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CAUCHY–BINET DETERMINANTAL IDENTITY AND ENUMERATION OF PLANE PARTITIONS IN A HIGH BOX

ABSTRACT. The amplitudes of the leading asymptotics of the $XX0$ Heisenberg spin chain depend on the generating function of plane partitions with the additional conditions. In our paper we apply the Cauchy–Binet determinantal identity for derivation of the generating function of plane partitions with the fixed conjugate trace in a high box.

§1. INTRODUCTION

The representation of the Schur functions make it possible to apply the well-developed theory of the symmetric functions to calculation of the correlation functions [1–3] for quantum integrable models. In turn, the correlation functions allow to obtain and to interpret connections between symmetric functions [4], plane partitions [5], and enumerative combinatorics [6]. The correlation function of strongly correlated bosons [7], as well as the partition function of inhomogeneous four-vertex model [8], lead to a new approach to derivation of the norm-trace generating functions of plane partitions with the fixed traces [10, 11]. The Cauchy–Binet determinantal identity, which is widely used in calculations of auto-correlation function [9], finds the answer for the generating function of watermelons with deviation. The generating function of watermelons with deviation provides the norm-trace generating function of plane partitions in a high box with fixed values of the conjugate trace [10]. The application of generating function of plane partitions is then revealed, namely, the amplitudes of the leading asymptotics of the $XX0$ Heisenberg spin chain depend on them.

We start by the deriving of the q-represented Cauchy–Binet type determinantal identity in Section 2. In Section 3 the star configurations with

deviation are expressed in terms of the Schur functions. The attention is paid to the generating function of watermelons with deviation in Section 4. The Cauchy–Binet type identities are expressed in terms of the watermelon configurations of nests of lattice paths (Section 4.1). The norm-trace generating function of plane partitions in a high box with fixed values of conjugate trace is obtained (Section 4.2).

§2. The Cauchy–Binet determinantal identities

2.1. Notations and definitions. Boldface notations like $\mathbf{u}_N$, as well as $\mathbf{u}$, stand for N-tuples $(u_1, u_2, \ldots, u_N)$ of N (complex) numbers, and so on. We also use N-tuples like $\mathbf{M}_N = (M, M, \ldots, M)$ or $\mathbf{k}_N = (k, k, \ldots, k)$. The notation $[N] \equiv \{1, 2, \ldots, N\}$ implies that the elements of the set are ordered.

Let us introduce the notation $\bar{\mathbb{N}} \equiv \{0\} \cup \mathbb{N} = \{0, 1, 2, \ldots\}$. An N-tuple of strictly decreasing numbers $\mu_i \in \mathbb{N}$, $1 \leqslant i \leqslant N-1$, $\mu_N \in \bar{\mathbb{N}}$ is called *strict partition* $\boldsymbol{\mu} \equiv (\mu_1, \mu_2, \ldots, \mu_N)$. The elements of $\boldsymbol{\mu}$ are called *parts*, and they respect

$$M \geqslant \mu_1 > \mu_2 > \cdots > \mu_N \geqslant n. \tag{1}$$

The *length* of a partition, say, $\boldsymbol{\mu}$ is equal to the number of its parts, $l(\boldsymbol{\mu}) = N$. The *weight* $|\boldsymbol{\mu}|$ of partition is equal to the sum of its parts, $|\boldsymbol{\mu}| = \sum_{i=1}^{N} \mu_i$.

An N-tuple of weakly decreasing non-negative integers provides another important partition $\boldsymbol{\lambda} \equiv (\lambda_1, \lambda_2, \ldots, \lambda_N)$, where the parts $\lambda_j \in \bar{\mathbb{N}}$ respect

$$\mathcal{L} \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_N \geqslant n, \qquad \mathcal{L}, n \in \bar{\mathbb{N}}. \tag{2}$$

The relationship $\boldsymbol{\lambda} = \boldsymbol{\mu} - \boldsymbol{\delta}_N$, where $\boldsymbol{\delta}_N$ is the "staircase" partition

$$\boldsymbol{\delta}_N \equiv (N-1, N-2, \ldots, 1, 0), \tag{3}$$

enables to connect the partitions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ so that $\mathcal{L} = M - N + 1$ in (2) since $\lambda_i = \mu_i - N + i$, $1 \leqslant i \leqslant N$.

Let us consider a partition $\boldsymbol{\lambda}_{N-k}$ of the length $l(\boldsymbol{\lambda}_{N-k}) = N-k$, where $k \leqslant N$, $N, k \in \bar{\mathbb{N}}$. Proceeding with $\boldsymbol{\lambda}_{N-k}$, we shall use the notation $\widehat{\boldsymbol{\lambda}} \equiv \widehat{\boldsymbol{\lambda}}_N$ for a partition of the length $l(\widehat{\boldsymbol{\lambda}}) = N$, which can be viewed as $\boldsymbol{\lambda}_{N-k}$ "elongated" by k zero's as follows:

$$\widehat{\boldsymbol{\lambda}} \equiv (\lambda_1, \lambda_2, \ldots, \lambda_{N-k}, 0, 0, \ldots, 0) \equiv (\boldsymbol{\lambda}_{N-k}, 0, 0, \ldots, 0). \tag{4}$$

It is appropriate to introduce a strict partition $\widehat{\boldsymbol{\mu}}$, $l(\widehat{\boldsymbol{\mu}}) = N$:

$$\widehat{\boldsymbol{\mu}} \equiv \widehat{\boldsymbol{\lambda}} + \boldsymbol{\delta}_N = (\boldsymbol{\lambda}_{N-k} + \boldsymbol{\delta}_{N-k} + \mathbf{k}_{N-k}, \boldsymbol{\delta}_k) = (\boldsymbol{\mu}_{N-k} + \mathbf{k}_{N-k}, \boldsymbol{\delta}_k), \quad (5)$$

where $\boldsymbol{\delta}_N$, $\boldsymbol{\delta}_{N-k}$, and $\boldsymbol{\delta}_k$ are the staircase partitions (3) of the lengths N, $N - k$, and k, respectively, and $\mathbf{k}_{N-k}$ is a "constant" partition $(k, k, \ldots, k)$ of the length $N - k$.

Let us introduce k-tuple $\mathbf{i} \equiv \mathbf{i}_k \equiv (i_1, i_2, \ldots, i_k)$ consisting of strictly increasing integers $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant N$, $1 \leqslant k \leqslant N$. It is appropriate to introduce a relative *complement* of $\mathbf{i}$ in $[N]$ as $(N - k)$-tuple $\complement_N \mathbf{i}$:

$$\complement_N \mathbf{i} \equiv [N] \backslash \mathbf{i} = (1, 2, \ldots, \breve{i}_1, \ldots, \breve{i}_2, \ldots, \breve{i}_k, \ldots, N), \qquad (6)$$

where $\breve{i}_l$ implies that the sequence $1, 2, \ldots, N$ is missing the element i_l.

Let N-tuple of complex numbers $\mathbf{x}_N$ be given. Fixing k-tuple $\mathbf{i}$ and its complement $\complement_N \mathbf{i}$ (6), we introduce k-tuple $\mathbf{x_i}$ and $(N - k)$-tuple $\mathbf{x}_{\complement_N \mathbf{i}}$:

$$\mathbf{x_i} \equiv (x_{i_1}, x_{i_2}, \ldots, x_{i_k}), \qquad (7)$$

$$\mathbf{x}_{\complement_N \mathbf{i}} \equiv (x_1, x_2, \ldots, \breve{x}_{i_1}, \ldots, \breve{x}_{i_2}, \ldots, \breve{x}_{i_k}, \ldots, x_N) \equiv \overline{\mathbf{x_i}}, \qquad (8)$$

where $\breve{x}_{i_l}$ implies that x_{i_l} is dropped out of N-tuple $\mathbf{x}_N$. The equivalent notation $\overline{\mathbf{x_i}} \equiv \mathbf{x}_N \backslash \mathbf{x_i}$ (8) is to express that $\mathbf{x}_{\complement_N \mathbf{i}}$ is also viewed as a relative complement of k-tuple $\mathbf{x_i}$ (7) in $\mathbf{x}_N$. Besides, we shall use the following notation:

$$[N \backslash k] \equiv [N] \backslash [k]. \qquad (9)$$

We shall also consider $\mathbf{x}_{[k]}$ and $\mathbf{x}_{[N \backslash k]}$, as notations at $\mathbf{i} \equiv [k]$ for particular cases of $\mathbf{x_i}$ and $\mathbf{x}_{[N] \backslash \mathbf{i}}$, respectively:

$$\mathbf{x}_{[k]} \equiv (x_1, x_2, \ldots, x_k), \qquad \mathbf{x}_{[N \backslash k]} \equiv (x_{k+1}, x_{k+2}, \ldots, x_N) \qquad (10)$$

(it turns out that $\mathbf{x}_{[k]}$ and $\mathbf{x}_k$ denote the same).

Let us introduce a *plane partition of shape* $\boldsymbol{\lambda}$ as a map $\boldsymbol{\pi}$: $(i, j) \to \pi_{ij}$, $(i, j) \in \mathbb{N}^2$, from the Young diagram of partition $\boldsymbol{\lambda}$ to $\bar{\mathbb{N}}$ such that π_{ij} is a non-increasing function of i and j. The entries π_{ij} are called *parts* of the plane partition, and $|\boldsymbol{\pi}| = \sum_{i,j} \pi_{ij}$ is its *volume*.

Three-dimensional Young diagram is a stack of unit cubes such that π_{ij} is the height of the column with coordinates (i, j). A *box* $\mathcal{B}(L, N, K)$ of size $L \times N \times K$ is a subset of three-dimensional integer lattice:

$$\mathcal{B}(L, N, K) \equiv \left\{ (l, n, k) \in \bar{\mathbb{N}}^3 \,\middle|\, 0 \leqslant l \leqslant L, \ 0 \leqslant n \leqslant N, \ 0 \leqslant k \leqslant K \right\}. \quad (11)$$

It is said that a plane partition $\boldsymbol{\pi}$ is contained in $\mathcal{B}(L, N, K)$ if $i \leqslant L$, $j \leqslant N$, and $\pi_{ij} \leqslant K$ for all cubes of the Young diagram.

The *generating function* $Z_q(L, N, K)$ of plane partitions $\boldsymbol{\pi}$ contained in $\mathcal{B}(L, N, K)$ is of the form [4]:

$$Z_q(L, N, K) = \prod_{j=1}^{L} \prod_{k=1}^{N} \prod_{i=1}^{K} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}} = \prod_{j=1}^{L} \prod_{k=1}^{N} \frac{1 - q^{K+j+k-1}}{1 - q^{j+k-1}}. \quad (12)$$

Right-hand side of (12) gives at $q \to 1$ the number $A(L, N, K)$ of plane partitions in the box $\mathcal{B}(L, N, K)$ (MacMahonвЪ™s formula):

$$A(L, N, K) = \prod_{j=1}^{L} \prod_{k=1}^{N} \prod_{i=1}^{K} \frac{i + j + k - 1}{i + j + k - 2} = \prod_{j=1}^{L} \prod_{k=1}^{N} \frac{K + j + k - 1}{j + k - 1}. \quad (13)$$

2.2. The Cauchy–Binet identities. The *Schur functions* $S_{\boldsymbol{\lambda}}(\mathbf{x}_N)$ form a base in the ring of symmetric polynomials of N variables and are expressed in the ratio form [4,12]:

$$S_{\boldsymbol{\lambda}}(\mathbf{x}_N) \equiv S_{\boldsymbol{\lambda}}(x_1, x_2, \ldots, x_N) \equiv \frac{\det(x_j^{\lambda_k + N - k})_{1 \leqslant j, k \leqslant N}}{\mathcal{V}(\mathbf{x}_N)}, \quad (14)$$

where $\boldsymbol{\lambda}$ is a partition, and $\mathcal{V}(\mathbf{x}_N)$ is the Vandermonde determinant,

$$\mathcal{V}(\mathbf{x}_N) \equiv \det(x_j^{N-k})_{1 \leqslant j, k \leqslant N} = \prod_{1 \leqslant m < l \leqslant N} (x_l - x_m). \quad (15)$$

Bearing in mind the notations (6), (7), (8), (9), (10), we turn to

Proposition 1. *Let us choose k-tuple $\mathbf{i} = (i_1, i_2, \ldots, i_k)$ and a partition $\boldsymbol{\lambda}_{N-k}$. The following limiting relation for the Schur function $S_{\boldsymbol{\Lambda}}(\mathbf{x}_N)$ labelled by partition $\boldsymbol{\Lambda}$, $l(\boldsymbol{\Lambda}) = N$, holds true,*

$$\lim_{\mathbf{x_i} \to 0} S_{\boldsymbol{\Lambda}}(\mathbf{x}_N) \equiv \lim_{x_{i_1} \to 0} \lim_{x_{i_2} \to 0} \cdots \lim_{x_{i_k} \to 0} S_{\boldsymbol{\Lambda}}(\mathbf{x}_N) = S_{\boldsymbol{\lambda}_{N-k}}(\overline{\mathbf{x_i}}), \quad (16)$$

provided that $\boldsymbol{\Lambda}$ is of the form $\widehat{\boldsymbol{\lambda}}$ (4), and $\overline{\mathbf{x_i}}$ (10) is used in (16).

Proof. The relation (14) enables to prove (16). $\qquad \square$

The present section is concerned with the *Cauchy–Binet* type determinantal identity for the Schur functions [3]:

$$\mathcal{P}_{\mathcal{L}/n}(\mathbf{x}_N, \mathbf{y}_N) \equiv \sum_{\boldsymbol{\lambda} \subseteq \{(\mathcal{L}/n)^N\}} S_{\boldsymbol{\lambda}}(\mathbf{x}_N) S_{\boldsymbol{\lambda}}(\mathbf{y}_N) = \left(\prod_{l=1}^{N} x_l^n y_l^n \right) \frac{\det T(\mathbf{x}_N, \mathbf{y}_N)}{\mathcal{V}(\mathbf{x}_N) \mathcal{V}(\mathbf{y}_N)},$$

$$(17)$$

where the summation goes over all partitions $\boldsymbol{\lambda}$ satisfying (2), and $T(\mathbf{x}_N, \mathbf{y}_N) \equiv (T_{ij}(\mathbf{x}_N, \mathbf{y}_N))_{1 \leqslant i,j \leqslant N}$ in (17) is given by

$$T_{ij}(\mathbf{x}_N, \mathbf{y}_N) \equiv T_{ij} = \frac{1 - (x_i y_j)^{N + \mathcal{L} - n}}{1 - x_i y_j}. \tag{18}$$

Proposition 1 enables to go from (17) to the sum $\mathcal{P}_{\mathcal{L}}(\overline{\mathbf{x_i}}, \mathbf{y}_N)$, where one of the arguments is $(N - k)$-tuple $\overline{\mathbf{x_i}} \equiv \mathbf{x}_{[N] \setminus \mathbf{i}}$ (8). Indeed, let us proceed:

$$\mathcal{P}_{\mathcal{L}}(\overline{\mathbf{x_i}}, \mathbf{y}_N) \equiv \lim_{\mathbf{x_i} \to 0} \mathcal{P}_{\mathcal{L}}(\mathbf{x}_N, \mathbf{y}_N) = \sum_{\boldsymbol{\lambda} \subseteq \{\mathcal{L}^{N-k}\}} S_{\boldsymbol{\lambda}}(\overline{\mathbf{x_i}}) S_{\widehat{\boldsymbol{\lambda}}}(\mathbf{y}_N), \tag{19}$$

where $\mathbf{i}$ is fixed k-tuple, $\widehat{\boldsymbol{\lambda}}$ is given by (4), and summation is over $\boldsymbol{\lambda}$ of length $N - k$.

The q-parametrization

$$\mathbf{y}_N = \mathbf{q}_N \equiv (q, q^2, \ldots, q^N), \qquad \mathbf{x}_N = \mathbf{q}_N / q = (1, q, \ldots, q^{N-1}) \tag{20}$$

enables to represent $\mathcal{P}_{\mathcal{L}}(\overline{\mathbf{x_i}}, \mathbf{y}_N)$ (19) in the form:

$$\mathcal{P}_{\mathcal{L}}\left(\frac{\overline{\mathbf{q_i}}}{q}, \mathbf{q}_N\right) = \sum_{\boldsymbol{\lambda} \subseteq \{\mathcal{L}^{N-k}\}} S_{\widehat{\boldsymbol{\lambda}}}(\mathbf{q}_N) S_{\boldsymbol{\lambda}}\left(\frac{\overline{\mathbf{q_i}}}{q}\right), \tag{21}$$

where

$$\overline{\mathbf{q_i}} \equiv (q, q^2, \ldots, \underbrace{q^{i_1}}, \ldots, \underbrace{q^{i_2}}, \ldots, \underbrace{q^{i_k}}, \ldots, q^N),$$

and the underscored terms are missed. One arrives to the following

Theorem I. *The Cauchy–Binet type identity is valid for* $\mathcal{P}_{\mathcal{L}}(\frac{\overline{\mathbf{q_i}}}{q}, \mathbf{q}_N)$ *(21) under the q-parametrization* (20):

$$\mathcal{P}_{\mathcal{L}}\left(\frac{\overline{\mathbf{q_i}}}{q}, \mathbf{q}_N\right) = \frac{(-1)^{|\mathbf{i}| + kN - \frac{k}{2}(k-1)}}{q^{k(k-|\mathbf{i}|) + \frac{kN}{2}(N-1)}} \times \frac{\det \overline{\mathsf{T}}\left(\frac{\overline{\mathbf{q_i}}}{q}, \mathbf{q}_N\right)}{\mathcal{V}\left(\frac{\overline{\mathbf{q_i}}}{q}\right) \mathcal{V}(\mathbf{q}_N)}, \tag{22}$$

where $N \times N$ matrix $\overline{\mathsf{T}}(\frac{\overline{\mathbf{q_i}}}{q}, \mathbf{q}_N)$ consists of the entries $\overline{\mathsf{T}}_{ij}$:

$$
\begin{aligned}
\overline{\mathsf{T}}_{ij} &= \frac{[(\mathcal{L} + N)(j + i - 1)]}{[j + i - 1]}, & i &\in [N] \setminus \mathbf{i}, & 1 &\leqslant j \leqslant N, \\
\overline{\mathsf{T}}_{ij} &= q^{j(k-l)}, & i &\in \{i_l\}_{1 \leqslant l \leqslant k}, & 1 &\leqslant j \leqslant N,
\end{aligned}
\tag{23}
$$

and q-numbers $[n]$ $(n \in \bar{\mathbb{N}})$ are defined [13]:

$$[n] \equiv \frac{1 - q^n}{1 - q}. \tag{24}$$

Proof. Applying *Proposition* 1 in (17), one arrives, under q-parametrization (20), at (22). □

Two limits of (16) are of interest. In the first case, Eq. (16) is specified:

$$\lim_{\mathbf{x}_{[k]} \to 0} S_{\widehat{\boldsymbol{\lambda}}}(\mathbf{x}_N) = S_{\boldsymbol{\lambda}_{N-k}}(\mathbf{x}_{[N \backslash k]}), \tag{25}$$

where (10) is taken into account. In the second one, Eq. (16) reads:

$$\lim_{\mathbf{x}_{\mathbf{N}_k \backslash \boldsymbol{\delta}_k} \to 0} S_{\widehat{\boldsymbol{\lambda}}}(\mathbf{x}_N) = S_{\boldsymbol{\lambda}_{N-k}}(\mathbf{x}_{N-k}), \tag{26}$$

where $\mathbf{x}_{\mathbf{N}_k \backslash \boldsymbol{\delta}_k} \equiv (x_{N-k+1}, x_{N-k+2}, \ldots, x_{N-1}, x_N)$.

§3. The Schur functions and stars with deviation

Let a set of *semi-standard Young tableaux* T of shape $\boldsymbol{\lambda} = \boldsymbol{\lambda}_N$ with the entries taken from the set $[m] \equiv \{1, 2, \ldots, m\}$, $m \geqslant N$ is given, [12]. Each semi-standard tableau of shape $\boldsymbol{\lambda}$ with entries not exceeding N may be represented as a *star*, i.e., a nest of self-avoiding lattice paths with the prescribed starting and ending points [14, 15]. The *star* $\mathcal{C}_k$ *with deviation* k is the nest of paths introduced by

Definition 1. *The star* $\mathcal{C}_k$ *with deviation* k *is the nest of* N *self-avoiding lattice paths that connect, Figure 1, the starting points* $C_i = (i, N - i)$ *with the non-equidistant ending points* $(N, \widehat{\mu}_i)$, *where the parts of strict partiton* $\widehat{\mu}_i \in \widehat{\boldsymbol{\mu}}$ (5), *while upward steps are absent along the lines* $x_1, x_2, \ldots, x_k$. *The star* $\mathcal{C}_k$ *is characterized by such semi-standard Young tableau of shape* $\boldsymbol{\lambda}_{N-k}$ *that the number inside the upper left corner cell is greater than* k.

The Schur function associated with the star $\mathcal{C}_k$ is represented:

$$S_{\boldsymbol{\lambda}}(\mathbf{x}_{[N \backslash k]}) = \sum_{\{\mathcal{C}_k\}} \prod_{j=k+1}^{N} x_j^{c_j}, \tag{27}$$

where $\boldsymbol{\lambda} \equiv \boldsymbol{\lambda}_{N-k}$ and $\mathbf{x}_{[N \backslash k]}$ (10) is used.

We introduce an *extended volume* $|\mathcal{C}_k|_{\mathrm{w}}$ of the star $\mathcal{C}_k$:

$$|\mathcal{C}_k|_{\mathrm{w}} \equiv (N + k + 1)|\boldsymbol{\lambda}_{N-k}| - \sum_{j=k+1}^{N} j c_j, \tag{28}$$
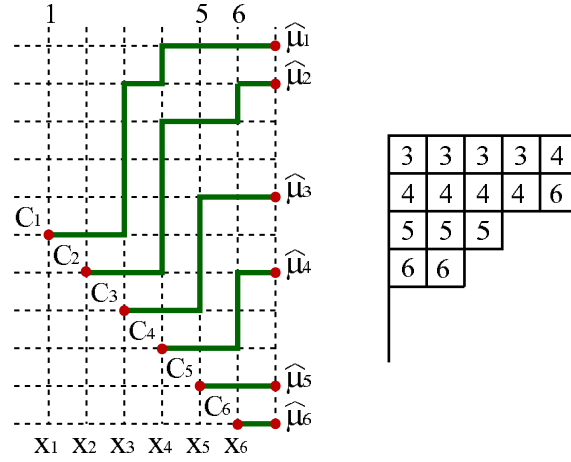
Figure 1. The star $\mathcal{C}_k$ with deviation $k = 2$ given by $\widehat{\boldsymbol{\mu}} = (10, 9, 6, 4, 1, 0)$ represents the semi-standard tableau of shape $\widehat{\boldsymbol{\lambda}} = (5, 5, 3, 2, 0, 0)$.

where $\sum_{j=k+1}^{N} c_j = |\boldsymbol{\lambda}_{N-k}|$ since $c_1 = c_2 = \ldots = c_k = 0$. We express the partition function of the stars $\mathcal{C}_k$:

$$\mathcal{Z}_{\{\mathcal{C}_k\}} = \sum_{\{\mathcal{C}_k\}} q^{|\mathcal{C}_k|_{\mathrm{w}}} = q^{(N+1)|\boldsymbol{\lambda}_{N-k}|} S_{\boldsymbol{\lambda}_{N-k}} \left(\frac{1}{\mathbf{q}_{N-k}} \right) = S_{\boldsymbol{\lambda}_{N-k}} (\mathbf{q}_{[N\setminus k]}).$$

(29)

The definition of the volume in the form

$$|\mathcal{C}_k|_{\overline{\mathrm{w}}} \equiv (N+1)|\boldsymbol{\lambda}_{N-k}| - \sum_{j=k+1}^{N} j c_j$$

(30)

leads to the partition function

$$\mathcal{Z}_{\{\mathcal{C}_k\}} = \sum_{\{\mathcal{C}_k\}} q^{|\mathcal{C}_k|_{\overline{\mathrm{w}}}} = S_{\boldsymbol{\lambda}_{N-k}} (\mathbf{q}_{N-k}).$$

(31)

Let us introduce the partition $\mathbb{M} \equiv (\mathcal{M}, \mathcal{M}, \ldots, \mathcal{M})$ of the length $l(\mathbb{M}) = N$, where $\mathcal{M} \equiv M - N + 1$. Let us advance

Definition 2. *The conjugate star $\mathcal{B}$, corresponding to the semi-standard skew Young tableau of shape $\mathbb{M}\setminus\boldsymbol{\lambda}$, is a configuration of N self-avoiding*

lattice paths that connect the non-equidistant points $(1, \mu_\alpha)$*, where* $\mu_\alpha = \lambda_\alpha + N - \alpha$*,* $1 \leqslant \alpha \leqslant N$*, with the equidistantly arranged points* $B_\alpha = (\alpha, M + 1 - \alpha)$*.*

We put $\mathbf{y} \equiv (y_1, y_2, \ldots, y_N)$ and obtain:

Proposition 2. *The following representation of the Schur function* $S_{\boldsymbol{\lambda}}(\mathbf{y})$ *is valid:*

$$S_{\boldsymbol{\lambda}}(\mathbf{y}) = \sum_{\{\mathcal{B}\}} \prod_{\alpha=1}^{N} y_\alpha^{\mathcal{M} - b_\alpha} . \tag{32}$$

We introduce a dual extended volume $|\mathcal{B}|_{\mathrm{w}}$,

$$|\mathcal{B}|_{\mathrm{w}} = \sum_{\alpha=1}^{N} (\mathcal{M} - b_\alpha)(\alpha - 1) . \tag{33}$$

The partition function of the star $\mathcal{B}$ is obtained from (32) and (33) under the parametrization $\mathbf{y} = \mathbf{q}_N / q$:

$$S_{\boldsymbol{\lambda}}\Big(\frac{\mathbf{q}_N}{q}\Big) = \sum_{\{\mathcal{B}\}} q^{\sum_{j=1}^{N}(j-1)(\mathcal{M} - b_j)} = \sum_{\{\mathcal{B}\}} q^{|\mathcal{B}|_{\mathrm{w}}} , \tag{34}$$

where summation is over all admissible stars $\mathcal{B}$.

§4. WATERMELONS AND THE GENERATING FUNCTIONS

4.1. Watermelons with deviation. *Watermelon* is the nest of self-avoiding lattice paths with equidistant starting C_l and ending B_l points $(1 \leqslant l \leqslant N)$. Only upward and rightward steps are allowed, and the nest is characterized by the paths with the total number of upward steps $\mathcal{M}$ and the total number of rightward steps N. The lth path in watermelon is contained within the rectangle such that C_l and B_l are its lower left and upper right vertices, respectively $(1 \leqslant l \leqslant N)$.

The nest of paths *watermelon with deviation* is introduced by means of

Construction. *Watermelon with deviation* k *(see Figure 2) is the nest of paths obtained by 'gluing' the stars* $\mathcal{C}_k$ *and* $\mathcal{B}$ *along the dissection line determined by* $\widehat{\boldsymbol{\mu}}$ *(5). with parts respecting (1) so that the points* $(N+1, \mu_i)$ *and the points* $(1, \mu_i)$ *are identified.*

It follows from *Construction* that the generating function of watermelons with deviation k is given by the sum:

$$\mathcal{P}_{\mathcal{M}}(\mathbf{x}_{N;k}, \mathbf{y}_N) \equiv \sum_{\boldsymbol{\lambda} \subseteq \{\mathcal{M}^{N-k}\}} S_{\boldsymbol{\lambda}}(\mathbf{x}_{N;k}) S_{\widehat{\boldsymbol{\lambda}}}(\mathbf{y}_N)$$

$$= \sum_{\boldsymbol{\lambda} \subseteq \{\mathcal{M}^{N-k}\}} \left(\sum_{\{\mathcal{C}_k\}} \prod_{x_j \in \mathbf{x}_{N;k}} x_j^{c_j} \right) \left(\sum_{\{\mathcal{B}\}} \prod_{\alpha=1}^N y_\alpha^{\mathcal{M}-b_\alpha} \right), \qquad (35)$$

where $\mathbf{x}_{N;k}$ implies either $\mathbf{x}_{N-k}$ or $\mathbf{x}_{[N \backslash k]}$.
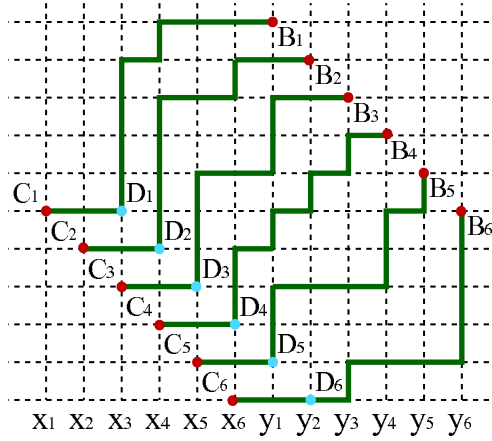

Figure 2. Watermelon with deviation $k = 2$ and $N = 6$.

Let us introduce the partition $\widehat{\mathbb{M}}$ of the length $l(\widehat{\mathbb{M}}) = N + L$ and the volume $|\widehat{\mathbb{M}}| = \mathcal{M}N$,

$$\widehat{\mathbb{M}} \equiv \widehat{\mathbb{M}}_{N+L} = (\mathbb{M}_N, \underbrace{0, 0, \ldots, 0}_{L}), \qquad \mathbb{M}_N \equiv (\underbrace{\mathcal{M}, \mathcal{M}, \ldots, \mathcal{M},}_{N}). \qquad (36)$$

The corresponding semi-standard Young tableau of shape $\widehat{\mathbb{M}}$ consists of cells arranged in N rows of length N (and L "rows" of zero length). Enumeration of the sets of numbers which "fill" the cells is equivalent to enumeration of $2N$-tuples $(m_1, m_2, \ldots, m_{2N})$.

Let us define the *volume of the path* as the number of cells below the path within the corresponding rectangle, and the *volume of watermelon*

as the volume of all paths constituting the watermelon. If so, the numbers of nests of the lattice paths constituting the watermelons are encoded by the *generating function of watermelons.* We introduce the generating function $W_q(N, L, \mathcal{M})$ of the watermelons with deviation k by means of the following.

Definition 3. *Let a nest* $\mathbf{w}_{NL\mathcal{M}}$ *of N lattice paths characterized by the total numbers L and $\mathcal{M}$ of steps along abscissa and ordinate axes to constitute the watermelon with deviation $N - L$. The generating function $W_q(N, L, \mathcal{M})$ of the nest* $\mathbf{w}_{NL\mathcal{M}}$ *is given by the polynomial*

$$W_q(N, L, \mathcal{M}) \equiv \sum_{\{\mathbf{w}_{NL\mathcal{M}}\}} q^{|\mathbf{w}_{NL\mathcal{M}}|_\delta}, \tag{37}$$

where summation goes over all admissible $\mathbf{w}_{NL\mathcal{M}}$. *Let the parameter δ to specify the volume of the star $\mathcal{C}_k$ used in Construction: the choice $\delta = k$ or $\delta = 0$ corresponds to the volume either (28) or (30). The corresponding volumes of the nests,* $|\mathbf{w}_{NL\mathcal{M}}|_\delta$, *are parameterized by δ:*

$$|\mathbf{w}_{NL\mathcal{M}}|_\delta \equiv \sum_{j=k+1}^{2N} (2N - j)m_j + \delta \sum_{j=k+1}^{N} m_j - \frac{\mathcal{M}N}{2}(N - 1). \tag{38}$$

The numbers of steps along the vertical lines with abscissae x_j, $k + 1 \leqslant j \leqslant 2N$, respect $\sum_{j=k+1}^{2N} m_j = \mathcal{M}N$.

In the case $N - L = k$, we define the Schur function labelled by $\widehat{\mathbb{M}}$ (36):

$$S_{\widehat{\mathbb{M}}}(\mathbf{x}_{[2N\setminus k]}) \equiv \sum_{\{\mathbf{w}_{NL\mathcal{M}}\}} \prod_{j=k+1}^{2N} x_j^{m_j}, \tag{39}$$

where the summation is over all admissible $\mathbf{w}_{NL\mathcal{M}}$. Then the graphical considerations enable to formulate

Proposition 3. *The generating function $W_q(N, L, \mathcal{M})$ (37) of watermelons with deviation $N - L = k$ respects the identities:*

$$W_q(N, L, \mathcal{M}) = \mathcal{P}_\mathcal{M}\left(\boldsymbol{q}_{N;k}, \frac{\boldsymbol{q}_N}{q}\right) \tag{40}$$

$$= q^{-\frac{\mathcal{M}N}{2}(N+1)} S_{\widehat{\mathbb{M}}}\left(\boldsymbol{q}_N, q^{\delta+N+1}, q^{\delta+N+2}, \ldots, q^{\delta+2N-k}\right), \tag{41}$$

where $\mathcal{P}_{\mathcal{M}}(\boldsymbol{q}_{N;k}, \boldsymbol{q}_N / q)$ and the Schur function $S_{\widehat{\mathbb{M}}}$ are given by (35) and (39), respectively, under the q-parametrization (20). The notation $\boldsymbol{q}_{N;k}$ implies $\boldsymbol{q}_{[N\setminus k]}$ at $\delta = k$ or $\boldsymbol{q}_{N;k} = \boldsymbol{q}_{N-k}$ at $\delta = 0$.

Proof. The volume $|\mathbf{w}_{NL\mathcal{M}}|_\delta$ (38) respects the relationship

$$|\mathcal{C}_k|_{\mathrm{w}}^\delta + |\mathcal{B}|_{\mathrm{w}} = |\mathbf{w}_{NL\mathcal{M}}|_\delta \,, \tag{42}$$

where $|\mathcal{B}|_{\mathrm{w}}$ is the volume (33), and the superscript δ in $|\mathcal{C}_k|_{\mathrm{w}}^\delta$ is to stress that either $|\mathcal{C}_k|_{\mathrm{w}}$ (28) or $|\mathcal{C}_k|_{\overline{\mathrm{w}}}$ (30) is used at $\delta = k$ or $\delta = 0$, respectively. Equations (37) and (42) lead us to the following relation:

$$W_q(N, L, \mathcal{M}) = \sum_{\boldsymbol{\lambda} \subseteq \{\mathcal{M}^L\}} \left(\sum_{\{\mathcal{C}_k\}} q^{|\mathcal{C}_k|_{\mathrm{w}}^\delta} \right) \left(\sum_{\{\mathcal{B}\}} q^{|\mathcal{B}|_{\mathrm{w}}} \right), \tag{43}$$

since the summation $\sum_{\boldsymbol{\lambda} \subseteq \{\mathcal{M}^L\}} \sum_{\{\mathcal{C}_k\}} \sum_{\{\mathcal{B}\}}$ replaces the sum $\sum_{\{\mathbf{w}_{NL\mathcal{M}}\}}$. Right-hand side of Eq. (43) is re-expressed:

$$\sum_{\boldsymbol{\lambda} \subseteq \{\mathcal{M}^L\}} S_{\boldsymbol{\lambda}}(\mathbf{q}_{N;k}) S_{\widehat{\boldsymbol{\lambda}}}\left(\frac{\mathbf{q}_N}{q} \right) = \mathcal{P}_{\mathcal{M}}\left(\mathbf{q}_{N;k}, \frac{\mathbf{q}_N}{q} \right). \tag{44}$$

Equation (40) is valid due to (43) and (44).

Let us turn to Eq. (41). We obtain from (37), (38) and (39):

$$W_q(N, L, \mathcal{M}) = q^{-\frac{N\mathcal{M}}{2}(N-1) + 2N^2\mathcal{M}} S_{\widehat{\mathbb{M}}}\left(\frac{q^{2\delta - k}}{\mathbf{q}_{N;k}}, \frac{1}{\mathbf{q}_{[2N \setminus N]}} \right). \tag{45}$$

We re-express $W_q(N, L, \mathcal{M})$ (45) at $\delta = 0$:

$$W_q(N, L, \mathcal{M}) = q^{-\frac{\mathcal{M}N}{2}(N+1)} S_{\widehat{\mathbb{M}}}(\mathbf{q}_{2N-k}). \tag{46}$$

Further, we obtain at $\delta = k$:

$$S_{\widehat{\mathbb{M}}}\left(\frac{1}{q}, \frac{1}{q^2}, \cdots, \frac{1}{q^{N-k}}, \frac{1}{q^{N+1}} \cdots, \frac{1}{q^{2N}} \right)$$

$$= q^{-\mathcal{M}N(2N+1)} S_{\widehat{\mathbb{M}}}\left(q^{2N}, q^{2N-1}, \ldots, q^{N+k+1}, q^N, q^{N-1}, \ldots, q^2, q \right), \tag{47}$$

where the homogeneity of the Schur functions is accounted for. Equation (41) is thus valid due to (45) and (47). $\qquad\square$

Corollary. *The Schur function $S_{\widehat{\mathbb{M}}}$ (41) acquires the determinantal representation due to Proposition 3 provided that (21), (22), (23) are taken*

into account:

$$S_{\widehat{\mathbb{M}}}\big(\boldsymbol{q}_N, q^{\delta+N+1}, q^{\delta+N+2}, \ldots, q^{\delta+N+L}\big)$$

$$= q^{\frac{\mathcal{M}N}{2}(N+1)-\frac{L}{2}(N-L)(2\delta+L-1)} \frac{\det \overline{T}^{\delta}}{\mathcal{V}(\mathbf{q}_N/q)\mathcal{V}(\mathbf{q}_{N;k})} , \qquad (48)$$

where the entries of $N \times N$ matrix $\overline{T}^{\delta}$ are given:

$$\overline{T}^{\delta}_{ij} \equiv \frac{[(\mathcal{M}+N)(\delta+j+i-1)]}{[\delta+j+i-1]}, \quad 1 \leqslant i \leqslant N-k, \qquad 1 \leqslant j \leqslant N,$$

$$\overline{T}^{\delta}_{ij} \equiv q^{j(N-i)}, \qquad\qquad\qquad N-k+1 \leqslant i \leqslant N, \quad 1 \leqslant j \leqslant N. \tag{49}$$

4.2. The generating function $W_q(N, L, \mathcal{M})$ as the norm-trace generating function. The Schur function representation (41) for $W_q(N, L, \mathcal{M})$ depends on δ, and the corresponding limits at $q \to 1$ coincide for $\delta = 0$ and $\delta = k$.

Let us recall the statement due to R. P. Stanley, which is expressed by

Theorem II. ([10]). *We have*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} T^*_{NLn}(m)\, a^n q^m = \prod_{i=1}^{L} \prod_{j=1}^{N} \frac{1}{1 - aq^{i+j-1}} , \qquad (50)$$

where $T^*_{NLn}(m)$ is the number of plane partitions of m with $\leqslant N$ rows and largest part $\leqslant L$, and conjugate trace n.

By definition, *conjugate trace* of plane partition $\boldsymbol{\pi}$ is the number of parts π_{ij} respecting $\pi_{ij} \geqslant i$ [10].

The double product in right-hand side of (50) is the norm-trace generating function of plane partitions with fixed trace along main diagonal (at $N = L$) [7]. At $N \neq L$, right-hand side contributes into the generating function of plane partitions with fixed trace along secondary diagonal [8].

The matrix $\overline{T}^{\delta}$ (49) at $\delta = k$ is simplified in the limit $M \to \infty$ so that the corresponding determinant is tractable. As a result, the partition function $W_q(N, L, \mathcal{M})$ of the watermelon with deviation acquires the form proportional to the norm-trace generating function of plane partitions (50). The statement is given by the following

Theorem III. *The partition function $W_q(N, L, \mathcal{M})$ of the watermelon with deviation $N - L = k$ expressed by (41) with $\delta = k$ takes the form at*

$M \to \infty$:

$$W_q(N, L, \mathcal{M}) \underset{M\to\infty}{=} q^{\frac{N-L}{2}(N+L-1)} \prod_{i=1}^{L} \prod_{j=1}^{N} \frac{1}{1-q^{k+i+j-1}} \,. \qquad (51)$$

Proof. The entries (49) at $\delta = k$ are simplified in the limit $M \to \infty$ provided that $q^{kM} \ll 1$:

$$\overline{T}_{ij}^{\delta=k} = \frac{1}{1-q^{k+j+i-1}} \,, \qquad 1 \leqslant i \leqslant N-k, \quad 1 \leqslant j \leqslant N \,.$$

In order to evaluate $\det \overline{T}^{(\delta=k)}$, one firstly combines neighboring rows i^{th} and $(i+1)^{\text{th}}$ in (49), $L+1 \leqslant i < N$, as required to calculate the Vandermonde determinant. After this, the columns j^{th} and N^{th}, $1 \leqslant j < N$, are combined to obtain $N-1$ zeros in N^{th} row. After $N-L$ steps one obtains:

$$\det \overline{T}^{(\delta=k)} = q^{\frac{N}{2}(N-1)+\frac{L}{2}(2N-L-1)(N-L)} \prod_{i=1}^{L} \prod_{j=L+1}^{N} \frac{1}{1-q^{k+i+j-1}}$$

$$\times \frac{\mathcal{V}(\mathbf{q}_N/q)}{\mathcal{V}(\mathbf{q}_L)} \det \left(\frac{1}{1-q^{k+i+j-1}} \right)_{1\leqslant i,j\leqslant L} \,. \qquad (52)$$

The Cauchy determinant in (52) is evaluated, [7], and one obtains (51) from (41) provided that (52) is used in (48). $\qquad\square$

The expansion (50) is valid for the double products (51) at $a = q^k$. The representation (52) enables the limiting form of the Schur function (48). The generating function is given by $q^{\frac{k}{2}(k+1)-kN} W_q(N, L, \mathcal{M})|_{M\to\infty}$:

$$q^{|\boldsymbol{\delta}_{N-k}|-|\boldsymbol{\delta}_N|} \sum_{\boldsymbol{\lambda}\subseteq\{\mathcal{M}^{N-k}\}} q^{k|\boldsymbol{\lambda}_{N-k}|} S_{\boldsymbol{\lambda}}(\mathbf{q}_{N-k}) S_{\widehat{\boldsymbol{\lambda}}}\left(\frac{\mathbf{q}_N}{q}\right)$$

$$\underset{M\to\infty}{=} \prod_{i=1}^{N} \prod_{j=1}^{N-k} \frac{1}{1-q^{k+i+j-1}} \,.$$

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