Записки научных
семинаров ПОМИ
Том 509, 2021 г.

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DEFORMATION OF THE POISSON STRUCTURE OF A POINT PARTICLE DUE TO GRAVITATIONAL BACK REACTION

Abstract. The dynamics of a massive particle in a frame of a test particle in 3+1 spacetime dimensions is considered with gravitational interaction taken into account. The total action (gravity+particles) collapses to a boundary separating the massive particle and the test particle, and is further reduced to a finite dimensional action depending only on relative particle coordinates and momenta. It turns out that the momentum space is a coadjoint orbit of the Lorentz group. The momentum space is thus curved and its curvature falls off with the particle relative distance according to the Newton law. This defines the modified form of the Poisson brackets. At the quantum level, this results in non-commutativity and partial discreteness in coordinate space.

§1. Introduction

It has long been believed that quantum theory of gravity could regularize its singularities and, possibly, divergences appeared in perturbation theory. The first argument for that dates back to Bronstein [1], who showed that there could be no measurable distances smaller than the Planck length. In the absence of a full theory of gravity there is still no decisive answer whether this is indeed the case or not.

This question can be answered for gravity in $2+1$ spacetime dimensions which is an integrable model [2], but it is non-ivial only in the presence of matter sources, e.g. point particles. It was first shown in [3, 4] and then studied in details in [5] that the momentum space of such particles due to gravitational back reaction is the Lorentz group manifold. It is curved and

Key words and phrases: quantum gravity, hamiltonian reduction, Poisson manifold.

The work of A.S. was done under the support of Leonard Euler Saint-Petersburg international mathematical institute (Mathematical center), and also under the support of a grant for creation and development of International Mathematical Centers, agreement no. 075-15-2019-1620 of November 8, 2019, between Ministry of Science and Higher Education of Russia and PDMI RAS.

has a compact dimension, which leads to non-commutativity and partial discreteness in coordinate space.

Extending these results to 3+1 dimensional gravity is problematic because it is not an integrable model.

On the other hand one can study reduced models for General Relativity in which all but a few degrees of freedom are removed. Of special interest are those which contain black hole solutions, which are essential for Bronstein's argument.

The simplest model possible is a homogenous universe with a matter field. Quantization of this model has been extensively studied, and the overall conclusion is that quantum theory does't cure the singularity in this case [6], unless we include some exotic matter [7].

The second simplest model is spherically symmetric spacetime in which matter is represented by one or more dust shells. Unlike homogenous model, it accomodates black hole like solutions, which results in a nontrivial phase space, with branching of the solution to the constraints.

There is a variety of works studying such models both on classical [8–10] and quantum [11–13] level. In some of the versions of quantum theory the central singularity is removed [12, 13]. However the above results do not always agree with each other. Apart from quantization ambiguity, the other possible reason for that is a complicated phase space structure of the model. Different quantum theories could arise on different sectors of such phase space.

In such situation the common wisdom is that the wavefunction of a quantum theory has to be defined on all possible configurations, independently of whether they are classically accessible or not. In a particular way it was realized in [13] where by making use of complex coordinates different sectors of the phase space were assembled into one Riemann surface, the branching point identified with a horizon.

Another possibility is to try to find a real global chart for the phase space (if it exists). The above mentioned results on 2+1 dimensional gravity coupled to a point particle [3–5] are of this kind. Different branches of the solution to the constraints result from different way of intersecting Lorenz group manifold by a plane. These results were extended to a spherically symmetric shell in [20, 21], but in $2 + 1$ dimensional case.

One can try to extend the results of [20, 21] to $3 + 1$ dimension, as in spherically symmetric case there is no gravitational waves and the number of degrees of freedom is finite. However there is an even simpler model to study. This is $3+1$ dimensional gravity coupled to a massive point particle and a test particle whose gravitational field can be neglected. Then the gravitational field field can be considered spherically symmetric, centered at the location of the massive particle. The degrees of freedom will be the coordinates of massive particle with respect to the test particle. The massive particle will be freely moving and thus emitting no gravitational waves. This model is simpler than the spherical shell model, because there is no contribution of inter shell movement of dust energy to the gravitational field. There have been attempts to study this model in [22]. Here we provide a more systematic way of deriving the symplectic structure of the model.

In section 2, we rederive the results of [3, 4] and [5] for point particles coupled to 2+1 dimensional gravity closely following the approach of [17] and [18].

In section 3, we describe 3+1 dimensional gravity action coupled to point particles and its symmetries. Gauge and diffeomorphism parameters at the location of the particle provide its degrees or freedom.

In section 4, we find boundary conditions which lead to a well defined variational principle and could be satisfied by a gauge choice with no restrictions on physical degrees of freedom. The corresponding boundary term is also found.

In section 5, we solve the constraints and substitute the solution back into the action. It turns out that the action in the bulk, including the particle contribution, cancels out, and all that remains is the boundary action. This is a 3-dimensional analog of WZW action [15, 16]. Its degrees of freedom are gauge and diffeomorphism transformation parameters on the boundary.

In section 6, we combine the above boundary actions from inner and outer space, and show that this combination eventually reduces to a mechanical model, containing a finite number of degrees of freedom.

In section 7, we study the above action in the vicinity of the horizon, where it becomes particularly simple. We find that in this region the coordinate space is linear, while the momentum space is given by a coadjoint orbit of the Lorenz group. We show that this momentum space cover the entire Penrose diagram of the black hole created by this particle. The curvature of momentum space results in Poisson bracket structure, where the particle coordinates are non-commutative.

In section 8, we perform quantization in momentum representation on ADS^3. Apart from non-commutativity of the coordinates it results in discreteness of the spectrum of one of them (time). The invariant distance to the horizon has discrete spectrum inside the black hole and continuous, but separated from zero outside.

In section 9, we discuss the possible implications of this model, its limitations, and how it could possibly be extended beyond these limits.

§2. Example: 2+1 dimensional case

In this section, we consider the simplest example in which the gravitational back reaction on kinetic action of a point particle were derived. This is 2+1 dimensional case. This result is not new, it was obtained previously in [3–5]. Here we present a derivation which closely follow [17, 18] and also used in [20, 21]. This will be a starting point for generalizing it to 3+1 dimensions.

We start with action for 2+1 gravity coupled to point particles.

The basic variables are the iad $e_\mu = e^a_\mu \gamma_a$ and the connection $\omega^{ab}_\mu \gamma_a \gamma_b$, where γ^a are generators of sl(2)-algebra. The action reads:

$$S = \int\limits_M d^3x \epsilon^{\mu\nu\rho} \mathrm{Tr}(e_\mu R_{\nu\rho}) + S_{\text{particles}} \tag{1}$$

where $R_{\nu\rho}$ is the curvature of ω_ρ.

$$S_{\text{particles}} = \sum_i^N \int\limits_{l_i} \mathrm{Tr}(K_i e_\mu) dx^\mu \tag{2}$$

where l_i is i-th particle worldline and $K_i = m_i \gamma_0$ – a fixed element of sl(2)-algebra.

Gravity action is invariant with respect to gauge transformations:

$$\omega_\mu \to g^{-1}(\partial_\mu + \omega_\mu) g \quad e_\mu \to g^{-1}(e_\mu + \partial_\mu \xi) g \tag{3}$$

where g is an SL(2) group element, and ξ is an sl(2) algebra element.

The shell action is not invariant. The i-th particle term transforms as

$$\int\limits_{l_i} \mathrm{Tr}(K_i e_\mu) dx^\mu \to \int\limits_{l_i} \mathrm{Tr}(\tilde{K}_i e_\mu) dx^\mu + \int\limits_{l_i} \mathrm{Tr}(\tilde{K}_i \dot{\xi}) d\tau \tag{4}$$

where $\tilde{K}_i = gK_ig^{-1}$, τ is a parameter along the particle worldline and dot is the derivative with respect to it.

In the last term in the r.h.s. of 4 one can recognize the standard particle action as it has the form of $p_a\dot{x}^a$, where $p_a = \mathrm{Tr}(\gamma_a\tilde{K}_i)$, $x^a = \mathrm{Tr}(\gamma^a\xi)$, and given the definition of $\tilde{K}_i$ p^a satisfies the constraint $p^ap_a = m^2$. Thus the particles degrees of freedom are represented as gauge degrees of freedom evaluated at the location of the particles.

To obtain a reduced action for this model we have to solve the constraints and plug the solution back into the initial action. We choose slicing so that particle worldlines move along the time coordinate and obtain the constraints by varying action (1) with respect to ω_0 and e_0:

$$\epsilon^{0\mu\nu}\nabla_\mu e_\nu = 0 \quad \epsilon^{0\mu\nu}R_{\mu\nu} = \sum_i^N \tilde{K}_i\delta^2(x,x_i) \tag{5}$$

where x_i is the location of the i-th particle. The first constraint (5) generates the first of the transformations (3) and the second generates the second.

By using transformations (3) one can put to zero simultaneously one component of ω and one component of e. This automatically linearizes the constraints (5). However, such a gauge choice cannot be made globally, because the model has a non-trivial moduli space, containing for example the gauge parameter evaluated at the location of one particle with respect to another. Following [17,18] we divide the spacial slice into regions in each of which the above gauge choice could be made. Each such region should contain no more than one particle. Around each particle we draw a circle, so that the circle are connected to a common origin, but have no common boundaries. By making cuts along the circles the manifold is divided into N discs, each containing a particle, and a polygon containing no particles, but connected to infinity.

For the discs it is convenient to write down the solution in polar coordinates with the origin at the location of the particles. We choose the gauge in which the radial components of e and ω equal zero, solve the constraints, and put the solution back into an arbitrary gauge:

$$\begin{aligned} \omega_{r,i} &= g_i^{-1}\partial_r g_i \quad \omega_{\phi,i} = g_i^{-1}\nabla_\phi g_i \\ e_{r,i} &= g_i^{-1}\partial_r\xi_i g_i \quad e_{\phi,i} = g_i^{-1}\nabla_\phi\xi_i g_i \end{aligned} \tag{6}$$

where $\nabla_\phi\xi_i = \partial_\phi\xi_i + [\xi_i, K_i]$.

And similar for polygon, for which the gauge parameters will be denoted h and ζ.

Now this solutions have to be put back into the kinetic term of the action which reads (for i-th disk):

$$S_{D_i} = \int\limits_{D_i} d^3x \epsilon^{0\mu\nu} \mathrm{Tr}(e_\mu \dot{\omega}_\nu) + \int\limits_{l_i} \mathrm{Tr}(\tilde{K}_i \dot{\xi}_i) d\tau. \quad (7)$$

By using the identity (notice that K_i does not depend on time, so ∇_ϕ commutes with time derivative)

$$g_i^{-1} \dot{\nabla}_\mu g_i = g_i^{-1} \nabla_\mu (\dot{g}_i g_i^{-1}) g_i$$

we find that in the first term of (7) there is a δ-functional contribution which cancels the second term, plus another term which is a total derivative. Thus the action for the disk collapses to its boundary:

$$S_{D_i} = \int\limits_{\partial D_i} d^2x \mathrm{Tr}(\nabla_\phi \xi_i \dot{g}_i g_i^{-1}). \quad (8)$$

Similar for polygon, whose boundary,however, consists of N edges E_i, and the resulting action is a sum of contributions from every edge:

$$S_P = \sum_i^N \int\limits_{E_i} d^2x \mathrm{Tr}(\partial_\phi \zeta \dot{h}_i h_i^{-1}). \quad (9)$$

The next step is to assemble all the above pieces of the action together and apply the condition of continuity of metric and connection across the boundary between discs and polygon.

First we convert the covariant derivative in (6) into ordinary derivative by a gauge transformation

$$\tilde{g}_i = \exp(K\phi) g_i, \quad \tilde{\xi}_i = \exp(K\phi) \xi_i \exp(-K\phi).$$

This condition violates the periodicity, so the boundary of the disk is no longer a circle but an interval. Then disc action (8) changes to

$$S_{D_i} = \int\limits_{\partial D_i} d^2x \mathrm{Tr}(\partial_\phi \tilde{\xi}_i \dot{\tilde{g}} \tilde{g}_i^{-1}), \quad (10)$$

and continuity conditions for metric and connection (6) take a simple form:

$$\tilde{g}_i = C_i h\Big|_{E_i} \quad \tilde{\xi}_i = C_i \left(\zeta\Big|_{E_i} + \chi_i \right) C_i^{-1}, \quad (11)$$

where C_i and χ_i are functions only of time. Substituting this into (9) and (10), and combining them one obtains

$$S_{full} = S_P + \sum_i^N S_{D_i} = \sum_i^N \int_{E_i} \mathrm{Tr}(\partial_\phi \zeta C_i^{-1} \dot{C}_i) = -\sum_i^N \int_{\partial D_i} \mathrm{Tr}(\partial_\phi \tilde{\xi}_i \dot{C}_i C_i^{-1}). \tag{12}$$

The integrands are total derivatives, so the result contains contributions only from the vertices of the polygon or endpoints of discs boundaries.

$$S_{\mathrm{full}} = \sum_i^N \int_R \mathrm{Tr}((\zeta_{i+1} - \zeta_i) C_i^{-1} \dot{C}_i) = -\sum_i^N \int_R \mathrm{Tr}((\tilde{\xi}_i(2\pi) - \tilde{\xi}_i(0)) \dot{C}_i C_i^{-1}), \tag{13}$$

where ζ_i is the value of ζ at the i-th vertex of the polygon.

Introduce new variables

$$u_i = C_i^{-1} \exp(2\pi K) C_i, \quad \text{and} \quad \bar{\xi}_i = C_i^{-1} \tilde{\xi}_i(0) C_i. \tag{14}$$

Then taking into account that

$$\tilde{\xi}_i(2\pi) = \exp(2\pi K) \tilde{\xi}_i(0) \exp(-2\pi K)$$

and

$$u_i^{-1} \dot{u}_i = C_i^{-1} \dot{C}_i - C_i^{-1} \exp(-2\pi K) \dot{C}_i C_i^{-1} \exp(2\pi K) C_i$$

we can rewrite the second equation in (13) as

$$S_{\mathrm{full}} = \sum_i^N \int_R \mathrm{Tr}(\bar{\xi}_i u_i^{-1} \dot{u}_i). \tag{15}$$

Here $\bar{\xi}_i$ plays the role of coordinate and u_i – the role of momentum of i-th particle. u_i satisfies the constraint

$$\mathrm{Tr}(u_i) = \cos(\pi G m_i). \tag{16}$$

Because momentum space is curved, it results in coordinate non-commutativity.

Below, we will try to obtain an analog of action (15) to 3+1 spacetime dimensions. Because in 3+1 gravity the N-body problem is not solvable, we will restrict ourselves to the case of two particles one of which is massive and the other is a test particle, whose gravitational field is negligible.

§3. Action principle for gravity coupled to point particle in 3+1 dimensions

Gravity action depends on tetrad e^a_μ and Lorenz connection ω^{ab}_μ. We shall write it in spinor representation using Dirac matrices, $e_\mu = e^a_\mu \gamma_a$, $\omega_\mu = \omega^{ab}_\mu \gamma_a \gamma_b$.

$$S_{GR} = \frac{1}{8\pi G} \int_M d^4x \epsilon^{\mu\nu\alpha\beta} \mathrm{Tr}(\gamma_5 e_\mu e_\nu R(\omega)_{\alpha\beta}), \tag{17}$$

where $R(\omega)_{\alpha\beta}$ is the curvature of connection ω. This action is invariant with respect to local Lorentz transformation

$$e \to g^{-1} e g, \quad \omega \to g^{-1}(d+\omega)g, \tag{18}$$

where $g \in SL(2,C)$, and four-dimensional diffeomorphism transformations, which in infinitesimal form can be written as

$$\begin{aligned} \delta e^a_\mu &= \nabla_\mu \delta\xi^a + \delta\xi^\nu \omega^{ab}_\nu e_{\mu,b} \\ \delta\omega^{ab}_\mu &= \delta\xi^\nu R^{ab}_{\nu\mu} + \nabla_\mu(\delta\xi^\nu \omega^{ab}_\nu) \end{aligned} \tag{19}$$

one can notice that the second terms in the r.h.s. of eqs. (19) represent the infinitesimal version of gauge transformations (18) and can be combined with the later. So, when transformations (19) and (18) are applied simultaneously, the second terms in (19) can be ignored.

As the next step we include point particles in the action. Particle can be included as a point source of metric and connection fields. A spinless particle (in the present paper we consider only spinless case) is a source of metric field only:

$$S = S_{GR} + S_{\text{particle}}, \quad S_{\text{particle}} = M \int_\gamma \mathrm{Tr}(\gamma^0 e_\mu) dx^\mu, \tag{20}$$

where γ is the particle worldline. In the action above γ is not specified to be whether timelike or spacelike from the beginning. This is possible only after a specific metric of spacetime is given. Metric appears as a solution of Einstein's equations, in particular the spherically symmetric solution with a point mass source is the Schwarzschild solution. With respect to the Schwarzschild metric the worldline of the point particle in the center is spacelike.

The particle action in (20) is not invariant with respect to transformations (18) and (19). To restore invariance, one has to perform transformations and to include the transformation parameters in the action as extra fields:

$$S_{\text{particle}} = M \int_\gamma \text{Tr}(\gamma^0 g^{-1}(e_\mu + \nabla_\mu \xi) g) dx^\mu$$

$$= M \int_\gamma \text{Tr}(g\gamma^0 g^{-1} e_\mu) dx^\mu + S_{\text{particlekinetic}}. \tag{21}$$

The first term above is gauge invariant coupling of the particle to gravity and the second term is the particle kinetic term. One can show that in the absence of gravity the particle kinetic term reduces to the ordinary relativistic particle action in flat space. To see this, choose γ to be in the time direction and $\omega_0 = 0$. Then

$$S_{\text{particlekinetic}} = M \int_\gamma \text{Tr}(g\gamma^0 g^{-1} \dot{\xi}) dt = \int_\gamma p_a \dot{\xi}^a dt, \tag{22}$$

where $\xi^a = \text{Tr}(\gamma^a \xi)$, $p_a = M\text{Tr}(g\gamma_0 g^{-1}\gamma_a)$. The r.h.s. is the ordinary particle action if we take into account that p_a from its definition satisfies the constraint $p^a p_a = M^2$.

§4. Boundary term and boundary conditions

Variation of the action (17) in the presence of boundary result in both bulk and boundary terms.

$$\delta S_{GR} = \frac{1}{4\pi G} \int_M d^4x \epsilon^{\mu\nu\alpha\beta} \text{Tr}(\gamma_5 (e_\mu R(\omega)_{\alpha\beta} \delta e_\nu + \nabla_\alpha (e_\mu e_\nu) \delta\omega_\beta)) \tag{23}$$

$$+ \frac{1}{8\pi G} \int_{\partial M} d^3x \epsilon^{\mu\nu\alpha\beta} n_\alpha \text{Tr}(\gamma_5 e_\mu e_\nu \delta\omega_\beta),$$

where n_α is a unit normal to the boundary. The equations following from vanishing of the boundary variation has to be compatible with the bulk equations. Otherwise it is said that “The action has no extremum”. In our case the boundary equation is $[e_\mu, e_\nu]\big|_{\partial M} = 0$, i.e. the metric on the boundary has to be degenerate, which is not compatible with the most of interesting bulk solutions.

In order to fix the situation one has to impose boundary conditions on the fields and add a corresponding boundary term to the action. For example the Gibbons-Hawking term corresponds to the boundary conditions where the metric on the boundary is fixed. For our purposes another choice of boundary conditions is suitable. We place boundary arbitrary to divide space into regions in each of which static gauge condition could be applied. This means that boundary conditions has to impose no restrictions on physical degrees of freedom and could be satisfied by a gauge choice. A boundary term resulting in such boundary conditions does exist and has the following form:

$$S_{\text{boundary}} = \frac{1}{8\pi G} \int\limits_{\partial M} d^3x \epsilon^{\mu\nu\alpha 0} n_\alpha \mathrm{Tr}(\gamma_5 e_\mu e_\nu \omega_0). \tag{24}$$

Its variation combined with boundary term in (23) has the form

$$\delta S_{\text{boundary}} + \delta S_{GR}\big|_{\partial M} = \frac{1}{8\pi G} \int\limits_{\partial M} d^3x \epsilon^{ijk} n_i \mathrm{Tr}(\gamma_5 (e_j e_0 \delta\omega_k - 2 e_j \omega_0 \delta e_k)). \tag{25}$$

It could be made vanishing if one fixes $e_0 = 0$ and $\omega_0 = 0$. Such conditions can be satisfied by an arbitrary field configuration after applying transformations (19) and (18), i.e. by a gauge choice.

The resulting total action

$$S = S_{GR} + S_{\text{boundary}} + S_{\text{particle}} \tag{26}$$

is now a sum of kinetic terms of gravity and particles and a linear combination of constraints

$$S = \frac{1}{8\pi G} \int\limits_{M} d^4x \epsilon^{ijk} \mathrm{Tr}(\gamma_5 (e_i e_j \dot{\omega}_k + \omega_0 \nabla_i (e_j e_k) \tag{27}$$
$$+ e_0 (e_i R(\omega)_{jk} - 8\pi G M \gamma_0 \delta^3(x)))) + S_{\text{particlekinetic}}.$$

§5. Reduction to the boundary

We consider a system in which gravity is coupled to a massive particle, creating gravitational field, and a test particle, whose gravitational field can be neglected. We cannot apply a static gauge in the entire space, because the massive particle and the test particle can move with respect to each other. So we divide space into two regions, each containing one particle. In each of the two regions we can apply static gauge, so that the solution will be time independent.

The constraint equations following from action (27)

$$\epsilon^{ijk}\nabla_i(e_j e_k) = 0, \quad \epsilon^{ijk} e_i R(\omega)_{jk} = 8\pi G M \gamma_0 \delta^3(x) \tag{28}$$

together with requirement of spherical symmetry fixes the solution uniquely up to a gauge choice. This is the Schwarzschild solution which in spherical coordinates in static gauge can be written as

$$\begin{aligned} \bar{e}^1_r = 1/N, \quad \bar{e}^2_\theta = R, \quad \bar{e}^3_\phi = R\sin\theta, \\ \bar{\omega}^{12}_\theta = N, \quad \bar{\omega}^{13}_\phi = N\sin\theta, \quad \bar{\omega}^{23}_\phi = \cos\theta, \end{aligned} \tag{29}$$

where $N = \sqrt{1 - \frac{2GM}{R}}$, and bar on top of e and ω denotes the background Schwarzschild solution in static gauge. We locate in the neighborhood of the horizon, $R = 2GM$, so that it could be deformed to the horizon by a small diffeomorphism transformation with $\delta\xi \ll 2GM$, and all the calculations will be done to the first order of $\delta\xi$. The test particle should be in the outside region, so we will study the dynamics when the test particle is near horizon.

Then we take the solution to the constraints in arbitrary gauge

$$e_i = g^{-1}(\bar{e}_i + \bar{\nabla}_i \delta\xi)g, \quad \omega_i = g^{-1}(\partial_i + \bar{\omega}_i + \delta\xi^\nu \bar{R}_{\nu i})g \tag{30}$$

and plug it back into the action (27). The result is that the action collapses to the boundary, and the bulk contribution, including that of particles, cancels out.

This can be shown by direct calculation using the identity

$$\frac{\partial}{\partial t}(g^{-1}\bar{\nabla}_i g) = g^{-1}\bar{\nabla}_i(\dot{g}g^{-1})g, \quad \text{if} \quad \frac{\partial}{\partial t}\bar{\omega}_i = 0, \tag{31}$$

constraint equations and the Bianchi identity. There is also a simple argument why it should be so. If we plug the solution to the constraints in the static gauge (29) into the action (27) it disappears identically: constraint terms because it is a solution and kinetic terms because it is static. Transformations (19) and (18) do not change the action (27) except the boundary term (24).

The resulting boundary action depends on gauge and diffeomorphism transformation parameters.

$$S = \frac{1}{8\pi G}\int\limits_{\partial M} d^3x \epsilon^{ij} \mathrm{Tr}(\gamma_5 g e_i e_j g^{-1}\dot{g}g^{-1}), \tag{32}$$

where $e_i = g^{-1}(\bar{e}_i + \bar{\nabla}_i \delta\xi)g$. Here and below indices i, j label the directions on the spacial slice of the boundary ($i, j = \theta, \phi$ in spherical coordinates).

Introduce a background value of ξ, such that $\bar{e}_i = \bar{\nabla}_i \bar{\xi}$. It could be combined with diffeomorphism transformation parameter $\xi = \bar{\xi} + \delta\xi$. Then $g e_i g^{-1} = \nabla_i \xi$. Then the action (32) can be rewritten as:

$$S = \frac{1}{8\pi G} \int\limits_{\partial M} d^3x \epsilon^{ij} \mathrm{Tr}(\gamma_5 \nabla_i \xi \nabla_j \xi \dot{g} g^{-1}). \tag{33}$$

Notice that while the above action contains non-linear contributions in ξ, it was derived only in linear approximation in $\delta\xi$. Whether this action is valid beyond this approximation is an open question. In the present paper we use it in linear approximation only.

The action (33) is a three-dimensional analog of Wess-Zumino-Witten action obtained from Chern-Simons theory. This action is a field theory with infinitely many degrees of freedom. On the other hand the model in question (massive point particle in a frame of a test particle, with no gravitational radiation) has only finitely many degrees of freedom. To extract relevant degrees of freedom from the action obtained one has to combine it with the action on the same boundary, but induced by the bulk action in the outer region (so far we considered the action in the inner region). Then the extra degrees of freedom will cancel out. This is considered in the next section.

§6. Further reduction

The action for entire space is a sum of two actions of the form (33), one from inner region, S_{in}, and the other from outer region, S_{out}. The background field, $\bar{e}_i$, $\bar{\omega}_i$, is the same in both pieces of the action, but the gauge parameters are different. The gauge parameters at the location of particles play the role of particle degrees of freedom, and they are chosen so, that in each region the corresponding particle is at rest.

To combine the terms in the total action,

$$S = S_{\mathrm{in}}[\xi_{\mathrm{in}}, g_{\mathrm{in}}] + S_{\mathrm{out}}[\xi_{\mathrm{out}}, g_{\mathrm{out}}], \tag{34}$$

one has to apply the condition of continuity of metric and connection across the boundary (because there is no sources of gravitational field on the boundary one can always choose a gauge in which both metrics and connection are continuous there):

$$e_{i,\mathrm{in}} = e_{i,\mathrm{out}}, \quad \omega_{i,\mathrm{in}} = \omega_{i,\mathrm{out}}. \tag{35}$$

Because the background field is the same on both sides of the boundary, the condition (35) is a condition on gauge parameters ξ and g. This condition written in a form

$$g_{\rm in}^{-1}\bar{\nabla}_i\xi_{\rm in}g_{in} = g_{\rm out}^{-1}\bar{\nabla}_i\xi_{\rm out}g_{\rm out}, \quad g_{\rm in}^{-1}\bar{\nabla}_i g_{\rm in} = g_{\rm out}^{-1}\bar{\nabla}_i g_{\rm out}, \qquad (36)$$

result in the following relation between gauge parameters inside and outside the boundary:

$$g_{\rm out} = h g_{\rm in}, \quad \xi_{\rm out} = h(\xi_{\rm in} + \zeta)h^{-1}, \qquad (37)$$

where the fields ζ and h are covariantly constant on the boundary, i.e.

$$\bar{\nabla}_i\zeta = 0, \quad \bar{\nabla}_i h = 0. \qquad (38)$$

Because the connection $\bar{\omega}$ has non-zero curvature on the boundary, the condition (38) results in purely algebraic conditions on ζ and h:

$$h^{-1}\bar{R}_{ij}h = \bar{R}_{ij}, \quad [\bar{R}_{ij}, \zeta] = 0, \qquad (39)$$

i.e. ζ is in the direction of stability axis of R_{ij} and h is an exponential of algebra element which is in the same Cartan subalgebra as R_{ij}.

Now this has to be substituted in the total action (34) with

$$S_{\rm in,out} = \frac{1}{8\pi G}\int\limits_{\partial M} d^3x\epsilon^{ij}{\rm Tr}\Big(\gamma_5\nabla_i\xi\nabla_j\xi\dot{g}g^{-1}\Big)_{\rm in,out}. \qquad (40)$$

Applying conditions (37,38), the action (34) is reduced to

$$S = S_{\rm in} + S_{\rm out} = \frac{1}{8\pi G}\int\limits_{\partial M} d^3x\epsilon^{ij}Tr\Big(\gamma_5\nabla_i\xi_{\rm out}\nabla_j\xi_{\rm out}h^{-1}\dot{h}\Big). \qquad (41)$$

It does not depend on ζ, but depends on h. h is a solution of continuity conditions (38), which can be written as

$$h = h_\theta C(\phi)h_\theta^{-1}, \qquad (42)$$

where h_θ is a transformation which converts covariant derivative w.r.t. θ into ordinary one, $h_\theta^{-1}\bar{\nabla}_\theta h_\theta = \partial_\theta$. C is a function only on ϕ and satisfies the condition $\tilde{\nabla}_\phi C(\phi) = 0$, where $\tilde{\nabla}_\phi$ is a covariant derivative with connection $\tilde{\omega}_\phi = h_\theta^{-1}\omega_\phi h_\theta$.

The transformation $h_\theta = \exp(-\omega_\theta\theta)$ cannot be defined globally on the boundary, because it is a non simply connected manifold. One can divide the boundary into upper and lower hemispheres and on each of them transformation h_θ can be defined:

$$h_{\theta,u} = \exp(-\omega_\theta\theta), \quad h_{\theta,d} = \exp(-\omega_\theta\theta - \pi(1-N)\gamma^1\gamma^2). \qquad (43)$$

In general, $h_{\theta,u}$ and $h_{\theta,d}$ do not coincide at the equator of the sphere, $\theta = \pi/2$. Only when $N = 1$ in (29), i.e. there is no mass source and spacetime curvature is zero, h_θ is continuous across the equator. Because of this, the integral in (41) the integral has to be taken not on entire sphere ∂M, where h is not defined, but on unity $(\partial M)_u \bigcup (\partial M)_d$

Using Cartan equation for the connection on the boundary $\epsilon^{ij}\nabla_i\nabla_j\xi = \epsilon^{ij}\nabla_i e_j = 0$ and the continuity condition $\nabla_i h = 0$ equation (41) can be rewritten as

$$S = \frac{1}{8\pi G} \int\limits_{(\partial M)_u \bigcup (\partial M)_d} d^3x \epsilon^{ij} \mathrm{Tr}\Big(\gamma_5 \nabla_i (\xi_{\mathrm{out}} \nabla_j \xi_{\mathrm{out}} h^{-1} \dot{h})\Big). \tag{44}$$

There is a total derivative under the integral, so the action collapses to its boundary, $\partial(\partial M)_u \bigcup \partial(\partial M)_d$, the equator as a boundary of upper hemisphere, and the equator as a boundary of lower hemisphere. Two above contributions do not coincide, and the resulting action is their difference:

$$\begin{aligned} S = &\frac{1}{8\pi G} \int\limits_{S^1} dt d\phi \mathrm{Tr}\Big(\gamma_5 \xi_u \nabla_\phi \xi_u h_u^{-1} \dot{h}_u\Big) \\ &- \frac{1}{8\pi G} \int\limits_{S^1} dt d\phi Tr\Big(\gamma_5 \xi_d \nabla_\phi \xi_d h_d^{-1} \dot{h}_d\Big), \end{aligned} \tag{45}$$

where $\xi_{u,d}$ and $h_{u,d}$ are the values of diffeomorphism and gauge transformation parameters on upper and lower hemispheres respectively.

The next step is to apply the condition of continuity of the metric and the connection across the equator. Before doing this, it is convenient to perform a gauge transformation which converts covariant derivative ∇_ϕ into ordinary one

$$h_\phi = \exp(\omega_\phi \phi), \quad \tilde{h} = h h_\phi, \quad \tilde{\xi} = h_\phi \xi h_\phi^{-1}. \tag{46}$$

Notice that this transformation is not periodic in ϕ on the equator. So, in (45) one has to replace circle S_1 as integration region by an interval I_1, where ϕ varies from 0 to 2π, but the points $\phi = 0$ and $\phi = 2\pi$ are not identified. In this gauge, the continuity condition of metric and connection across the equator read

$$\tilde{g}_d = C\tilde{g}_u, \quad \tilde{\xi}_d = C(\tilde{\xi}_u + \zeta)C^{-1}, \quad \partial_\phi C = 0, \quad \partial_\phi \zeta = 0. \tag{47}$$

From these conditions one can deduce that

$$h_d = C h_u C^{-1}. \tag{48}$$

Substituting all this into (45), one obtains

$$S = \frac{1}{8\pi G}\int\limits_{I^1} dtd\phi \mathrm{Tr}\Big(\gamma_5 \tilde{(}\xi_u + \zeta)\partial_\phi \tilde{\xi}_u (\dot{C}C^{-1} - Ch_u^{-1}C^{-1}\dot{C}h_u C^{-1})\Big)$$
$$+ \frac{1}{8\pi G}\int\limits_{I^1} dtd\phi \mathrm{Tr}\Big(\gamma_5 \tilde{\partial}_\phi (\zeta\tilde{\xi}_u) h_d^{-1}\dot{h}_d\Big). \tag{49}$$

One can see that the first term of this action does not contain time derivative of h_u, i.e. h_u is not dynamical. It was obtained from the condition that it commutes with the curvature on the sphere, $R_{\theta\phi} = h_u^{-1} R_{\theta\phi} h_u$, and being not dynamical, it could be fixed as $h_u = \exp(\bar{R}_{\theta\phi})$. Then the first term of the action can be rewritten as

$$S_1 = \frac{1}{8\pi G}\int\limits_{I^1} dtd\phi \mathrm{Tr}\Big(\gamma_5 \tilde{\xi}_u \partial_\phi \tilde{\xi}_u h_d^{-1}\dot{h}_d\Big), \tag{50}$$

h_d is now not a general Lorenz group element, but an orbit $h_d = Ch_uC^{-1}$ with a fixed conjugacy class h_u. It has four independent parameters instead of six.

The second term in (49) is readily a total derivative, so it reduces to an action depending only on a finite number of parameters

$$S_1 = \frac{1}{4\pi G}\int dt \mathrm{Tr}\Big(\gamma_5 \tilde{\xi}\zeta\dot{u}u^{-1}\Big), \tag{51}$$

where $u = h_d^{-1}\exp(\pi\omega_\phi)h_d$. In the first order, $\tilde{\xi}$ will play the role of the background field, which is not dynamical, and ζ will play the role of canonical coordinate.

A bit more complicated situation with the first term in (49). The form $\tilde{\xi}_u d\tilde{\xi}_u$ entering (49) is closed, because $\tilde{\xi}_u$ depends only on ϕ. Because it is defined on a simply connected interval I^1, it is also exact. There has to exist a form X^{ab} such that

$$dX^{ab} = \tilde{\xi}_u^{[a} d\tilde{\xi}_u^{b]}. \tag{52}$$

Then the expression under the integral in the first term (49) is again total derivative and it reduces to the boundary of the interval I^1:

$$S = \frac{1}{4\pi G}\int dt \mathrm{Tr}\Big(\gamma_5 X h_d^{-1}\dot{h}_d\Big)\Big|_0^{2\pi}, \tag{53}$$

where $X = X^{ab}\gamma_a\gamma_b$. Because $X(2\pi) = \exp(\pi\omega_\phi)X(0)\exp(-\pi\omega_\phi)$, the action (53) can be rewritten as

$$S = \frac{1}{4\pi G}\int dt \mathrm{Tr}\Big(\gamma_5 \tilde{X}\dot{u}u^{-1}\Big), \tag{54}$$

where $u = h_d^{-1}\exp(\pi\omega_\phi)h_d$, and $\tilde{X} = h_d X(0) h_d^{-1}$.

The variable $\tilde{X}$ is related to coordinates of the test particle ξ with respect to the horizon by (52). To this point, we do not have an explicit expression of $\tilde{X}$ in terms of ξ in general. However, throughout this paper we are using linear approximation in variation of ξ, $\xi = \bar{\xi} + \delta\xi$, where $\bar{\xi}$ is a background field and $|\delta\xi| \ll |\bar{\xi}|$. In this approximation, and under the condition that $Tr(\bar{\xi}\delta\xi) = 0$, we have an explicit expression for $\tilde{X}$:

$$X^{ab} = \bar{\xi}^{[a}\delta\xi^{b]}. \tag{55}$$

This approximation we will study in the next section.

Thus, ξ will play the role of canonical coordinate. The role of momentum, canonically conjugate to ξ will be played by a group valued variable u. From its definition, and given the expression for ω_ϕ in (29), it satisfies the following constraint

$$\mathrm{Tr}(u) = \cos\Big(\pi\sqrt{1-\frac{2GM}{R}}\Big). \tag{56}$$

As we will see in the subsequent, this is a deformed mass shell constraint for a particle.

§7. Particle on a plane

From the action (54), using (55), one obtains a symplectic form:

$$\Omega = \frac{1}{\pi G}\epsilon_{\mathrm{abcd}}\bar{\xi}^a\delta\xi^b \wedge \mathrm{Tr}\Big(\gamma^c\gamma^d\dot{u}u^{-1}\Big), \tag{57}$$

$\bar{\xi}^a$ is a fixed vector, whose absolute value combines with the coupling constant and the direction defines the plane in which the dynamics of the particles under study unfolds. This is not in contradiction with what one knows from solutions of General Relativity. The relative movement of two particles, coupled to gravity, can always be confined to a plane fixed according to initial conditions.

The degrees of freedom of this reduced model will be coordinates ξ^a satisfying $\bar{\xi}_a\xi^a$ and momenta given by a group element u, such that $u^{-1}\bar{\xi}^a\gamma_a u = \bar{\xi}^a\gamma_a$, i.e. from the stability subgroup of the direction $\bar{\xi}^a$, which is $SO(2,1)$ or $SL(2)$ group, as $\bar{\xi}^a$ is spacelike. The absolute value of $\bar{\xi}^a$ is the areal radius R of the sphere at which the test particle is located. To the linear approximation in $\delta\xi^a$, this is the Schwarzschild radius $R = 2GM$.

One can show that in the no-gravity limit, $G \to 0$ or $R \to \infty$ the symplectic form (57) reduces to that of an ordinary particle. Take

$$u = \exp(\pi G \epsilon^{\text{abcd}} p_a \bar{\xi}_b \gamma_c \gamma_d / R^2) \tag{58}$$

and assume that $G p_a / R \ll 1$. Then

$$\delta u u^{-1} = \pi G \epsilon^{\text{abcd}} \delta p_a \bar{\xi}_b \gamma_c \gamma_d / R^2. \tag{59}$$

Substituting this into (57) one obtains

$$\Omega = \delta\xi^a \wedge \delta p_a. \tag{60}$$

The same anzatz (58) substituted in the constraint equation (56) results in

$$p_a p^a = M^2. \tag{61}$$

The last two equations contain only those components p_a and ξ_a which are orthogonal to $\bar{\xi}^a$.

The Poisson brackets resulting from (57) are:

$$\begin{aligned} \{\xi^a, u\} &= \pi G \epsilon^{\text{abcd}} \frac{\bar{\xi}_b}{|\bar{\xi}|^2} \gamma_c \gamma_d u \\ \{u, \otimes u\} &= 0 \\ \{\xi^a, \xi^b\} &= G \epsilon^{\text{abcd}} \frac{\bar{\xi}_c}{|\bar{\xi}|^2} \xi_d. \end{aligned} \tag{62}$$

Because of commutativity of momenta, quantization will be convenient in momentum representation.

Momentum of the massive particle is given by holonomy around the equator of a sphere, whose radius is specified by the test particle location. One can show that this holonomy provides a real global parametrization of the Penrose diagram of the black hole created by the massive particle. Recall that the solution (29) has four distinct branches. First $\sqrt{1 - \frac{2GM}{R}}$ could be real or imaginary. Secondly, the sign in front of square root can be "+" or "-". The corresponding four regions are shown on Fig. 1.

U is also always real From the constraint (56) one can see that u is elliptic when $1 - \frac{2GM}{R} > 0$ and hyperbolic when $1 - \frac{2GM}{R} < 0$. In elliptic

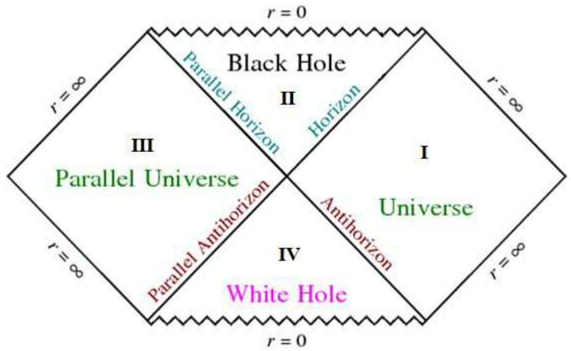

Figure 1. Penrose diagram and its four regions.

case, $u = g^{-1}\exp([\gamma_i,\gamma_j]\phi)g$, when $\sqrt{1-\frac{2GM}{R}} > 0$ then $0 < \phi < \pi$ (region I in Fig.1), and when $\sqrt{1-\frac{2GM}{R}} < 0$ then $\pi < \phi < 2\pi$ (region III in Fig. 1). And similar for the hyperbolic case: $u = g^{-1}\exp([\gamma_i,\gamma_j]\chi)g$, when $i\sqrt{1-\frac{2GM}{R}} > 0$ then $\chi > 0$ (region II in Fig.1), and when $i\sqrt{1-\frac{2GM}{R}} < 0$ then $\chi < 0$ (region VI in Fig.1). In other words, u provides a real global parametrization of the momentum space of the model.

All these four regions can be seen on Fig.2 as different ways of crossing of the group manifold of u by $\mathrm{Tr}(u) = \mathrm{const}$ plane.

Global parametrization of momentum space then will be given by the Euler angles on $SL(2)$-manifold. Let γ_a, $a = 0,1,2$ (three directions which are orthogonal to $\bar{\xi}$) be the generators of $sl(2)$. Then an $SL(2)$ element could be parameterized as

$$u = \exp\Big(\frac{\phi}{2}\gamma_0\Big)\exp\Big(\frac{\rho}{2}\gamma_0\Big)\exp\Big(\chi\gamma_1\Big)\exp\Big(\frac{\rho}{2}\gamma_0\Big)\exp\Big(-\frac{\phi}{2}\gamma_0\Big). \qquad (63)$$

Here ϕ is an axial angle in spacial plane, χ is a boost parameter related to the spatial momentum, and ρ is an angular variable related to energy. These variables will be used for quantization in the next section.

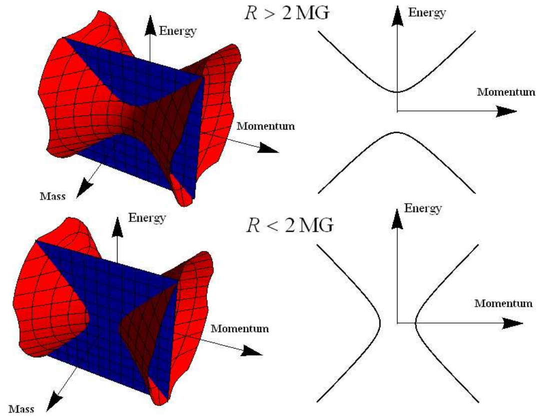

Figure 2. Solutions to the constraint. The momentum space is depicted as a hyperboloid (only time and radial directions are depicted). The solution to the constraint is crossing of the hyperboloid by $M = const$ plane. It results either in mass shell of ordinary particle (outside the horizon) or that of a tachion (inside the horizon).

§8. Quantization

In this section we shall briefly describe quantization of the model using momentum (Euler angle) representation. The exposition will closely follow [5], but take into account that the Poisson brackets now include Newtonian potential, and that the constraint describing dynamics is now different.

We define the kinematical states of the model as functions of u from (63)

$$= \Psi(u) = \Psi(\rho, \chi, \phi), \tag{64}$$

single-valued functions on the entire momentum space. From the requirement of Ψ being single-valued it immediately follows the periodicity property,

$$\Psi(\rho + 2\pi, \chi, \phi) = \Psi(\rho, \chi, \phi), \tag{65}$$

and

$$\Psi(\rho, \chi, \phi + 2\pi) = \Psi(\rho, \chi, \phi), \tag{66}$$

which will have an important consequence on the spectra of coordinates.

Next, for defining the scalar product, we need a Lorenz-invariant measure on our momentum space. It can be inferred from the Haar measure on SL(2):

$$dU = \frac{1}{\pi}\sinh(2\chi)d\rho d\chi d\phi, \tag{67}$$

and the scalar product is thus:

$$\langle\Phi,\Psi\rangle = \frac{1}{\pi}\int \sinh(2\chi)d\rho d\chi d\phi \Phi(\rho,\chi,\phi)^*\Psi(\rho,\chi,\phi). \tag{68}$$

Easiest of all is to calculate the spectrum of time coordinate, ξ^0, which is canonically conjugate to p_0, which is related to the dimensionless parameter ρ as $\rho = Gp_0/R$, and the corresponding operator is

$$\hat{\xi}^0|\rho,\chi\rangle = i\frac{\hbar G}{R}\frac{\partial}{\partial\rho}|\rho,\chi\rangle \tag{69}$$

its eigenstates are

$$|t;\psi\rangle = \frac{1}{\pi}\int \sinh(2\chi)d\rho d\chi \exp(it\rho)\psi(\chi)|\rho,\chi\rangle, \tag{70}$$

where t is an integer. Thus, time operator has a discrete spectrum:

$$\hat{\xi}^0|t;\psi\rangle = t\frac{\hbar G}{R}|t;\psi\rangle. \tag{71}$$

Notice that it is quantized not in units of the Planck length, but in units of the Planck length squared over the black hole size. The later is the result of the presence of Newtonian potential in the Poisson brackets. The smaller is the black hole the scarcer is the spectrum of time operator near its horizon.

A more interesting observable is the invariant distance to the horizon, $X^2 = \xi_a\xi^a$, which is spacelike outside the horizon and timelike inside. Because ξ_a in (62) is defined as the left-invariant derivative on the group, its square is the Beltrami–Laplace operator on our momentum space:

$$\hat{X}^2|t;\psi\rangle = 2\pi|t;\Delta\psi\rangle, \tag{72}$$

where

$$\Delta = \hbar^2\left(\frac{1}{\sinh(2\chi)}\frac{\partial}{\partial\chi}\sinh(2\chi)\frac{\partial}{\partial\chi} + \frac{t^2}{\cosh^2(2\chi)}\right). \tag{73}$$

This operator was shown in [5] to have two series of eigenvalues. One is continuous, but separated from zero, corresponds to positive,i.e. spacelike,

X^2

$$\hat{X}^2|t,\lambda\rangle = 2\pi(\lambda^2+1)\frac{\hbar^2 G^2}{R^2}|t,\lambda\rangle, \tag{74}$$

where λ is a real number. The other is discrete, but containing zero, corresponds to negative, i.e. timelike, X^2

$$\hat{R}^2|t,l\rangle = -2\pi l(l+2)\frac{\hbar^2 G^2}{R^2}|t,l\rangle, \tag{75}$$

where l is a non-negative integer, subject to the condition $l \leqslant t$. This two series of eigenvalues correspond to principle and supplementary series of unitary representations of $SL(2)$ [19]. Only unitary infinite-dimensional representations of $SL(2)$ result in normalizible states.

The discreteness of the radial variable inside the black hole provide a chance for resolution of the central singularity, $R=0$. But this point is located far away from the horizon and not reachable within approximation used in the present paper.

The Hamiltonian constraint in terms of the Euler angle variables has the form

$$\cos(\rho)\cos(\chi) = \cos\left(\pi\sqrt{1-\frac{2GM}{R}}\right). \tag{76}$$

It can be solved with respect to $\cos(\rho)$ and imposed on states in ξ_0, χ representation. As a result one obtains an evolution equation, which is not differential< but a finite difference equation in ξ_0

$$\Psi(\xi_0+\frac{G\hbar}{R},\chi)+\Psi(\xi_0-\frac{G\hbar}{R},\chi) = \frac{\cos\left(\pi\sqrt{1-\frac{2GM}{R}}\right)}{\cosh(\chi)}\Psi(\xi_0,\chi). \tag{77}$$

Due to presence of R in the hamiltonian the momenta, unlike in 2+1 dimensional gravity do not conserve. This is expectable on the physical ground, because a test particle used as a reference frame will be attracted by Newtonian force towards massive particle, the momentum of the later in this frame will change. There are still conserved quantities, but they are related to ADM momenta at infinity, which remain undeformed.

§9. Conclusion

So far we do not have the full quantum theory of the model studied in this paper. What remains to do is to describe the dynamics, in particular calculate the transition amplitudes between different eigenvalues of the invariant distance to the horizon.

What was shown is that the spectrum of the invariant distance to the horizon is discrete inside the black hole with the eigenvalue spacing increasing in the depth of the black hole. This provide a chance that the central singularity of the black hole will be avoided in quantum theory is avoided. However, this is not certain, because in the approximation used in this paper, the invariant distance to the horizon is much smaller than the black hole size, the center of the black hole is not reachable.

It is not clear how to go beyond this approximation. The problem is that the distance to the horizon in the present approach is represented by diffeomorphism parameters. This parameters are taken to be small, because only in this case they act on the fields locally. We are not aware of a well developed technique for working with non-local field transformations.

However in equation (33) small diffeomorphism parameters combined with the background field could be viewed as finite diffeomorphism parameters. Whether this can lead to a consistent model is a subject of further study.

Acknowledgements. We are also grateful to V. A. Berezin for extensive conversations on his work, which was crucial for understanding this model, and A. A. Andrianov and Y. Elmahalawy for collaboration on a closely related topic.

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Поступило 18 ноября 2021 г.

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