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I. N. Burenev

FIVE-VERTEX MODEL AND LOZENGE TILINGS OF A HEXAGON WITH A DENT

Abstract. We consider the five-vertex model on a regular square lattice of the size $L \times M$ with boundary conditions fixed in such a way that configurations of the model are in one-to-one correspondence with the lozenge tilings of the hexagon with a dent. We obtain two determinant representations for the partition function. In the free-fermionic limit, this result implies some summation formulae for Schur functions.

§1. Introduction

Exactly solvable lattice models of statistical mechanics are of interest from the point of view of both theoretical and mathematical physics. One of the peculiar features of such models is that in the limit of large domains, thermodynamic properties in the bulk may strongly depend on boundary conditions. This was first recognized in [1] for the dimer model on a square lattice. As an important example, we should mention the famous "Arctic Circle phenomenon" proven rigorously for the dimer model on the Aztec Diamond region [2]. Another reason to study exactly solvable lattice models lies in its connection with problems of enumerative combinatorics [3]. The famous example is Kuperberg's proof [4] of Mills, Robbins and Rumsey conjecture on the number of alternating sign matrices [5].

One of the most well-studied examples of exactly solvable lattice models is the so-called six-vertex model [6–9]. This model in its simplest form was introduced by Pauling in [10] to calculate the residual entropy of water ice.

The five-vertex model is a special case of the six-vertex model with one vertex being frozen out. It first appeared in the context of two-dimensional crystal growth [11, 12]. This model can be interpreted as a model of the interacting dimers on a honeycomb lattice [13] or as a generalization of

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lozenge tilings. Recent interest in the study of the five-vertex model is due to its close connection with some families of symmetric functions [14, 15].

In this paper, we consider the five-vertex model on a square lattice with boundary conditions fixed in such a way that the configurations of the model are in one-to-one correspondence with lozenge tilings of a hexagon with a dent. These boundary conditions are a natural generalization of ones considered in [16].

We obtain several determinant representations for the partition function of the model (Theorem 1). The proof is based on the technique of Quantum Inverse Scattering Method (QISM) [17]. Actually, our results hold for any integrable model with the same R-matrix. Two particular examples are the four-vertex model [18] and the non-Hermitian phase model [19].

Since the five-vertex model at the free-fermion point can be used for the definition of the Schur functions (see e.g. [20]), the determinant representations provided by Theorem 1 implies some summation formulae for the Schur functions.

In Section 2 we properly define the model and the boundary conditions. In Section 3 we obtain determinant representation for the partition function of the inhomogeneous five-vertex model. In Section 4 we discuss the model at free-fermion point.

§2. The model

Consider a regular square lattice with an arrow pointing along each edge. If we impose the arrow conservation low at each lattice site (number of incoming arrows is equal to the number of outgoing arrows) then only six out of the sixteen vertex configurations are possible (see Fig 1). This is the so-called six-vertex model (ice model). We follow the convention of [8] and use another equivalent graphical representation in terms of paths flowing through the lattice. Namely, we draw a line on each edge with an arrow pointing down or left.

The five-vertex model is a special case of the six-vertex model with one vertex configuration (second in the standard order) being frozen out. In this paper we consider the five-vertex model on a square lattice of the size $L \times M$. The boundary conditions are fixed as follows: there are $n+m$ paths flowing through the lattice entering it in the left corner of the bottom boundary and exiting on the top boundary at the edges from 1 to n and from $n+l+1$ to $n+m+l$ (we count lines from right to left). For an example of such boundary conditions see Fig. 2.

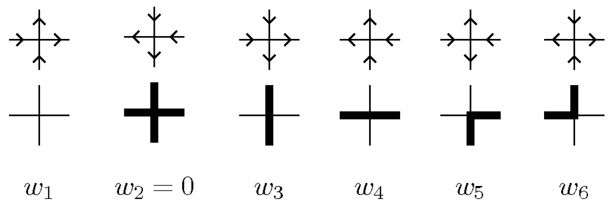

Figure 1. The vertices of the six-vertex model in terms of arrows (first row) or lines (second row) and their Boltzmann weights in the five-vertex model (third row).

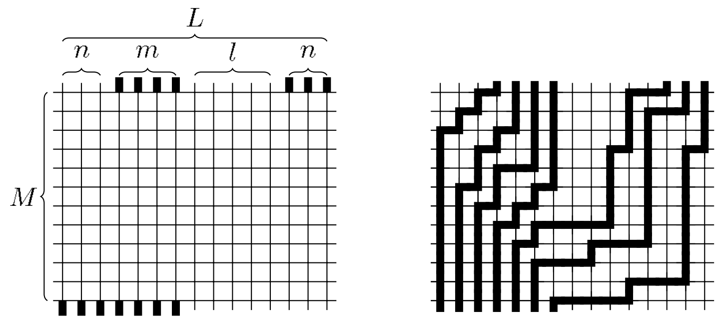

Figure 2. Boundary conditions (left) and one of the configurations (right) with $M = 12$, $n = 3$, $m = 4$, $l = 5$ ($L = 2n + m + l = 15$).

The configurations of the five-vertex model with considered boundary conditions can be interpreted as a pile of cubes fitting in a box without an L-shaped part (see Fig 3). In this correspondence, the lines of the vertex model play the role of gradient lines. There is also natural one-to-one correspondence between the configurations of the model and the lozenge tilings of the hexagon with a dent (see Fig 4).

The partition function of the model is defined in the usual manner by

$$Z = \sum_{\text{conf}} \prod_{k=1}^{M} \prod_{\mu=1}^{L} W_{k,\mu}(\text{conf}),$$

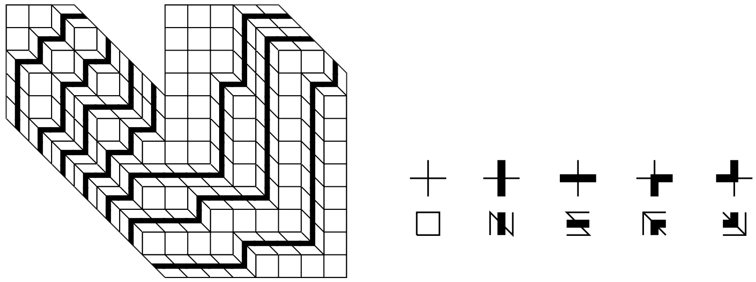

Figure 3. The configuration of the five-vertex model shown in Fig 2 as a pile of cubes fitting in a box without an L-shaped part (left) and the rule for the vertex transformation (right).

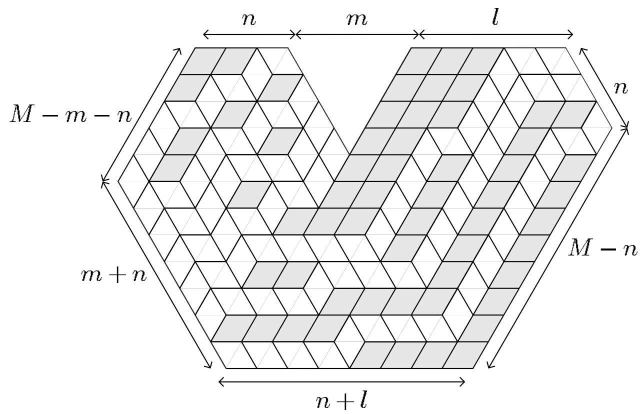

Figure 4. Lozenge tiling corresponding to the configuration of the five-vertex model shown in Fig 2.

where we perform summation over all possible configurations of the model and $W_{k,\mu}(\text{conf})$ stands for the Boltzmann weight of the vertex at the intersection of kth horizontal and μth vertical line, $W_{k,\mu}(\text{conf}) = w_i$ ($i = 1, 3, 4, 5, 6$ depending on the configuration). We are interested in the case of the inhomogeneous model when Boltzmann weights are site dependent. We associate the parameter u_μ, where $\mu = 1, \ldots, L$ to each vertical line and parameter ξ_k, where $k = 1, \ldots, M$ to each horizontal line (we count lines from right to left and from top to bottom) and set $W_{k,\mu}(\mathrm{conf}) = w_i(u_\mu/\xi_k)$, where the value of i is determined by the configuration. We consider the model with functions $w_i(u)$ defined by

$$w_1(u) = \frac{\alpha}{\Delta}\left(u - \frac{1}{u}\right), \quad w_3(u) = \frac{u}{\alpha}, \quad w_4(u) = \alpha u, \quad w_5(u) = w_6(u) = 1. \tag{2.1}$$

§3. Determinant representations for the partition function

Let us recall some basics of the Quantum Inverse Scattering Method (QISM) [17] and show how to utilize it to obtain determinant representations for the partition function of the inhomogeneous five-vertex model.

We first assign a copy of the vector space $\mathbb{C}^2$ to each vertical and horizontal line of the lattice. Depending on wheter the edge is empty or there is a path flowing through it, we denote the corresponding state by the vectors $|0\rangle$ (empty edge) or $|1\rangle$ (edge with path) defined in a standard way as

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \quad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}.$$

The Boltzmann weight of the vertex at the intersection of the μth vertical and kth horizontal line can be represented as a matrix element of the operator $L_{\mu,k}$, which acts non-trivially on the direct product of two spaces: "vertical" space $\mathcal{V}_\mu$ and "horizontal" space $\mathcal{H}_k$. The operator $L_{\mu,k}$ has the form

$$\begin{aligned} L_{\mu,k} = w_1 \left(\frac{1+\tau_\mu^z}{2}\right)\left(\frac{1+\sigma_k^z}{2}\right) + w_3 \left(\frac{1-\tau_\mu^z}{2}\right)\left(\frac{1+\sigma_k^z}{2}\right) \\ + w_4 \left(\frac{1+\tau_\mu^z}{2}\right)\left(\frac{1-\sigma_k^z}{2}\right) + w_5\, \tau_\mu^- \sigma_k^+ + w_6\, \tau_\mu^+ \sigma_k^-, \end{aligned}$$

where τ_μ^k and σ_k^i $(i = +,-,z)$ denote operators acting as Pauli matrices in $\mathcal{V}_\mu$ and $\mathcal{H}_k$ respectively. The explicit form of the operator $L_{\mu,k}$ in the standard tensor product basis $|0\rangle \otimes |0\rangle$, $|0\rangle \otimes |1\rangle$, $|1\rangle \otimes |0\rangle$, $|1\rangle \otimes |1\rangle$ reads

$$L_{\mu,k} = \begin{pmatrix} w_1 & 0 & 0 & 0 \\ 0 & w_4 & w_6 & 0 \\ 0 & w_5 & w_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{[\mathcal{V}_\mu \otimes \mathcal{H}_k]}.$$

Recall that the Boltzmann weights w_i depend on the ratio of two parameters: u_μ (associated with μth vertical line) and ξ_k (associated with kth horizontal line).

The integrability of the model is based on the fact that the L-operator satisfies the intertwining relation (RLL-relation), that is

$$R_{\mu,\nu}(u,v)L_{\mu,k}(u,\xi)L_{\nu,k}(v,\xi) = L_{\mu,k}(v,\xi)L_{\nu,k}(u,\xi)R_{\mu,\nu}(u,v), \tag{3.1}$$

where $R_{\mu,\nu}(u,v)$ is an operator acting non-trivially in $\mathcal{V}_\mu \otimes \mathcal{V}_\nu$. In the tensor product basis this operator reads

$$R_{\mu,\nu}(u,v) = \begin{pmatrix} f(v,u) & 0 & 0 & 0 \\ 0 & g(v,u) & 1 & 0 \\ 0 & 0 & g(v,u) & 0 \\ 0 & 0 & 0 & f(v,u) \end{pmatrix}_{[\mathcal{V}_\mu\otimes\mathcal{V}_\nu]}, \tag{3.2}$$

where

$$f(v,u) = \frac{\Delta v^2}{v^2-u^2}, \qquad g(v,u) = \frac{\Delta vu}{v^2-u^2}. \tag{3.3}$$

Next we define the quantum monodromy matrix as an ordered product of L-operators over auxiliary space (in our case this is the space associated with the vertical line)

$$T_\mu(u_\mu;\xi_1,\dots,\xi_M) = L_{\mu,M}(u_\mu,\xi_M)\cdots, L_{\mu,2}(u_\mu,\xi_2)L_{\mu,1}(u,\xi_1), \tag{3.4}$$

or as a matrix in the space $\mathcal{V}_\mu$

$$T_\mu(u_\mu;\xi_1,\dots,\xi_M) = \begin{pmatrix} A(u_\mu;\xi_1,\dots,\xi_M) & B(u_\mu;\xi_1,\dots,\xi_M) \\ C(u_\mu;\xi_1,\dots,\xi_M) & D(u_\mu;\xi_1,\dots,\xi_M) \end{pmatrix}_{[\mathcal{V}_\mu]}.$$

The entries $A,\dots,D$ are operators acting in the space $\mathcal{H} = \bigotimes_{k=1}^M \mathcal{H}_k = \left(\mathbb{C}^2\right)^{\otimes M}$. For the graphical representation of this operators see Fig 5.

Applying (3.1) to (3.4) M times we find that the monodromy matrix also satisfies the intertwining relation

$$R_{\mu,\nu}(u,v)T_{\mu,k}(u)T_{\nu,k}(v) = T_{\mu,k}(v)T_{\nu,k}(u)R_{\mu,\nu}(u,v). \tag{3.5}$$

To shorten the notation here and subsequently in this section we omit the arguments $\xi_1,\dots,\xi_M$. The equation (3.5) defines commutation relations between the A-, B-, C- and D-operators. The most important are

$$X(u)X(v) = X(v)X(u), \quad X \in \{A,B,C,D\}, \tag{3.6a}$$

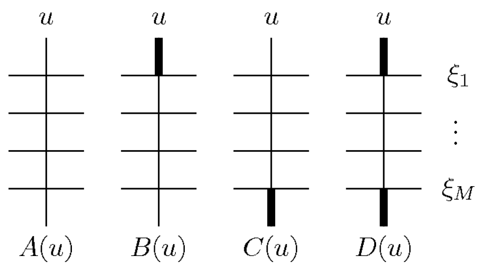

Figure 5. The graphical representation of the entries of the quantum monodromy matrix.

and

$$\begin{aligned}
C(v)D(u) &= f(v,u)D(u)C(v) + g(u,v)D(v)C(u), && (3.6b)\\
C(v)A(u) &= f(u,v)A(u)C(v) + g(v,u)A(v)C(u), && (3.6c)\\
A(u)B(v) &= f(u,v)B(v)A(u) + g(v,u)B(u)A(v), && (3.6d)\\
D(u)B(v) &= f(v,u)B(v)D(u) + g(u,v)B(u)D(v). && (3.6e)
\end{aligned}$$

With the help of (3.6) we find how A- and D-operators acts on an off-shell Bethe state. We formulate this result as the following Lemma.

Lemma 1. *Let $a(u)$ and $d(u)$ be vacuum eigenvalues of the A- and D-operators respectively, i.e.,*

$$\begin{aligned}
A(u)\left|\Omega\right\rangle &= a(u)\left|\Omega\right\rangle, \quad \left\langle\Omega\right| A(u) = \left\langle\Omega\right| a(u),\\
D(u)\left|\Omega\right\rangle &= d(u)\left|\Omega\right\rangle, \quad \left\langle\Omega\right| D(u) = \left\langle\Omega\right| d(u).
\end{aligned}$$

Then the following relations are valid:

$$\begin{aligned}
&A(u_{n+1}) \prod_{j=1}^{n} B(u_j)\left|\Omega\right\rangle\\
&\qquad = \sum_{i=1}^{n+1} a(u_i) \frac{g(u_i,u_{n+1})}{f(u_i,u_{n+1})} \prod_{\substack{j=1\\ j\neq i}}^{n+1} f(u_i,u_j) \prod_{\substack{j=1\\ j\neq i}}^{n+1} B(u_j)\left|\Omega\right\rangle \qquad (3.7a)
\end{aligned}$$

and

$$\langle\Omega|\prod_{j=1}^{n} C(u_j) D(u_{n+1}) \\ = \sum_{i=1}^{n+1} d(u_i) \frac{g(u_{n+1}, u_i)}{f(u_{n+1}, u_i)} \prod_{\substack{j=1\\ j\neq i}}^{n+1} f(u_j, u_i)\, \langle\Omega| \prod_{\substack{j=1\\ j\neq i}}^{n+1} C(u_j). \tag{3.7b}$$

Proof. Let us consider relation (3.7a). The commutation relation (3.6d) allows us to "move" the A-operator through the B-operators, and after using the standard technique of algebraic Bethe Ansatz we get

$$A(u_{n+1}) \prod_{j=1}^{n} B(u_j)\,|\Omega\rangle = a(u_{n+1}) \prod_{j=1}^{n} f(u_{n+1}, u_j) \prod_{j=1}^{n} B(u_j)\,|\Omega\rangle \\ + \sum_{i=1}^{n} a(u_i) g(u_i, u_{n+1}) \prod_{\substack{j=1\\ j\neq n}}^{n} f(u_i, u_j) B(u_{n+1}) \prod_{\substack{j=1\\ j\neq i}}^{n} B(u_j)\,|\Omega\rangle\,.$$

Then we note that $f(u,v)/g(u,v) = u/v$ and hence $f(u,u)/g(u,u) = 1$. Thus the terms on the right hand side can be combined into a single sum, which is exactly (3.7a).

The relation (3.7b) follows from (3.6b) in a similar way. □

Let us now state the main result.

Theorem 1. *Let $A(u)$, $B(u)$, $C(u)$ and $D(u)$ be operators satisfying commutation relations* (3.6) *and $a(u)$ and $d(u)$ be vacuum eigenvalues of the operators $A(u)$ and $D(u)$ respectively,*

$$A(u)\,|\Omega\rangle = a(u)\,|\Omega\rangle\,, \quad D(u)\,|\Omega\rangle = d(u)\,|\Omega\rangle\,,$$

then the matrix element $S_{n,m,l}$ defined by

$$S_{n,m,l}(u_1,\dots,u_L) \\ = \langle\Omega| \prod_{j=1}^{n} C(u_{n+l+m+j}) \prod_{j=1}^{m} D(u_{n+l+j}) \prod_{j=1}^{l} A(u_{n+j}) \prod_{j=1}^{n} B(u_j)\,|\Omega\rangle \tag{3.8}$$

admits two equivalent determinant representations. The first one reads

$$S_{n,m,l}(u_1,\dots,u_L)=\Delta^{(n+m+l)n}\prod_{j=1}^{l}u_{n+j}\prod_{j=1}^{m}u_{n+l+j}^{-1}\times\prod_{j=1}^{n+l}u_j^{-2m}\prod_{1\leqslant i<j\leqslant L}\frac{1}{u_j^2-u_i^2}\det V_{n,m,l},\quad(3.9)$$

where $V_{n,m,l}$ is an $L\times L$ matrix, $L=2n+m+l$, with entries

$$(V_{n,m,l})_{ij}=\begin{cases}d(u_j)u_j^{2i-1}\prod\limits_{k=1}^{n+l}\left(u_k^2-u_j^2\right), & i=1,\dots,m,\\ d(u_j)u_j^{2i-1}, & i=m+1,\dots,n+m,\\ a(u_j)u_j^{2i-3}, & i=n+m+1,\dots,L.\end{cases}\quad(3.10)$$

The second representation reads

$$S_{n,m,l}(u_1,\dots,u_L)=\Delta^{(n+m+l)n}\prod_{j=1}^{l}u_{n+j}\prod_{j=1}^{m}u_{n+l+j}^{-1}\times\prod_{j=1}^{n+m}u_{n+l+j}^{2l}\prod_{1\leqslant i<j\leqslant L}\frac{1}{u_j^{-2}-u_i^{-2}}\det W_{n,m,l},\quad(3.11)$$

where $W_{n,m,l}$ is an $L\times L$ matrix, $L=2n+m+l$, with entries

$$(W_{n,m,l})_{ij}=\begin{cases}a(u_j)u_j^{-2i+1}\prod\limits_{k=1}^{n+m}\left(u_{n+l+k}^{-2}-u_j^{-2}\right), & i=1,\dots,l,\\ a(u_j)u_j^{-2i+1}, & i=l+1,\dots,n+l,\\ d(u_j)u_j^{-2i+3}, & i=n+l+1,\dots,L.\end{cases}\quad(3.12)$$

Before proceeding with the proof we note (see Fig. 5) that the partition function of the five-vertex model with considered boundary conditions (see Fig 2) can be represented as

$$Z(u_1,\dots,u_L;\xi_1,\dots,\xi_M)=S_{n,m,l}(u_1,\dots,u_L),$$

with functions $a(u)$ and $d(u)$ defined in accordance with (2.1) as

$$a(u)=\frac{\alpha^M}{\Delta^M}\prod_{j=1}^{M}\left(\frac{u}{\xi_j}-\frac{\xi_j}{u}\right),\quad d(u)=\frac{1}{\alpha^M}\prod_{j=1}^{M}\frac{u}{\xi_j}.$$

We emphasize that Theorem 1 holds for any operators satisfying the algebra defined by (3.5) with R-matrix (3.2). For example the same determinant representations are valid for the four-vertex model [18] and the non-Hermitian phase model [19] (for more examples of such models we refer the reader to [21]).

The rest of the section is devoted to the proof.

We first recall the results of [16] where determinant representations for the matrix elements $S_{n,m,0}$ and $S_{n,0,l}$ were obtained. Namely

$$S_{n,m,0} = \Delta^{(n+m)n} \prod_{j=1}^{m} u_{n+j}^{-1} \prod_{1\leqslant i<j\leqslant L} \frac{1}{u_j^2 - u_i^2} \det Q_{n+m,n} \tag{3.13}$$

and

$$S_{n,0,l} = \Delta^{(n+l)n} \prod_{j=1}^{l} u_{n+j} \prod_{1\leqslant i<j\leqslant L} \frac{1}{u_j^2 - u_i^2} \det Q_{n,n+l}, \tag{3.14}$$

where

$$(Q_{ab})_{ij} = \begin{cases} d(u_j)u_j^{2i-1}, & i = 1, \dots, a, \\ a(u_j)u_j^{2i-3}, & i = a+1, \dots, a+b. \end{cases}$$

To shorten the notation here and subsequently we omit the arguments of $S_{n,m,l}(u_1, \dots, u_l)$.

We start with the representation (3.9). The proof is by induction on m. If $m = 0$ then (3.9) and (3.14) are exactly the same. Therefore the base case is verified and we can proceed with the inductive step and prove that (3.9) holds for some n, m, l assuming that it is valid for n, $m-1$, l.

From (3.8) and (3.7b) we find that $S_{n,m,l}$ satisfy the following recurrence relation:

$$S_{n,m,l} = \sum_{i=L-n}^{L} d(u_i) \frac{g(u_{L-n}, u_i)}{f(u_{L-n}, u_i)} S_{n,m-1,l}(\backslash u_i) \prod_{\substack{j=L-n \\ j\neq i}}^{L} f(u_j, u_i). \tag{3.15}$$

Here $S_{n,m-1,l}(\backslash u_i)$ stands for the matrix element (3.8) with the operator $D(u_i)$ being excluded.

Now we need to prove that (3.9) is a solution of (3.15). Substituting (3.9) into (3.15) yields

$$S_{n,m,l} = \Delta^{(n+m+l-1)n} \prod_{j=1}^{l} u_{n+j} \prod_{j=1}^{m-1} u_{n+l+j}^{-1} \prod_{j=1}^{n+l} u_j^{-2(m-1)}$$
$$\times \sum_{i=L-n}^{L} d(u_i) \frac{g(u_{L-n}, u_i)}{f(u_{L-n}, u_i)} \prod_{\substack{j=L-n \\ j\neq i}}^{L} f(u_j, u_i) \prod_{\substack{1\leqslant k<j\leqslant L \\ j,k\neq i}} \frac{1}{u_j^2 - u_k^2} \det V_{n,m-1,l}(\backslash u_i). \tag{3.16}$$

where $V_{n,m-1,l}(\backslash u_i)$ is the $(L-1)\times(L-1)$ matrix obtained from the matrix $V_{n,m,l}$ by dividing every column by u_j^2, where $j = 1, \ldots, L$ denote the column number, and then removing the first row and the ith column.

Substituting the explicit formulae for the functions $f(u,v)$ and $g(u,v)$ (3.3) into (3.16) we get

$$S_{n,m,l} = \Delta^{(n+m+l)n} \prod_{j=1}^{l} u_{n+j} \prod_{j=1}^{m} u_{n+l+j}^{-1} \prod_{j=1}^{n+l} u_j^{-2(m-1)}$$
$$\times \sum_{i=L-n}^{L} u_i d(u_i) \prod_{\substack{j=L-n \\ j\neq i}}^{L} \frac{u_j^2}{u_j^2 - u_i^2} \prod_{\substack{1\leqslant k<j\leqslant L \\ j,k\neq i}} \frac{1}{u_j^2 - u_k^2} \det V_{n,m-1,l}(\backslash u_i). \tag{3.17}$$

After some simple algebraic manipulations we rewrite (3.17) as

$$S_{n,m,l} = \Delta^{(n+m+l)n} \prod_{j=1}^{l} u_{n+j} \prod_{j=1}^{m} u_{n+l+j}^{-1} \prod_{1\leqslant k<j\leqslant L} \frac{1}{u_j^2 - u_k^2} \prod_{j=1}^{n+l} u_j^{-2m}$$
$$\times \frac{\prod_{j=1}^{L} u_j^2}{\prod_{j=1}^{m-1} u_{n+l+j}} \sum_{i=L-n}^{L} (-1)^i \frac{d(u_i)}{u_i} \prod_{j=1}^{n+l+m-1} (u_j^2 - u_i^2) \det V_{n,m-1,l}(\backslash u_i). \tag{3.18}$$

Note that we can add to the sum in (3.18) all terms with $i < L$ since they are equal to zero. Thus this sum is nothing but a minor expansion of the determinant $\tilde{V}_{n,m,l}$ with elements

$$\left(\tilde{V}_{n,m,l}\right)_{ij} = \begin{cases} \frac{d(u_j)}{u_j} \prod\limits_{k=1}^{n+l+m-1} (u_k^2 - u_j^2), & i = 1, \\ d(u_j) u_j^{2i-3} \prod\limits_{k=1}^{n+l} (u_k^2 - u_j^2), & i = 2, \dots, m, \\ d(u_j) u_j^{2i-3}, & i = m+1, \dots, n+m, \\ a(u_j) u_j^{2i-5}, & i = n+m+1, \dots, L. \end{cases}$$

The elements of the first row are equal to $d(u_j)u_j^{-1}\prod\limits_{j=1}^{n+l}(u_j^2 - u_k^2)$ times polynomial of degree $m-1$ with respect to u_j^2. Therefore after rows substitution we can remove all terms except of the $\prod\limits_{j=1}^{m-1} u_{n+l+j}^2$. After factoring this term out of the determinant and then moving u_j^2 from the prefactor $\prod\limits_{j=1}^{L} u_j^2$ to the jth column of the matrix we see that

$$\frac{\prod_{j=1}^{L} u_j^2}{\prod_{j=1}^{m-1} u_{n+l+j}} \det \tilde{V}_{n,m,l} = \det V_{n,m,l}.$$

This completes the proof.

This method carries over to the representation (3.11) with minor changes. The proof is by induction on l, with (3.13) being the base; the inductive step is based on recurrent relation

$$S_{n,m,l} = \sum_{i=1}^{n+1} a(u_i) \frac{g(u_i, u_{n+1})}{f(u_i, u_{n+1})} S_{n,m,l-1}(\backslash u_i) \prod_{\substack{j=1 \\ j\neq i}}^{n+1} f(u_i, u_j),$$

which follows from (3.7a). The detailed verification is left to the reader.

Finally we note that the equivalence of the representations (3.9) and (3.11) can be verified explicitly, i.e. the identity

$$\prod_{1\leqslant i<j\leqslant L} \frac{1}{u_j^2 - u_i^2} \prod_{j=1}^{n+l} u_j^{-2m} \det V_{n,m,l} = \prod_{j=1}^{n+m} u_{n+l+j}^{-2l} \prod_{1\leqslant i<j\leqslant L} \frac{1}{u_j^{-2} - u_i^{-2}} \det W_{n,m,l} \qquad (3.19)$$

holds for arbitrary functions $a(u)$ and $d(u)$. To see this we first note that the both sides of (3.19) has the form

$$\sum_{\sigma\in S_n} C_\sigma a(u_{\sigma(1)})\dots a(u_{\sigma(n+l)})d(u_{\sigma(n+l+1)})\dots d(u_{\sigma(L)}),$$

where summation is performed over all elements of the symmetric group S_n, and the coefficients C_σ depend on the parameters u_1, ..., u_L. Then by straightforward calculation one can show that coefficients on the RHS and LHS of (3.19) are the same.

§4. The model at the free-fermion point

In this section we consider the special limit of the five-vertex model (the free-fermionic limit) [22]. Namely, we introduce new parameters x_j and ν_k defined by

$$u_j = e^{x_j\Delta/2}, \quad j = 1,\dots,L, \qquad \xi_k = e^{\nu_k\Delta/2}, \quad k = 1,\dots M.$$

and then take a limit $\Delta \to 0$. The Boltzman weight of the vertex at the intersection of kth horizontal and μth vertical line now equals $w_i^{\Delta=0}(x_\mu - \nu_k)$, where the value of i is determined by the configuration ($i = 1, 3, 4, 5, 6$) and the functions $w_i^{\Delta=0}(x)$ in accordance with (2.1) are given by

$$\begin{gathered} w_1^{\Delta=0}(x) = \alpha x, \quad w_3^{\Delta=0}(x) = \alpha^{-1}, \quad w_4^{\Delta=0}(x) = \alpha, \\ w_5^{\Delta=0}(x) = w_6^{\Delta=0}(x) = 1. \end{gathered} \tag{4.1}$$

The model with weights (4.1) at $\alpha = 1$ can be used for the definition of Schur functions [20]. Therefore the determinant representations for the partition function imply summation formulae for Schur functions (for the exact result see Proposition 1). To see this we set all inhomogeneity parameters associated with horizontal lines to be zeros $\nu_k = 0$, where $k = 1,\dots,M$.

The partition function of the partially homogeneous model at the free-fermion point reads

$$\begin{aligned} &Z^{\Delta=0}(x_1,\dots,x_L) \\ &\quad= \lim_{\Delta\to 0} Z\left(e^{x_1\Delta/2},\dots,e^{x_L\Delta/2}; e^{\nu_1\Delta/2},\dots,e^{\nu_M\Delta/2}\right)\Big|_{\nu_1=\dots=\nu_M=0}. \end{aligned}$$

The "classic" way to define Schur function corresponding to the integer partition $\lambda = (\lambda_1,\dots,\lambda_N)$, where $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_N$, is by the ratio of two determinants

$$s_\lambda(x_1,\dots,x_N) = \frac{\det_{1\leqslant i,j\leqslant N}\left[x_j^{\lambda_i+N-i}\right]}{\det_{1\leqslant i,j\leqslant N}\left[x_j^{N-i}\right]}. \tag{4.2}$$

To reproduce the "lattice" definition we recall the branching rule

$$s_\lambda(x_1,\dots,x_N) = \sum_{\mu\,:\,\mu\prec\ell} s_\mu(x_1,\dots,x_{N-1})x_N^{|\lambda|-|\mu|}, \tag{4.3}$$

where $|\lambda| = \sum_{i=1}^N \lambda_i$, $|\mu| = \sum_{i=1}^{N-1}\mu_i$ and summation is performed over all partitions μ interlaced with λ, i.e.

$$\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \mu_{N-1} \geqslant \lambda_N.$$

Applying (4.3) N times we obtain

$$s_\lambda(x_1,\dots,x_N) = \sum_{()=\lambda^0\prec\lambda^1\cdots\prec\lambda^N=\lambda}\prod_{j=1}^N x_j^{\left|\lambda^j\right|-\left|\lambda^{j-1}\right|}. \tag{4.4}$$

This summation can be represented graphically in terms of lozenge tilings, which are a tilings of a domain on a regular triangular lattice by rhombi of three different types (see Fig. 6). Given the partition $\lambda = (\lambda_1,\dots,\lambda_N)$ we define domain to be a trapezoid of the size $N\times\lambda_1$ with N "tooths" at the left boundary, with z-coordinates λ_i+N-i (see Fig 7). Note that each tiling of this domain can be encoded by the set of interlacing partitions $\lambda^1\prec\lambda^2\prec\ldots\prec\lambda^N=\lambda$. To do this we shift all numbers λ_k^j to λ_k^j+N-i and interpret them as a z-coordinates of the lozenges of the first type with y-coordinate being equal to j. It can be easily seen that $\left|\lambda^{j+1}\right|-\left|\lambda^j\right|$ is just the number of the lozenges of the third type with y-coordinates being equal to $j+1/2$. Therefore by assigning to this lozenges weight x_j we reproduce the branching rule (4.3).

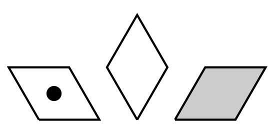

Figure 6. Three types of lozenges.

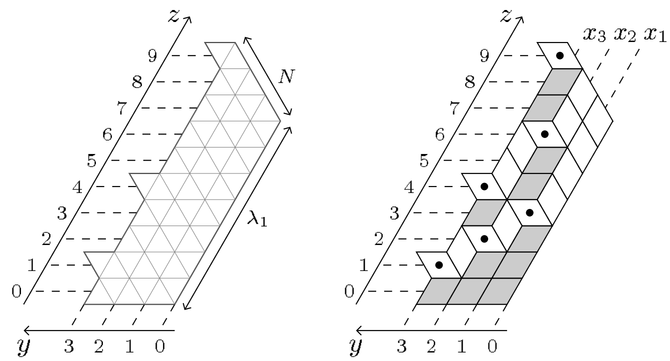

Figure 7. The domain corresponding to the partition $\lambda = (7, 3, 1)$ (left) and tiling encoded by $\lambda^1 = (3)$, $\lambda^2 = (5, 2)$, $\lambda^3 = (7, 3, 1)$ with the weight $x_1^3 x_2^4 x_3^4$ (right).

Finally we note that one can easily generalize (4.4) and introduce skew Schur functions

$$s_{\lambda/\mu}(x_k, \dots, x_N) = \sum_{\mu=\lambda^k \prec \cdots \prec \lambda^N = \lambda} \prod_{j=k}^{N} x_j^{|\lambda^j| - |\lambda^{j-1}|},$$

where $\mu = (\mu_1, \dots, \mu_k)$ is a partition of the length k. In terms of lozenge tilings this corresponds to the tilings of trapezoidal domain with "tooths" on both sides (k on the right and N on the left).

The discussed graphical representation allows us to represent the partition function of the model as a sum of Schur functions. Note that in all configurations, the number of vertices of the first type is fixed (equal to $M(l+n) - n(l+m+n)$), as well as the difference between the number of vertices of 3th and 4th type (equal to $ln - (M-n)(l+n)$). Hence the dependence of the partition function on α is reduced to the overall factor $\alpha^{M(l-m)}$. Therefore for the partition function we have

$$Z^{\Delta=0} = \alpha^{M(l-m)} \sum_{\mu \in \mathcal{M}} s_\mu(x_1, \dots, x_{n+l}) s_{\lambda/\mu}(x_{n+l+1}, \dots, x_L), \tag{4.5}$$

where $\lambda = (\lambda_1, \dots, \lambda_L)$ is a partition with elements

$$\lambda_j = \begin{cases} M - m - n, & j \leqslant n + l, \\ 0, & j > n + l, \end{cases} \tag{4.6}$$

and the summation is is performed over set $\mathcal{M}$ of all partitions $\mu = (\mu_1, \dots, \mu_{n+l})$ such that first l elements of the partition μ are equal to $M-n$ and the rest n elements are not greater than $M-m-n$, i.e.

$$\mu_j = \begin{cases} M-n, & j \leqslant l, \\ \tilde{\mu}_{j-l} & j > l, \end{cases} \tag{4.7}$$

where $\tilde{\mu} = (\tilde{\mu}_1, \dots, \tilde{\mu}_n)$ is a partition with elements $\tilde{\mu}_j \leqslant M-m-n$. The configuration shown in Fig 4 corresponds to the term with $\mu = (9,9,9,9,9,2,1,0)$.

On the other hand the partition function can be evaluated by taking the limit $\Delta \to 0$ in Theorem 1.

Proposition 1. *Let $\lambda = (\lambda_1, \dots, \lambda_L)$ and be a partition with elements defined by* (4.6) *and $\mathcal{M}$ be a set of all partitions $\mu = (\mu_1, \dots, \mu_{n+l})$ satisfying* (4.7)*. Then the following summation formulae are valid*

$$\sum_{\mu \in \mathcal{M}} s_\mu(x_1, \dots, x_{n+l}) s_{\lambda/\mu}(x_{n+l+1}, \dots, x_L)$$
$$= \prod_{1 \leqslant i < j \leqslant L} \frac{1}{x_i - x_j} \det \mathcal{V}_{n,m,l} = \prod_{1 \leqslant i < j \leqslant L} \frac{1}{x_i - x_j} \det \mathcal{W}_{n,m,l},$$

where $\mathcal{V}_{n,m,l}$ and $\mathcal{W}_{n,m,l}$ are $L \times L$ matrices ($L = 2n+m+l$) with elements

$$(\mathcal{V}_{n,m,l}) = \begin{cases} x_j^{M+n+l-i}, & i = 1, \dots, n+l, \\ x_j^{L-m-i}, & i = n+l+1, \dots, L-m, \\ x_j^{L-i} \prod_{k=1}^{n+l} (x_k - x_j), & i = L-m+1, \dots, L, \end{cases} \tag{4.8}$$

and

$$(\mathcal{W}_{n,m,l})_{ij} = \begin{cases} x_j^{M+l-i} \prod_{k=1}^{n+m} (x_j - x_{n+l+k}), & i = 1, \dots, l, \\ x_j^{M+l+n-i}, & i = l+1, \dots, n+l, \\ x_j^{L-i}, & i = n+l+1, \dots, L. \end{cases} \tag{4.9}$$

Proof. The proof is by straightforward calculation. First we expand the entries of (3.10) in Taylor series as $\Delta \to 0$, set all the inhomogeneity parameters ν_k to be equal to 0, then perform row substitution and "reflect" the matrix ($i \mapsto L-i+1$). After that transformations we find that (3.9)

reads
$$Z^{\Delta=0} = \alpha^{M(l-m)} \prod_{1\leqslant i<j\leqslant L} \frac{1}{x_i - x_j} \det \mathcal{V}_{n,m,l},$$
where the matrix $\mathcal{V}_{n,m,l}$ is exactly the one defined by (4.8).

In the similar way from (3.11) and (3.12) we obtain
$$Z^{\Delta=0} = \alpha^{M(l-m)} \prod_{1\leqslant i<j\leqslant L} \frac{1}{x_i - x_j} \det \mathcal{W}_{n,m,l},$$
where the matrix $\mathcal{W}_{n,m,l}$ is defined by (4.9).

To complete the proof we use (4.5). □

Note that if $m = 0$ then $\det \mathcal{V}_{n,m,l}$ is exactly the numerator in the classical definition of the Schur function (4.2) corresponding to the partition λ defined by (4.6). Therefore we are left with the well-known decomposition formula
$$s_\lambda(x_1, \dots, x_L) = \sum_{\mu} s_\mu(x_1, \dots, x_{n+l}) s_{\lambda/\mu}(x_{n+l+1}, \dots, x_L),$$
where summation is performed over all partitions μ of the size $n + l$. The condition (4.7) is satisfied automatically due to combinatoric constraints.

References

1. P. W. Kasteleyn, *The statistics of dimers on a lattice: I. The number of dimer arrangements on a quadratic lattice.* — Physica **27** (1961), 1209–1225.
2. N. Elkies, G. Kuperberg, M. Larsen, and J. Propp, *Alternating sign matrices and domino tilings. Part II.* — J. Algebraic Combin. **1** (1992), 219–234.
3. N. M. Bogolyubov, K. L. Malyshev, *Integrable models and combinatorics.* — Russian Math. Surveys, **70**,No. 5 (2015) 789–856.
4. G. Kuperberg, *Another proof of the alternating sign matrix conjecture.* — Int. Math. Res. Notices, **1996**, No. 3 (1996), 139–150.
5. W. H. Mills, D. P. Robbins, H. Rumsey, *Alternating sign matrices and descending plane partitions.* — J. Comb. Th. A, **34**No. 3 (1983), 340–359 .
6. E. H. Lieb, *The residual entropy of square ice.* — Phys. Rev. **162** (1967), 162–172.
7. B. Sutherland, *Exact solution of a two-dimensional model for hydrogen-bonded crystals.* — Phys. Rev. Lett. **19** (1967), 103–104.
8. R. J. Baxter, *Exactly Solvable Models in Statistical Mechanics.* San Diego, CA, Academic Press, 1982.
9. A. G. Izergin, D. A. Coker, and V. E. Korepin, *Determinant formula for the six-vertex model.* — J. Phys. A **25** (1992), 4315–4334.
10. L. Pauling, *The Structure and Entropy of Ice and of Other Crystals with Some Randomness of Atomic Arrangement.* — J. Am. Chem. Soc. **57**, No. 12 (1935), 2680–2684.

11. C. Garrod, *Stochastic models of crystal growth in two dimensions.* — Phys. Rev. A **41** (1990), 4184–4194.
12. C. Garrod, A. C. Levi, and M. Touzani, *Mapping of crystal growth onto the 6-vertex model*, — Solid State Comm. **75** (1990), 375–382.
13. H. Y. Huang, F. Y. Wu, H. Kunz, D. Kim, *Interacting dimers on the honeycomb lattice: an exact solution of the five-vertex model.* — Physica A **228**, No. 1 (1996), 1–32.
14. K. Motegi, K. Sakai, *Vertex models, TASEP and Grothendieck polynomials.* — Journal of J. Phys. A: Math. Theor. **46**, No. 35 (2013), 355201.
15. B. Brubaker, V. Buciumas, D. Bump, and H. P. A. Gustafsson, *Colored five-vertex models and demazure atoms.* — J. Combin. Theory A, **178** (2021), 105354.
16. I. N. Burenev and A. G. Pronko, *Determinant formulae for the five-vertex model.* — J. Phys. A: Math. Theor. **54** (2021), 055008.
17. V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions.* Cambridge University Press, Cambridge, 1993.
18. N. M. Bogoliubov, *Four-vertex model and random tilings.* — Theor. Math. Phys. **155** (2008), 523–535.
19. N. M. Bogoliubov, T. Nasar, *On the spectrum of the non-Hermitian phase-difference model.* — Phys. Lett. A **234** (1997), 345–350.
20. I. G. Macdonald, *Symmetric Functions and Hall Polynomials, 2nd edition.* — Oxford University Press, Oxford, 1995.
21. I. N. Burenev, A. G. Pronko, *Quantum Hamiltonians generated by the R-matrix of the five-vertex model.* — Zap. Nauchn. Sem. POMI **494** (2020), 103–124.
22. A. G. Pronko, *The Five-Vertex Model and Enumerations of Plane Partitions.* — J. Math. Sci. **213** (2016), 756–768.

St.Petesburg Department
of Mathematics Institute,
Fontanka 27, St.Petersburg 191011, Russia
E-mail: inburenev@pdmi.ras.ru

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