane picking up the residues at the poles of the gamma functions which are located at $\{-\beta_j - k, k = 0, 1, \dots, 1 \leq j \leq N\}$. Taking into account the symmetry of the integrand under permutations of the arguments one derives in this way

$$\begin{aligned} \mathcal{R}_{\pm}(a, \alpha, \beta) &= N! e^{\mp i\pi \mathcal{B}} \sum_{n_1, \dots, n_N=0}^{\infty} \frac{1}{n_1! \dots n_N!} \\ &\times \prod_{k < j} \frac{\sin \pi(\beta_j - \beta_k)}{\pi} (-1)^{n_j + n_k} (\beta_k + n_k - \beta_j - n_j) \\ &\times \prod_{j=1}^N \frac{\prod_{k=1}^{N+1} \Gamma(\alpha_k + \beta_j + n_j)}{\Gamma(a + \beta_j + n_j)} \prod_{1 \leq k \neq j \leq N} \Gamma(\beta_k - \beta_j - n_j) \\ &= N! e^{\mp i\pi \mathcal{B}} \frac{\prod_{j=1}^N \prod_{k=1}^{N+1} \Gamma(\alpha_k + \beta_j)}{\prod_{j=1}^N \Gamma(a + \beta_j)} \\ &\times \sum_{n_1, \dots, n_N=0}^{\infty} \prod_{k < j} \frac{\beta_k + n_k - \beta_j - n_j}{\beta_k - \beta_j} \prod_{j=1}^N \prod_{k=1}^{N+1} \frac{(\alpha_k + \beta_j)_{n_j}}{(1 - \beta_k + \beta_j)_{n_j}}. \end{aligned} \quad (2.3)$$

Here $\beta_{N+1} \equiv 1 - a$ and $(a)_n$ stands for the Pochhammer symbol. Finally, evaluating the sum in the last line of Eq. (2.3) with the help of Milne's generalization of the Gauss summation theorem, see e.g. [9, Theorem 2.7], one obtains

$$\mathcal{R}_\pm(a, \alpha, \beta) = N! e^{\mp i\pi\mathcal{B}} \Gamma(a - \mathcal{B} - \mathcal{A}) \frac{\prod_{k=1}^N \prod_{j=1}^{N+1} \Gamma(\alpha_j + \beta_k)}{\prod_{j=1}^{N+1} \Gamma(a - \alpha_j)}, \quad (2.4)$$

where $\mathcal{B} = \sum_{k=1}^N \beta_k$ and $\mathcal{A} = \sum_{j=1}^{N+1} \alpha_j$. Note that the pole of the first gamma function at $\nu = a - \mathcal{B} - \mathcal{A} = 0$ signals that the integrals diverge for these values of the parameters. It is quite remarkable that the residues of \mathcal{R}_\pm at $\nu = 0$ can be itself represented as certain MB integrals. In order to find these integrals it is convenient to represent the original integrals (2.1) in a determinant form.

§3. DETERMINANT REPRESENTATION FOR \mathcal{R}_\pm

Taking into account that

$$\prod_{k < j} \frac{1}{\Gamma(z_k - z_j)\Gamma(z_j - z_k)} = \prod_{k < j} \frac{1}{\pi} (z_j - z_k) \sin \pi(z_k - z_j) \quad (3.1)$$

one gets the following representation for \mathcal{R}_\pm

$$\begin{aligned} \mathcal{R}_\pm(a, \alpha, \beta) &= \pi^{-N(N-1)/2} \frac{1}{(2\pi i)^N} \int_{-i\infty}^{i\infty} \dots \int_{-i\infty}^{i\infty} \prod_{k=1}^N (\cos \pi z_k)^{N-1} \\ &\times \prod_{k < j} (z_k - z_j) \left(\tan \pi z_j - \tan \pi z_k \right) Q_\pm(z_1) \dots Q_\pm(z_N) dz_1 \dots dz_N, \end{aligned} \quad (3.2)$$

where

$$Q_\pm(z) = \frac{e^{\pm i\pi z}}{\Gamma(a - z)} \prod_{k=1}^{N+1} \Gamma(\alpha_k - z) \prod_{j=1}^N \Gamma(z + \beta_j). \quad (3.3)$$

The factor $\prod_{k < j} (z_k - z_j) (\tan z_j - \tan z_k)$ is given by the product of two Vandermonde determinants

$$(-1)^{N(N-1)/2} \det \begin{vmatrix} 1 & \dots & 1 \\ z_1 & \dots & z_N \\ \vdots & \ddots & \vdots \\ z_1^{N-1} & \dots & z_N^{N-1} \end{vmatrix} \times \det \begin{vmatrix} 1 & \dots & 1 \\ t_1 & \dots & t_N \\ \vdots & \ddots & \vdots \\ t_1^{N-1} & \dots & t_N^{N-1} \end{vmatrix}, \quad (3.4)$$

where $t_k \equiv \tan \pi z_k$. Taking into account the symmetry of the integrand under permutations of the arguments z_1, \dots, z_N one can represent the integrals (3.2) as follows

$$\begin{aligned} \mathcal{R}_{\pm}(a, \alpha, \beta) = & (-\pi)^{\frac{N(N-1)}{2}} N! \int_{-i\infty}^{i\infty} \dots \int_{-i\infty}^{i\infty} \prod_{k=1}^N z_k^{k-1} (\cos \pi z_k)^{N-1} Q_{\pm}(z_k) \\ & \times \det |\mathbb{T}_N| \prod_{\ell=1}^N \frac{dz_{\ell}}{2\pi i}, \end{aligned} \quad (3.5)$$

where the matrix \mathbb{T}_N is the matrix of tangents in (3.4). Since the entries of the k -th column of the matrix \mathbb{T}_N depend only on one variable, z_k , one obtains the following (determinant) representation for the integrals in question

$$\mathcal{R}_{\pm}(a, \alpha, \beta) = (-\pi)^{N(N-1)/2} N! \det |\mathcal{Q}_{\pm}|. \quad (3.6)$$

Here \mathcal{Q}_{\pm} are $N \times N$ matrices with the entries

$$(\mathcal{Q}_{\pm})_{mk} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dz (\cos \pi z)^{N-1} (\tan \pi z)^{m-1} z^{k-1} Q_{\pm}(z). \quad (3.7)$$

Let $I_{mk}^+(z)$ and $I_{mk}^-(z)$ be the integrands in this expression. It is easy to see that $I_{mk}^+(iu)$ and $I_{mk}^-(-iu)$ decay exponentially ($\sim e^{-2\pi u}$) when $u \rightarrow \infty$ and power-like when $u \rightarrow -\infty$. The integrands in the expression (3.7), $I_{mk}^+(iu)$ and $I_{mk}^-(-iu)$, decay exponentially ($\sim e^{-2\pi u}$) when $u \rightarrow \infty$ and power-like when $u \rightarrow -\infty$

$$I_{mk}^{\pm}(\mp iu) \underset{u \rightarrow \infty}{=} 2\pi^N u^{\mathcal{A}+\mathcal{B}-a-N+k-1} \times (\mp i)^{\mathcal{B}-\mathcal{A}+a+m+k-2} (1+O(1/u)). \quad (3.8)$$

It is easy to see from Eqs. (3.7) and (3.8) that all matrix elements in the last row, $(Q_{\pm})_{mN}$, diverge when $a \rightarrow \mathcal{A} + \mathcal{B}$. Namely,

$$(Q_{\pm})_{mN} \underset{a \rightarrow \mathcal{A} + \mathcal{B}}{=} \pi^{N-1} \frac{e^{\mp i\pi \mathcal{B}} (\mp i)^{m+N-2}}{a - \mathcal{A} - \mathcal{B}} + O(1), \quad (3.9)$$

while the elements $(Q_{\pm})_{m,k}$ with $k < N$ are finite. Thus the singular part of the integrals \mathcal{R}_{\pm} in the limit $a \rightarrow \mathcal{A} + \mathcal{B}$ can be represented in the following form

$$\begin{aligned} \mathcal{R}_{\pm}(a, \alpha, \beta) \underset{a \rightarrow \mathcal{A} + \mathcal{B}}{=} & (-\pi)^{\frac{N(N-1)}{2}} N! \frac{e^{\mp i\pi \mathcal{B}} (\mp i\pi)^{N-1}}{a - \mathcal{A} - \mathcal{B}} \\ & \times \int_{-i\infty}^{i\infty} \cdots \int_{-i\infty}^{i\infty} \left(\prod_{k=1}^{N-1} z_k^{k-1} (\cos \pi z_k)^{N-1} Q_{\pm}(z_k) \right) \\ & \times \det |\widehat{\mathbb{T}}_N^{\pm}| \prod_{\ell=1}^{N-1} \frac{dz_{\ell}}{2\pi i} + O(1), \end{aligned} \quad (3.10)$$

where the matrices $\widehat{\mathbb{T}}_N^{\pm}$ read

$$\widehat{\mathbb{T}}_N^{\pm} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & (\mp i) \\ \vdots & \ddots & \vdots & \\ t_1^{N-1} & t_2^{N-1} & \cdots & (\mp i)^{N-1} \end{pmatrix}. \quad (3.11)$$

The determinants of these matrices can be written as follows

$$\det \widehat{\mathbb{T}}_N^{\pm} = (-1)^{N-1} \prod_{k=1}^{N-1} (t_k \pm i) \times \det \mathbb{T}_{N-1}, \quad (3.12)$$

where \mathbb{T}_{N-1} is the $(N-1) \times (N-1)$ Vandermonde matrix

$$\mathbb{T}_{N-1} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_{N-1} \\ \vdots & \ddots & \vdots & \vdots \\ t_1^{N-2} & t_2^{N-2} & \cdots & t_{N-1}^{N-2} \end{pmatrix}. \quad (3.13)$$

Since

$$t_k \pm i = \pm i \frac{e^{\mp i\pi z_k}}{\cos \pi z_k}$$

one can rewrite Eq. (3.10) as follows

$$\begin{aligned} \mathcal{R}_\pm(a, \alpha, \beta) & \underset{a \rightarrow \mathcal{A} + \mathcal{B}}{=} \frac{e^{\mp i\pi \mathcal{B}}}{a - \mathcal{A} - \mathcal{B}} (-\pi)^{(N-1)(N-2)/2} N! \\ & \times \int_{-i\infty}^{i\infty} \cdots \int_{-i\infty}^{i\infty} \left(\prod_{k=1}^{N-1} z_k^{k-1} (\cos \pi z_k)^{N-2} \tilde{Q}_\pm(z_k) \right) \\ & \times \det |T_{N-1}| \prod_{\ell=1}^{N-1} \frac{dz_\ell}{2\pi i} + O(1), \end{aligned} \tag{3.14}$$

where the functions \tilde{Q}_\pm are given by the product of gamma functions

$$\begin{aligned} \tilde{Q}_\pm(z) & = e^{\mp i\pi z} Q_\pm(z) \Big|_{a=\mathcal{A}+\mathcal{B}} \\ & = \frac{1}{\Gamma(\mathcal{A} + \mathcal{B} - z)} \prod_{k=1}^{N+1} \Gamma(\alpha_k - z) \prod_{j=1}^N \Gamma(z + \beta_j). \end{aligned} \tag{3.15}$$

Since the integral in (3.14) coincides, up to changes $N \rightarrow N - 1$ and $Q \rightarrow \tilde{Q}$, with the integral (3.5) it can be written in the form of the MB integral

$$\mathcal{R}_\pm \underset{a \rightarrow \mathcal{A} + \mathcal{B}}{=} N \frac{e^{\mp i\pi \mathcal{B}}}{a - \mathcal{A} - \mathcal{B}} \times \mathcal{T}_{N-1} + \text{regular terms}, \tag{3.16}$$

where

$$\mathcal{T}_{N-1} = \int_{-i\infty}^{i\infty} \cdots \int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^{N-1} \prod_{k=1}^{N+1} \Gamma(\alpha_k - z_j) \prod_{m=1}^N \Gamma(z_j + \beta_m)}{\prod_{k=1}^{N-1} \Gamma(a - z_k) \prod_{k < j} \Gamma(z_k - z_j) \Gamma(z_j - z_k)} \prod_{k=1}^{N-1} \frac{dz_k}{2\pi i} \tag{3.17}$$

and $a = \mathcal{A} + \mathcal{B}$.

Comparing residues at $a = \mathcal{A} + \mathcal{B}$ on the both sides of (2.4) one gets the following equation

$$\mathcal{T}_{N-1} = (N - 1)! \frac{\prod_{k=1}^N \prod_{j=1}^{N+1} \Gamma(\alpha_j + \beta_k)}{\prod_{j=1}^{N+1} \Gamma(\mathcal{A} + \mathcal{B} - \alpha_j)}, \tag{3.18}$$

which is, up to renumeration $N \rightarrow N - 1$, nothing but the second Gustafson integral (1.2).

§4. SUMMARY

The Mellin - Barnes integrals and their q - and elliptic generalizations play an important role in many topics in physics and mathematics, see e.g. refs. [4,10,11]. Many important results generalizing first and second Barnes' lemmas to certain Lie algebras were obtained by R.A. Gustafson [1,2]. In this article, we have presented an alternative proof of the integral (1.2), which is based on the study of the analytic properties of some auxiliary integrals as functions of external parameters. This technique is not restricted to the case under consideration. It has already been applied to the complex $SL(2, \mathbb{C})$ MB integrals, see ref. [8]. Finally we note that the same analysis, applied to the MB integral

$$\int_{-i\infty}^{i\infty} \dots \int_{-i\infty}^{i\infty} \frac{\prod_{k=1}^{2N+1} \prod_{j=1}^N \Gamma(\alpha_k \pm z_j)}{\prod_{k=1}^N \Gamma(\beta \pm z_k) \Gamma(\pm 2z_k) \prod_{k<j} \Gamma(\pm z_k \pm z_j)} \prod_{n=1}^N \frac{dz_n}{4\pi i}$$

$$= N! \Gamma(\beta - A) \frac{\prod_{k<j} \Gamma(\alpha_k + \alpha_j)}{\prod_{k=1}^{2N+1} \Gamma(\beta - \alpha_k)},$$

where $\Gamma(\pm a) = \Gamma(a)\Gamma(-a)$, $\Gamma(a \pm b) = \Gamma(a + b)\Gamma(a - b)$, etc., allows one to obtain the MB integral [1, Theorem 5.3].

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