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A GENERAL VECTOR FIELD COUPLED TO A STRONGLY COMPRESSIBLE TURBULENT FLOW

ABSTRACT. We consider the model of a transverse vector (e.g. magnetic) field with the most general form of the nonlinearity, known as the $\mathcal{A}$ model, passively advected by a strongly compressible turbulent flow, governed by the randomly stirred Navier-Stokes equation. The full stochastic problem is equivalent to a certain renormalizable field theoretic model with an infrared- attractive fixed point. Thus, the scaling behaviour for the large-scale, long-distance behaviour is established. However, the question whether the parameter $\mathcal{A}$ tends to a certain fixed-point value of the renormalization group equations or remains arbitrary, cannot be answered within the one-loop approximation of our study.

§1. INTRODUCTION

In a few last decades of the past Millennium, intermittent interest had been attracted to the problem of intermittency and anomalous scaling in fluid turbulence; see e.g. Refs. [1, 2] and the literature cited therein.

The term "anomalous scaling" reminds of the critical scaling in models of equilibrium phase transitions. In those, the field theoretic methods were successfully employed to establish the existence of asymptotic scaling regimes and to construct regular perturbative schemes (the famous ε expansion and its relatives) for the corresponding exponents, scaling functions, etc; see e.g. [3] and references therein.

In turbulence, the phenomenon manifests itself in singular (arguably power-like) behavior of various statistical quantities as functions of the integral turbulence scales, with infinite sets of independent anomalous exponents [1].

Within the framework of numerous semi-heuristic approaches, the anomalous exponents were related to statistical properties of the local dissipation rate, the fractal dimension of structures formed by the small-scale turbulent eddies, the characteristics of nontrivial structures (vortex filaments), and so on; see Refs. [1, 4–6] for a review and further references. In spite of this undeniable achievement[1], the problem still remains open to further investigation.

The common feature of such models is that they are only intuitively related to underlying dynamic equations, involve arbitrary adjusting parameters and, therefore, cannot be considered to be the firm basis for of a systematic quantitative expansion in a certain small parameter. No regular calculation scheme, based on an underlying dynamical model and reliable perturbation expansion (hopefully similar to the famous ε expansion for critical exponents) had yet been constructed for the anomalous exponents of the turbulent velocity field. Thus, some doubt still remains about the universality of anomalous exponents and the very existence of deviations from the classical K41 theory.

In this direction of study, an extremely inspiring progress was achieved for the simplified Kraichnan's rapid-change model of a passively advected scalar field [8], where the advecting velocity field was taken Gaussian, not correlated in time, and having a power-like correlation function of the form $\sim \delta(t - t')/k^{d+\xi}$, where d is the dimension of space, k being the wave number and ξ an arbitrary exponent. There, for the first time, the existence of anomalous scaling was firmly established on the basis of a dynamic model; the corresponding anomalous exponents were calculated in controlled approximations [9–11] and, eventually, within a systematic perturbation expansion in a formal small parameter ξ [12, 13] up to the order ξ^3; see [14, 15] and the references.

In the original Kraichnan's "rapid-change model," the velocity ensemble was taken Gaussian, not correlated in time, isotropic, and the fluid was assumed to be incompressible. More realistic models should take into account finite correlation time and non-Gaussianity of the velocity ensemble, anisotropy, compressibility of the fluid, etc; see the discussion in [16].

The most efficient way to study anomalous scaling is provided by the field theoretic renormalization group (RG) combined with the operator product expansion (OPE); see [3, 17, 18] for the detailed exposition of these

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[1]Giorgio Parisi was awarded the Nobel Prize in Physics 2021 "for the discovery of the interplay of disorder and fluctuations in physical systems" [7].

techniques and the references. In the RG+OPE scenario for the anomalous scaling in turbulence, proposed in [19], the singular dependence on the integral scales emerges as a consequence of the existence in the corresponding models of composite fields with negative dimensions (termed in [19] "dangerous operators").

In the RG+SDE treatment of the Kraichnan's model, the anomalous exponents are identified with the scaling dimensions ("critical dimensions" in the terminology of the critical state theory) of certain *individual* Galilean-invariant composite operators [12]. This allows one to give a self-consistent derivation of the anomalous scaling, to construct a systematic perturbation expansion for the anomalous exponents in ξ, and to calculate the exponents in the second [12, 13] and in the third [14, 15] orders. The RG approach can be generalized to the case of finite correlation time [20] and to the non-Gaussian advecting velocity field, governed by the stochastic Navier–Stokes (NS) equation [22]. A general overview of the RG approach to Kraichnan's model and its descendants and more references can be found in [16].

A next important step toward the real NS turbulence is to consider the turbulent advection of passive *vector* fields. The latter can have different physical meaning: magnetic field in the Kazantsev-Kraichnan model of magnetohydrodynamical turbulence in the kinematic approximation; perturbation in the linearized NS equation with prescribed statistics of the background field; density of an impurity with internal degrees of freedom, etc. Despite the obvious practical significance of these physical situations, the passive vector problem is especially interesting because of the insight it offers into the inertial-range behavior of the NS turbulence.

Owing to the coupling between different components of the vector field (both by the dynamical equation and the incompressibility condition) and to the presence of a new stretching term in the dynamical equation, that couples the advected quantity to the gradient of the advecting velocity, the behavior of the passive vector field appears much richer than that of the scalar field: "...there is considerably more life in the large-scale transport of vector quantities," (p. 232 of Ref. [1]). Indeed, passive magnetic fields reveal anomalous scaling already on the level of the pair correlation function [25, 26].

The general vector "$\mathcal{A}$ model" introduced in [27], and further studied in [28–30], includes as special cases the kinematic magnetic model ($\mathcal{A} = 1$), linearized NS equation ($\mathcal{A} = -1$) and the special model without

the stretching term ($\mathcal{A} = 0$). Thus, in particular, it allows one to control the nonlocal pressure contribution ($\sim (\mathcal{A} - 1)$) and to quantitatively study its effects on the anomalous scaling. The generalized model also naturally arises within the multiscale technique, as a result of the vertex renormalization [1].

Other important issues are the mixing of composite operators, responsible for the anomalous scaling, and the effects of pressure on the inertial-range behavior, especially in anisotropic sectors. In the scalar case, the anomalous exponents for all structure functions are given by a single expression which includes n, the order of the function, as a parameter [12]. This remains true for the general vector model with the stretching term ($\mathcal{A} \neq 0$), including the magnetic case [31–35] with $\mathcal{A} = 1$.

In the special vector model without the stretching term ($\mathcal{A} = 0$), introduced and studied in [36–42], the number and the form of the operators entering into the relevant family depend essentially on n, and different structure functions should be studied separately. As a result, no general expression valid for all n exists in the model, and the anomalous exponents are related by finite families of composite operators rather than by individual operators. In this respect, such models are closer to the nonlinear NS equation, where the inertial-range behavior of structure functions is believed to be related with the Galilean-invariant operators, which form infinite families that mix heavily in renormalization.

It was argued that similar mechanism is responsible for the origin of anomalous scaling in the real fluid turbulence, see, e.g. [17–19].

Numerous studies were devoted to the joint effects of compressibility and intermittency on the anomalous scaling. Analysis of various simplified models suggests that compressibility strongly affects the passively advected fields. In particular, the anomalous exponents become nonuniversal due to dependence on the compressibility parameters, such that the anomalous scaling is enhanced.

From the RG viewpoints, the key problem is the renormalizability (in plain words, the self-consistency) of the model in question. The straightforward RG approach by [43] suffered by the lack of renormalizability. Another approach, proposed in [44], with the price of certain reasonable approximations and natural extensions of the original set-up, lead to an internally consistent renormalizable model in terms of an effective density field. The model [44] exhibits an unique IR fixed point of the RG equation,

which allows for detailed one-loop analysis of the IR anomalous scaling behaviour within the RG+SDE approach for the scalar [45] and magnetic [46] fields[2].

In this paper, we study the model of a transverse vector field (e.g. magnetic) with the most general form of the nonlinearity, known as the $\mathcal{A}$ model, passively advected by a strongly compressible turbulent flow. The latter is governed by the randomly stirred NS equation and treated according to the approach proposed [44] and developed in [45, 46]. Thus, the special case $\mathcal{A} = 1$ corresponds to the kinematic magnetic model [46].

The full stochastic problem is equivalent to a renormalizable field theoretic model with an infrared attractive fixed point. Thus, the scaling behaviour for the large-scale, long-distance behaviour is established. However, the question whether the parameter $\mathcal{A}$ tends to a certain fixed-point value or remains free, cannot be answered within our one-loop approximation.

We shall not describe the contents of this paper here, because it is clear from the following text.

§2. DESCRIPTION OF THE MODEL

Following [44–46], we describe the stochastic dynamics of a compressible fluid by the set of the equations:

$$\nabla_t v_i = \nu_0[\delta_{ik}\partial^2 - \partial_i\partial_k]v_k + \mu_0\partial_i\partial_k v_k - \partial_i\phi + f_i \tag{1}$$

$$\nabla_t \phi = -c_0^2\partial_i v_i, \tag{2}$$

which are derived from the momentum balance equation and the continuity equation [51] with the two assumptions: the kinematic viscosity coefficients ν_0 and μ_0 are assumed to be constant, that is, independent of $x = \{t, \mathbf{x}\}$, and the equation of state is taken in the simplest form of the linear relation $(p - \bar{p}) = c_0^2(\rho - \bar{\rho})$ between the deviations of the pressure $p(x)$ and the density $\rho(x)$ from their mean values; then the constant c_0 has the meaning of the (adiabatic) speed of sound. Here and below, we use the subscript "0" to denote the bare (unrenormalized) parameters to distinguish them from their renormalized (ultraviolet finite) analogs.

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[2]The crossover between the turbulent advection and "shaking" requires special analysis in the vicinity of $d = 4$; see [48–50]. Another special value is $d = 2$, where the NS turbulence becomes close to the equilibrium distribution. The vicinity of $d = 2$ requires additional analysis even for the incompressible case; see, e.g. [18]. In the following, we assume $d > 2$.

As the basic (primary) fields we use the velocity field $\boldsymbol{v} = \{v_i(x)\}$ and, instead of the density, the scalar field defined as $\phi(x) = c_0^2 \ln(\rho(x)/\bar{\rho})$. Furthermore,

$$\nabla_t = \partial_t + v_k \partial_k \tag{3}$$

is the Lagrangian (Galilean covariant) derivative, $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x_i$, and $\partial^2 = \partial_i \partial_i$ is the Laplace operator. The problem is studied in the d-dimensional (for generality) space $\mathbf{x} = \{x_i\}$, $i = 1 \ldots d$, and the summations over the repeated Latin indices are always implied.

In the NS equation (1), f_i is the density of the external force, which models the energy input into the system from the large-scale stirring. In order to apply the standard RG to the problem, and to ensure the Galilean symmetry, the force is taken to be Gaussian with zero mean, not correlated in time, with the given covariance

$$\langle f_i(x) f_j(x') \rangle = \delta(t - t') \int\limits_{k > m} \frac{d\mathbf{k}}{(2\pi)^d} \, D_{ij}^f(\mathbf{k}) \exp\{i\mathbf{k} \cdot \mathbf{x}\},$$

$$\tag{4}$$

where

$$D_{ij}^f(\mathbf{k}) = D_0 \, k^{4-d-y} \left\{ P_{ij}^\perp(\mathbf{k}) + \alpha P_{ij}^\parallel(\mathbf{k}) \right\}. \tag{5}$$

Here $P_{ij}^\perp(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ and $P_{ij}^\parallel(\mathbf{k}) = k_i k_j / k^2$ are the transverse and the longitudinal projectors, respectively, $k = |\mathbf{k}|$ is the wave number (momentum), D_0 and α are positive amplitudes. The parameter $g_0 = D_0/\nu_0^3$ plays the part of the coupling constant (expansion parameter in the perturbation theory); the relation $g_0 \sim \Lambda^y$ defines the typical ultraviolet (UV) momentum scale. The parameter $m \sim L^{-1}$, reciprocal of the integral turbulence scale, provides IR regularization; its precise form is unessential and the sharp cut-off is merely the simplest choice for the calculation reasons.

The exponent $0 < y \leqslant 4$ plays the role analogous to $\varepsilon = 4 - d$ in the RG theory of critical state [3]: it provides UV regularization (so that the ultraviolet divergences have the form of the poles in y) and various scaling dimensions are calculated as series in y. The most realistic (physical) value is given by the limit $y \to 4$: then the function (5) can be viewed as a power-like representation of the function $\delta(\mathbf{k})$ and it corresponds to the idealized picture of the energy input from infinitely large scales.

As already mentioned, more detailed justification and discussion of the compressible model (1)–(5) is given in [44–46].

In this paper, we confine ourselves with the case of a transverse (divergence-free) passive vector $\theta_i(x)$ field. Then the general advection-diffusion equation acquires the form

$$\nabla_t\theta_i - \mathcal{A}_0(\theta_k\partial_k)v_i + \partial_i\mathcal{P} = \kappa_0\partial^2\theta_i + \eta_i, \qquad \nabla_t \equiv \partial_t + (v_k\partial_k), \qquad (6)$$

where ∇_t is the Lagrangian (Galilean covariant) derivative 3, $\mathcal{P}(x)$ is the analog of the pressure $p(x)$, κ_0 is the diffusivity, ∂^2 is the Laplace operator and $\eta_i(x)$ is a transverse Gaussian stirring force with zero mean and covariance

$$\langle\eta_i(x)\eta_k(x')\rangle = \delta(t - t')\, C_{ik}(r/L). \qquad (7)$$

The parameter L is an integral scale related to the stirring, and C_{ik} is a dimensionless function with the condition $\partial_i C_{ik} = 0$, finite at $r = 0$ and rapidly decaying for $r \to \infty$; its precise form is unessential.

From the physics viewpoints most interesting is the special case $\mathcal{A}_0 = 1$, where the pressure term disappears: it corresponds to magnetohydrodynamic (MHD) turbulence. It was studied earlier within the context of anomalous scaling in numerous papers; see e.g. [25, 26], [31]- [35] and references therein.

According to the general theorem (see e.g. [3]), the full-scale stochastic problem (1), (2), (4), (6), (7), is equivalent to the field theoretic model of the doubled set of fields $\Phi = \{v, v', \phi', \phi', \theta, \theta'\}$ with the action functional

$$\mathcal{S}(\Phi) = \mathcal{S}_v(v', v) + \theta' D_\theta\theta'/2 + \theta'\left\{-\nabla_t - \mathcal{A}_0(\theta_k\partial_k)v_i + \kappa_0\partial^2\right\}\theta, \qquad (8)$$

where

$$\mathcal{S}_v(v', v, \phi', \phi) = \frac{1}{2}v_i' D_{ik}^f v_k' + v_i'\left\{-\nabla_t v_i + \nu_0\left[\delta_{ik}\partial^2 - \partial_i\partial_k\right]v_k - \partial_i\phi\right\}$$
$$+ \phi'\left[-\nabla_t\phi + v_0\nu_0\partial^2\phi - c_0^2(\partial_i v_i)\right]. \qquad (9)$$

The field theoretic formulation means that various correlation functions and response (Green) functions of the original stochastic problem are represented by functional averages over the full set of fields with weight $\exp\mathcal{S}(\Phi)$, and in this sense they can be viewed as the Green functions of the field theoretic model with action (8). The model corresponds to standard Feynman diagrammatic techniques with two vertices $-v'(v\partial)v$ and $-\phi'(v\partial)\phi$ and the free (bare) propagators, determined by the quadratic part of the action functional; in the frequency–momentum (ω–$\mathbf{k}$)

representation, they have the forms:

$$\langle vv'\rangle_0 = \langle v'v\rangle_0^* = P^\perp \epsilon_1^{-1} + P^\parallel \epsilon_3 R^{-1},$$

$$\langle vv\rangle_0 = P^\perp \frac{d^f}{|\epsilon_1|^2} + P^\parallel \alpha d^f \left|\frac{\epsilon_3}{R}\right|^2,$$

$$\langle \phi v'\rangle_0 = \langle v'\phi\rangle_0^* = -\frac{\mathrm{i}c_0^2 \mathbf{k}}{R}, \quad \langle v\phi'\rangle_0 = \langle \phi'v\rangle_0^* = \frac{\mathrm{i}\mathbf{k}}{R},$$

$$\langle \phi\phi'\rangle_0 = \langle \phi'\phi\rangle_0^* = \frac{\epsilon_2}{R}, \quad \langle \phi\phi\rangle_0 = \frac{\alpha c_0^4 k^2 d^f}{|R|^2},$$

$$\langle v\phi\rangle_0 = \langle \phi v\rangle_0^* = \frac{\mathrm{i}\alpha c_0^2 d^f \epsilon_3 \mathbf{k}}{|R|^2},$$

$$\langle \phi'\phi'\rangle_0 = \langle v'\phi'\rangle_0 = \langle v'v'\rangle_0 = 0,$$

$$\langle \theta_i'\theta_j\rangle_0 = \frac{P_{ij}^\perp(\mathbf{k})}{-\mathrm{i}\omega + w_0\nu_0 k^2}, \tag{10}$$

and the new vertex $V_{ijl}\theta_i'\theta_j v_l$ with the vertex factor

$$V_{ijl}(\mathbf{k}) = \mathrm{i}(\delta_{ij}k_l - \mathcal{A}_0\delta_{il}k_j), \tag{11}$$

where we have denoted

$$\epsilon_1 = -\mathrm{i}\omega + \nu_0 k^2, \quad \epsilon_2 = -\mathrm{i}\omega + u_0\nu_0 k^2,$$

$$\epsilon_3 = -\mathrm{i}\omega + v_0\nu_0 k^2, \quad R = \epsilon_2\epsilon_3 + c_0^2 k^2,$$

$$d^f = g_0\nu_0^3 k^{4-d-y} \tag{12}$$

and omitted the vector indices of the fields and the projectors.

In the limit $u_0 \to \infty$, the propagators $\langle vv'\rangle_0$ and $\langle vv\rangle_0$ become purely transverse, and we arrive at the case of incompressible fluid. Furthermore, the counterterms are polynomial in c_0 and can be calculated with $c_0 = 0$ in R which essentially simplifies the calculations; for more details, see [44–46].

§3. RENORMALIZATION AND COUNTERTERMS

The extended model (8) appears to be multiplicatively renormalizable, (see [44–46]). This means that all the UV divergences can be removed from the Green functions by the renormalization of the fields $\phi \to Z_\phi \phi$, $\phi' \to Z_{\phi'}\phi'$ and of the parameters:

$$g_0 = g\mu^y Z_g, \quad \nu_0 Z_\nu, \quad c_0 = cZ_c, \tag{13}$$

and so on. Here the renormalization constants Z_i absorb all the UV divergences, so that the Green functions are UV finite (that is, finite at $y = 0$) when expressed in terms of the renormalized parameters g, u, and so on; the reference scale (or the "renormalization mass") μ is an additional free parameter of the renormalized theory. No renormalization of the fields $\boldsymbol{v}', \boldsymbol{v}$ and of the parameters m, α is needed.

The renormalized analog of the action functional (8) has the form

$$\mathcal{S}^R(\Phi) = \mathcal{S}_v^R(\Phi) + \mathcal{S}_\theta^R(\Phi), \tag{14}$$

where $\mathcal{S}^R(\Phi)$ is the renormalized analog of the action $\mathcal{S}(\Phi)$, given in [44–46],

$$\mathcal{S}_\theta^R = \theta_i'\big\{-\partial_t\theta_i - \partial_k(v_k\theta_i - Z_A\mathcal{A}\theta_k v_i) + \kappa Z_\kappa \partial^2\theta_i\big\},$$

$$\begin{aligned} \mathcal{S}^R(\Phi) = {} & \frac{1}{2}v_i' D_{ik}^f v_k' \\ & + v_i'\big\{-\nabla_t v_i + Z_1\nu[\delta_{ik}\partial^2 - \partial_i\partial_k]v_k + Z_2 u\nu\partial_i\partial_k v_k - Z_4\partial_i\phi\big\} \\ & + \phi'\left[-\nabla_t\phi + Z_3 v\nu\partial^2\phi - Z_5 c^2(\partial_i v_i)\right]. \end{aligned} \tag{15}$$

The renormalized action (15) is obtained from the original one (8) by the renormalization of the fields $\phi \to Z_\phi\phi$, $\phi' \to Z_{\phi'}\phi'$ and the parameters

$$\begin{aligned} g_0 &= g\mu^y Z_g, \quad \nu_0 = \nu Z_\nu, \quad u_0 = u Z_u, \\ v_0 &= v Z_v, \quad c_0 = c Z_c, \\ \mathcal{A}_0 &= Z_A\mathcal{A}, \quad \kappa_0 = Z_\kappa\kappa. \end{aligned} \tag{16}$$

The renormalization constants in (15) and (16) are related as

$$\begin{aligned} Z_\nu &= Z_1, \quad Z_u = Z_2 Z_1^{-1}, \\ Z_v &= Z_3 Z_1^{-1}, \quad Z_\phi = Z_{\phi'}^{-1} = Z_4, \\ Z_c &= (Z_4 Z_5)^{1/2}, \quad Z_g = Z_\nu^{-3}. \end{aligned} \tag{17}$$

In this paper we present results of the one-loop calculation of the renormalization constants Z_κ and $Z_\mathcal{A}$.

§4. One-loop calculation and Galilean symmetry

4.1. Triangle diagrams. In order to calculate the renormalization constant $Z_\mathcal{A}$ we employed the MS scheme to eliminate the divergent parts for all the Feynman diagrams which correspond to the Green function $\langle\theta'\theta v\rangle$. In this manuscript we confined to the first order (one-loop approximation). For the readers who are interested in the calculations details we present

the one-loop answers for UV divergent parts (that are, poles in y) for the three diagrams:

$$= \frac{i\alpha}{2ud(u+w)^2}(p_3\delta_{12} - \mathcal{A}p_2\delta_{13})$$
$$+ i(\mathcal{A}-1)\frac{p_2\delta_{13} + p_3\delta_{12}}{d(d+2)}\Big(\frac{-\mathcal{A}}{2\nu^3(w+1)^2} + \frac{\alpha}{2u(u+w)^2}\Big), \quad (18)$$

$$= \frac{-i\mathcal{A}(1-\mathcal{A})(w+3)}{4\nu^3(w+1)^2}\frac{p_2\delta_{13} + p_3\delta_{12}}{d(d+2)}$$
$$+ \frac{-i\alpha(3u+w)}{4u^2(w+u)^2}\Big[p_3\delta_{12}\frac{\mathcal{A}+d+1}{d(d+2)} - p_2\delta_{13}\frac{\mathcal{A}d+\mathcal{A}+1}{d(d+2)}\Big], \quad (19)$$

$$= \frac{i\alpha}{4u^2\nu^3(u+w)}\Big[\frac{-p_2\delta_{13} + (d+1)p_3\delta_{12}}{d(d+2)} - \frac{(d+1)p_2\delta_{13} - p_3\delta_{12}}{d(d+2)}\Big]$$
$$+ \frac{i\mathcal{A}(\mathcal{A}-1)}{4\nu^3(w+1)}\frac{p_2\delta_{13} + p_3\delta_{12}}{d(d+2)}, \quad (20)$$

with the common overall factor $g\mu^y$.

It is easy to verify that the sum of the above expressions vanishes, that is, the sum of the one-loop diagrams is UV finite.

In the rapid-change version of our model, $Z_{\mathcal{A}} = 1$ and $\gamma_{\mathcal{A}} = 0$ identically because all nontrivial Feynman diagrams of the 1-irreducible Green function $\langle\theta'v\theta\rangle$ contain closed circuits of retarded propagators and therefore vanish; see, e.g., the discussion in [28].

In more realistic cases, the absence of the $O(g)$ term in $Z_{\mathcal{A}}$ and $\gamma_{\mathcal{A}}$ is a result of the cancellation of the (nontrivial) contributions from the three one-loop diagrams in the 1-irreducible Green function $\langle\theta'v\theta\rangle$. For

the counterterm $\theta'(v\partial)\theta$ such a cancellation is guaranteed by the Galilean symmetry (to all orders in g).

For the magnetic case $\mathcal{A} = 1$ we have $Z_1 = Z_2 = 1$ and, therefore, $Z_{\mathcal{A}} = 1$, to all orders in g. There, this is a consequence of the transversality relation $\partial_i V_i = 0$ for the vertex $V_i = \partial_k(v_i\theta_k - v_k\theta_i)$.

For general $\mathcal{A} \neq 1$ and incompressible fluid, the cancellation of the terms $\theta'(\theta\partial)v$ looks accidental and could be explained *a posteriori* by a rather simple form of the one-loop diagrams: the structures corresponding to the counterterm $\theta'(v\partial)\theta$ cancel each other due to the Galilean symmetry, while the structures corresponding to $\theta'(\theta\partial)v$ enter all the one-loop diagrams with the same coefficients and cancel out into the bargain [30]. However, this explanation can hardly pass beyond the one-loop approximation even for the incompressible fluid; thus nontrivial contributions of the order g^2 and higher in $Z_{\mathcal{A}}$ and $\gamma_{\mathcal{A}}$ are not forbidden.

Furthermore, in our compressible case, the expressions for the one-loop triangle diagrams become much more complicated, and their accidental cancellation raises serious doubts. However, a possible deep reason behind this effect remains uncovered.

4.2. Self-energy Σ. The constant Z_κ is found from the requirement for the 1-irreducible Green function $\langle\theta'\theta\rangle_{1-\text{ir}}$ be UV finite (that is, finite at $y \to 0$) when expressed in renormalized parameters. In the frequency–momentum representation it has the form:

$$\langle\theta'_1\theta_2\rangle_{1-\text{ir}}(\Omega, \mathbf{p}) = \left\{-\kappa_0 p^2 + \mathrm{i}\Omega\right\} P_{12}^{\perp}(\mathbf{p}) + \Sigma_{12}(\Omega, \mathbf{p}), \qquad (21)$$

where Σ_{12} is the "self-energy operator" given by infinite sum of 1-irreducible Feynman diagrams and $p = |\mathbf{p}|$. Because of the large number of tensor indices involved in our expressions, we use numbers (instead of latin letters) to denote them, with the standard convention on the summation over repeated indices.

The only one-loop self-energy diagram looks as follows:

$$\Sigma_{12} = \quad\text{}\quad, \qquad (22)$$

were the wavy line denotes the propagator $\langle vv\rangle$, the straight line crossed out at the right end denotes the propagator $\langle\theta\theta'\rangle$, and the vorticies composed of one wavy tip and two straight tips with one crossing correspond to the Green function $V_{123} = \langle v_1\theta_2\theta'_3\rangle$.

Here we do not present the details of the calculations and give only the result:

$$\Sigma_{12} = -\nu p^2 P_{12}(\mathbf{p}) \frac{\hat{g}}{2dy} \left(\frac{\mu}{m}\right)^y$$

$$\times \left[\frac{1}{1+w} \left\{ d-1 + (\mathcal{A}-1) \left[\frac{2w\mathcal{A}}{(1+w)(d+2)} + \frac{\mathcal{A}+d+1}{d+2} \right] \right\} \right.$$

$$\left. + \frac{\alpha}{u+w} \left\{ \frac{u-w}{u+w} + (\mathcal{A}-1) \left[\frac{-2w\mathcal{A}}{(u+w)(d+2)} - \frac{\mathcal{A}+\mathcal{A}d+1}{d+2} \right] \right\} \right]. \quad (23)$$

Here we passed to the new coupling constant

$$\hat{g} = gS_d/(2\pi)^d, \qquad S_d = 2\pi^{d/2}/\Gamma(d/2) \quad (24)$$

where S_d is the surface area of the unit sphere in d-dimensional space.

Then in the MS scheme the renormalization constant Z_κ that cancels the pole in the renormalized analog of the function (21) (that is, with the replacement $\kappa_0 \to \kappa Z_\kappa$ in the bare term) has the form:

$$Z_\kappa = 1 - \frac{g}{2dyw} \left[\frac{1}{1+w} \left\{ d-1 + (\mathcal{A}-1) \left[\frac{2w\mathcal{A}}{(1+w)(d+2)} + \frac{\mathcal{A}+d+1}{d+2} \right] \right\} \right.$$

$$\left. + \frac{\alpha}{u+w} \left\{ \frac{u-w}{u+w} + (\mathcal{A}-1) \left[\frac{-2w\mathcal{A}}{(u+w)(d+2)} - \frac{\mathcal{A}+\mathcal{A}d+1}{d+2} \right] \right\} \right], \quad (25)$$

while the corresponding anomalous dimension is

$$\gamma_\kappa = \frac{g}{2dw} \left[\frac{1}{1+w} \left\{ d-1 + (\mathcal{A}-1) \left[\frac{2w\mathcal{A}}{(1+w)(d+2)} + \frac{\mathcal{A}+d+1}{d+2} \right] \right\} \right.$$

$$\left. + \frac{\alpha}{u+w} \left\{ \frac{u-w}{u+w} + (\mathcal{A}-1) \left[\frac{-2w\mathcal{A}}{(u+w)(d+2)} - \frac{\mathcal{A}+\mathcal{A}d+1}{d+2} \right] \right\} \right] \quad (26)$$

with the corrections of the order $\hat{g}^2$ and higher.

§5. RG EQUATIONS, β FUNCTIONS AND FIXED POINTS

Multiplicative renormalizability of the field theoretic model allows one to derive, in a standard way, differential RG equations for the renormalized Green functions

$$G(e, \mu, \dots) = \langle \Phi \dots \Phi \rangle_R.$$

Here $e = \{g, \nu, u, v, w, c, m, \alpha\}$ is the full set of renormalized parameters, μ is the reference momentum scale and the ellipsis stands for the other arguments (times or frequencies and coordinates or momenta). For convenience, we introduced here three dimensionless ratios: $u_0 = \mu_0/\nu_0$ and

$v_0 = \chi_0/\nu_0$ are related to the viscosity and diffusivity coefficients of the (properly extended) model (1), (2), while $w_0 = \kappa_0/\nu_0$ is related to the magnetic diffusivity coefficient; u, v, w are their renormalized analogs.

The RG equation expresses the invariance of the renormalized Green function with respect to changing of the reference scale μ, when the bare parameters e_0 are kept fixed:

$$\left\{ \widetilde{\mathcal{D}}_\mu + \sum_\Phi N_\Phi \gamma_\Phi \right\} G(e, \mu, \dots) = 0. \tag{27}$$

Here and below we denote $\mathcal{D}_x \equiv x\partial_x$ for any variable x and $\widetilde{\mathcal{D}}_\mu$ is the operation $\mathcal{D}_\mu \equiv \mu\partial_\mu$ at fixed e_0. In terms of the renormalized variables, it takes the form

$$\widetilde{\mathcal{D}}_\mu = \mathcal{D}_\mu + \beta_g \partial_g + \beta_u \partial_u + \beta_v \partial_v + \beta_w \partial_w - \gamma_\nu \mathcal{D}_\nu - \gamma_c \mathcal{D}_c. \tag{28}$$

The anomalous dimension γ_F of a certain quantity F (a field or a parameter) is defined by the relation

$$\gamma_F = \widetilde{\mathcal{D}}_\mu \ln Z_F, \tag{29}$$

and the β functions for the dimensionless parameters ("coupling constants") are

$$\begin{aligned} \beta_g &= \widetilde{\mathcal{D}}_\mu g = g\left[-y - \gamma_g\right], \\ \beta_u &= \widetilde{\mathcal{D}}_\mu u = -u\gamma_u, \end{aligned} \tag{30}$$

and similarly for β_v, β_w. Here the second equalities result from the definitions and the relations of the type (13).

Note that from the definition of w_0 it follows that $Z_\kappa = Z_\nu Z_w$, so that $\beta_w = w[\gamma_\nu - \gamma_w]$, and

$$\begin{aligned} \beta_w = wy\Big[\frac{1}{3} &- \frac{1}{w(w+1)}\Big(\frac{2}{3} + \frac{\alpha(1-w)}{3(1+w)} \\ &+ (\mathcal{A}-1)(1-\alpha)\frac{2w\mathcal{A} + (w+1)(\mathcal{A}+4)}{15(w+1)} - \frac{\alpha}{5}(\mathcal{A}-1)^2\Big)\Big]. \end{aligned} \tag{31}$$

For better transparency, here and below we put $d = 3$, unless stated otherwise.

The possible types of IR asymptotic behavior are associated with IR attractive fixed points of the RG equations. The coordinates $g_* = \{g_{i*}\}$ of the fixed points are found from the equations

$$\beta_i(g_*) = 0, \tag{32}$$

where $g = \{g_i\}$ is the full set of coupling constants and $\beta_i = \widetilde{\mathcal{D}}_\mu g_i$ are their β functions. The character of a fixed point is determined by the matrix

$$\Omega_{ij} = \partial\beta_i/\partial g_j|_{g=g_*}. \tag{33}$$

For the IR fixed points the matrix Ω is positive (that is, positive are the real parts of all its eigenvalues).

The analysis performed in [44] (see also [45, 46]) on the base of the leading-order (one-loop) approximation has shown that the RG equations of the model $\mathcal{S}_v$, corresponding to the stochastic NS problem (1), (2), possess the only IR attractive fixed point in the physical region of parameters $(g, u, v > 0)$[3]:

$$\hat{g}_* = \frac{4dy}{3(d-1)} + O(y^2) = 2y + O(y^2), \quad u_* = 1 + O(y), \quad v_* = 1 + O(y). \tag{34}$$

From a certain exact relation between the renormalization constants [44], the exact result

$$\gamma_\nu^* = y/3 \tag{35}$$

follows (no corrections of the order y^2 and higher). Here and below, γ_i^* denotes the value of the anomalous dimension γ_i at the fixed point.

Now we substitute the one-loop expressions and the exact result (35) into Eq. (31). Then the equation $\beta_w = 0$ yields the equation

$$\frac{w}{3} - \frac{1}{(w+1)}\left(\frac{2}{3} + \frac{\alpha(1-w)}{3(1+w)}\right.$$
$$\left. + (\mathcal{A}-1)(1-\alpha)\frac{2w\mathcal{A} + (w+1)(\mathcal{A}+4)}{15(w+1)} - \frac{\alpha}{5}(\mathcal{A}-1)^2\right) = 0. \tag{36}$$

To find the coordinates of the fixed points one has to find the roots of the cubic in-w equation (36), the problem solved by the Cardano formula. In general case, the equation (36) has at least one real root, the other two can be real or complex depending on the model parameters. Furthermore, we are interested in IR attractive fixed points. The functions $\beta_{g,u,v}$ do not depend on w, so that the new eigenvalue of the matrix (33) coincides with the diagonal element $\partial\beta_w/\partial w|_{g=g_*}$.

Thus, the full description of the pattern of the fixed points (and their stability regions) in the full space of model parameters appears rather

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[3]It was argued in [52] that these first-order fixed-point coordinates [44] do not satisfy certain thermodynamic stability inequalities [51]. We avoid discussion of this interesting issue here and only refer to the two-loop results [53] which do not contradict the stability.

complicated. Below we will briefly touch only a few most interesting representative special cases.

- $\mathcal{A} = 1$: The only real root of the equation (36) is $w_* = 1$. It turns out that the point is IR-attractive ($\partial\beta_w/\partial w|_{g=g_*} > 0$). This case corresponds to the magnetic field passively advected by the compressible fluid (kinematic MHD approximation). This model was earlier considered in [46]; the are in agreement.

- $\mathcal{A} = 0$: The new symmetry appears: the model is invariant with respect to the shift $\theta \to \theta ++$ const. For $\alpha > 0$ there is always an IR attractive fixed point, with the coordinate depending on α. Mention an interesting special case within this special case: when $\alpha = 4$ and $\mathcal{A} = 0$, one obtains $w_* = 1$.

  It also turns out, that for some negative values of α (for example, $-1.1 < \alpha < -1.05$) there are two real fixed points, but only one of them is IR attractive. Of course, negative values of α require inventive physical interpretation.

- $\mathcal{A} = -1$: For $0 < \alpha < 4$ there is always an IR attractive fixed point, whose coordinates depend on α. This case can be interpreted as a linearized equation for the perturbation θ around the background velocity field v; see, e.g. discussion in [28].

Existence of an IR attractive fixed point in the physical region of the parameters implies existence of scaling behavior in the IR range. The critical dimension of some quantity F (a field or a parameter) is given by the relation (see [3, 17, 18])

$$\Delta_F = d_F^k + \Delta_\omega d_F^\omega + \gamma_F^*, \quad \Delta_\omega = 2 - \gamma_\nu^* = 2 - y/3. \tag{37}$$

Here d_F^k and d_F^ω are the canonical dimensions of F, γ_F^* is the value of the anomalous dimension γ_F at the fixed point, and Δ_ω is the critical dimension of the frequency.

The critical dimensions of the fields and parameters of the model described by the action $\mathcal{S}_v$ from Eq. (9) are presented in [44]; see also [45, 46]:

$$\Delta_v = 1 - y/3, \quad \Delta_{v'} = d - \Delta_v, \quad \Delta_\omega = 2 - y/3, \quad \Delta_m = 1 \tag{38}$$

(these results are exact due to $\gamma_\nu^* = y/3$ and $\gamma_{v,v',m}^* = 0$) and

$$\Delta_\phi = d - \Delta_{\phi'} = 2 - 5y/6 + O(y^2), \quad \Delta_c = 1 - 5y/12 + O(y^2). \tag{39}$$

In addition, our full model involves two more critical dimensions, whose meanings can be found in [47]:

$$\Delta_\theta = -1 + y/6, \quad \Delta_{\theta'} = d + 1 - y/6. \tag{40}$$

These expressions are exact for general d because the fields θ and θ' are not renormalized.

§6. Conclusion and open problems

We have studied a stochastic model of a transverse (divergence-free, e.g., magnetic) vector field θ, passively advected by a random non-Gaussian velocity field with finite correlation time, governed by the stochastic NS equations for a strongly compressible fluid. The model is described by an advection-diffusion equation with a random large-scale stirring force, non-local pressure term and the most general form of the inertial nonlinearity, "controlled" by the parameter $\mathcal{A}$. The NS model is treated according to the approach advocated in [44, 46, 47]. Within this approach, an extended full-scale model appears renormalizable, and the RG techniques can be applied to its IR behaviour.

The full model reveals a rather complicated manifold of the possible types of asymptotic behaviour and crossover regimes, governed by various fixed points of the RG equations. Existence, among them, of IR attractive points shows that the correlation functions of the model fields may exhibit scaling behaviour for certain areas of the parameters $d, y, \alpha, \mathcal{A}$.

The corresponding scaling dimensions for the basic fields and parameters are given by the one-loop order exactly (that is, to all orders in the y expansion).

The open question is whether the amplitude $\mathcal{A}$ in front of the "stretching term" $(\theta \partial) v$ in the advection-diffusion equation tends to some fixed-point value, or it survives as a free parameter which the anomalous dimensions depend upon. The solution remains elusive: in the one-loop approximation, the practical calculation shows that the fixed-point values appear arbitrary, as the consequence of the relation $Z_A = 1$. At the same time, any kind of identities, expressing the Galilean symmetry, impose no restriction on the exact fixed-point value of Z_A. Hopefully, the practical two-loop calculation could resolve this dilemma.

As the next steps, the two-loop renormalization of the models in question should be undertaken. Another following problem is the inclusion of

the relevant composite operators, responsible for the anomalous multiscaling, The work is already in progress.

Acknowledgments

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