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STABILITY OF THE ROTATION OF A TWO-PHASE DROP WITH SELF-GRAVITY

Abstract. A uniformly rotating finite mass consisting of two immiscible viscous incompressible self-gravitating fluids is governed by interface problem for Navier–Stokes system with mass forces in the right-hand side. Surface tension acts on the interface as well as on the exterior free boundary. The proof of stability is based on the analysis of an evolutionary problem for small perturbations of the equilibrium state of a rotating two-phase fluid with self-gravity. It is proved that under sufficient smallness of initial data, exponentially decreasing mass forces and angular velocity, as well as the positivity of the second variation of energy functional, the perturbation of the axisymmetric equilibrium figure exponentially tends to zero as $t \to \infty$, the motion of the drop going over to the rotation of liquid mass as a solid.

Dedicated to G. A. Seregin on the occasion of his 70th birthday

§1. Introduction

The problem on an isolated liquid mass rotating about a fixed axis as a rigid body was treated by many outstanding mathematicians most of which considered self-gravitating rotating fluids but without surface tension [1].

Mathematical treatment of the problem with capillarity effect was carried out by Charrueau [2, 3] in the begining of the 20th century. He gave a detailed analysis of the problem and considered some stability aspects. These results were presented in the book of Appell [4]. There one can find reasoning about the dominant effect and calculations of the sizes of rotating liquid masses which are affected by both self-gravity and capillarity.

The stability of equilibrium figures is one of the most important their characteristics. A. M. Lyapunov [1, 5] was the first who used analytical

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methods for studying the stability and instability of the forms of a rotating fluid mass. He analyzed the second variation of energy functional with respect to small perturbations of figure boundary. The positivity of this variation guarantees the stability of the system because the energy has a minimum at this state. One can find a review of Lyapunov's works on stability theory of equilibrium figures of celestial bodies in [6].

The Lyapunov method was developed for the case of a rotating capillary fluid by means of an analysis of the corresponding evolutionary free boundary problem in [7, 8]. We extend the above technique to the case of a finite mass of two immiscible liquids and treat stability problem for two rotating incompressible capillary self-gravitating fluids separated by an unknown interface close to the boundary of an equilibrium figure. The existence of equilibrium figures for a two-phase liquid was obtained in [9]. We adapt the proof of the global maximal regularity of two-fluid problem without rotation to our case [10–12]. One of the main steps of the consideration is global solvability of a linear problem. It is based on a priori exponential inequality for a generalized energy. The idea of constructing such an auxiliary energy potential was first given in [13]. The case of rotating two-phase drop without self-gravity and mass forces was studied by the authors in [14].

Let two viscous incompressible immiscible fluids of densities $\rho^{\pm}$ and viscosities $\mu^{\pm}$ be contained in a domain $\Omega_t \subset \mathbb{R}^3$ bounded by the free surface Γ_t^- and separated by the variable interface Γ_t^+. It is assumed that Γ_t^+ is the boundary of the domain Ω_t^+ filled with a fluid of the density ρ^+ which is surrounded by another fluid of the density ρ^- occupying the domain $\Omega_t^- = \Omega_t \setminus \overline{\Omega_t^+}$. This two-phase drop rotates about the vertical axis x_3 (see Fig. 1). At the initial instant $t = 0$, the surfaces Γ_0^-, Γ_0^+ are given. It is necessary to find Γ_t^-, Γ_t^+, as well as velocity vector field $\boldsymbol{v}(x,t)$ and pressure function $p(x,t)$ satisfying the interface problem for the Navier–Stokes system

$$
\begin{aligned}
&\rho^{\pm}\big(\mathcal{D}_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\big) - \mu^{\pm}\nabla^2\boldsymbol{v} + \nabla p = \rho^{\pm}(\boldsymbol{f} + \varkappa\nabla U),\\
&\qquad\qquad \nabla\cdot\boldsymbol{v} = 0 \quad \text{in } \cup\,\Omega_t^{\pm} = \Omega_t^{+}\cup\Omega_t^{-}, \quad t>0,\\
&\qquad\qquad\quad \boldsymbol{v}(x,0) = \boldsymbol{v}_0(x)\,\text{in } \cup\,\Omega_0^{\pm},\\
&\mathbb{T}(\boldsymbol{v},p)\boldsymbol{n}\big|_{\Gamma_t^-} = \sigma^- H^-\boldsymbol{n} \quad \text{on } \Gamma_t^-,\\
&[\boldsymbol{v}]\big|_{\Gamma_t^+} \equiv \lim_{\substack{x\to x_0\in\Gamma_t^+,\\ x\in\Omega_t^+}} \boldsymbol{v}(x,t) - \lim_{\substack{x\to x_0\in\Gamma_t^+,\\ x\in\Omega_t^-}} \boldsymbol{v}(x,t) = 0,\\
&[\mathbb{T}(\boldsymbol{v},p)\boldsymbol{n}]\big|_{\Gamma_t^+} = \sigma^+ H^+\boldsymbol{n} \;\text{ on } \Gamma_t^+,\\
&V_{\boldsymbol{n}} = \boldsymbol{v}\cdot\boldsymbol{n} \;\text{ on } \Gamma_t = \Gamma_t^+\cup\Gamma_t^-,
\end{aligned}
\tag{1.1}
$$

where $\mathcal{D}_t = \partial/\partial t$, $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$, $\boldsymbol{f}$ is the vector of mass forces,

$$U(x,t) = \int\limits_{\Omega_t} \frac{\rho^{\pm}\,\mathrm{d}z}{|x-z|},$$

$\varkappa \geqslant 0$ is self-attraction coefficient, $\boldsymbol{v}_0$ is initial velocity distribution,

$$\mathbb{T}(\boldsymbol{v},p) = -p + \mu^{\pm}\mathbb{S}(\boldsymbol{v})$$

is stress tensor, $\mathbb{S}(\boldsymbol{v}) = (\nabla\boldsymbol{v}) + (\nabla\boldsymbol{v})^T$ is doubled rate-of-strain tensor, the superscript T denotes the transposition, $\rho^{\pm}, \mu^{\pm} > 0$ are the step-functions of density and dynamical viscosity equal to ρ^-, μ^- in Ω_t^- and ρ^+, μ^+ in Ω_t^+; H^-, H^+ are twice the mean curvatures of the surfaces Γ_t^-, Γ_t^+ ($H^+ < 0$ at the points where Γ_t^+ is convex toward Ω_t^-); $\sigma^-, \sigma^+ > 0$ are the coefficients of the surface tension on Γ_t^-, Γ_t^+, respectively; $\boldsymbol{n}(x,t)$ is the outward normal to Γ_t^- and Γ_t^+, $V_{\boldsymbol{n}}$ is the velocity of evolution of the surfaces Γ_t^- and Γ_t^+ in the direction of $\boldsymbol{n}$. We suppose that a Cartesian coordinate system $\{x\}$ is introduced in $\mathbb{R}^3$. The centered dot means the Cartesian scalar product.

The summation is implied over the repeated indices from 1 to 3 if they are denoted by Latin letters. We mark the vectors and the vector spaces by boldface letters.

We assume that the domains Ω_0^+, Ω_0 differ little from equilibrium figures $\mathcal{F}^+$ and $\mathcal{F}$ such that

$$|\Omega_0^+| = |\mathcal{F}^+|, \quad |\Omega_0| = |\mathcal{F}|. \tag{1.2}$$

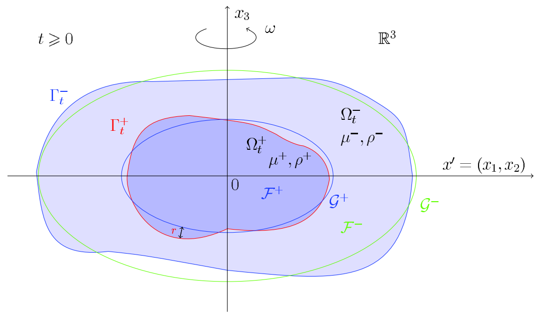

Figure 1

We denote $\mathcal{F}^- = \mathcal{F} \setminus \mathcal{F}^+$. Due to the incompressibility of the liquids, equalities (1.2) hold for any $t > 0$:

$$|\Omega_t^+| = |\mathcal{F}^+|, \quad |\Omega_t| = |\mathcal{F}|. \tag{1.3}$$

It implies the conservation of mass in view of constant densities of the fluids. In addition, we set $\mathcal{G}^+ = \partial\mathcal{F}^+$ and $\mathcal{G}^- = \partial\mathcal{F}$ (see Fig. 1).

If mass force $\boldsymbol{f}$ is orthogonal to all the vectors of rigid motion, i. e.,

$$\int_{\Omega_t} \rho^{\pm} \boldsymbol{f}(x,t)\,\mathrm{d}x = 0, \qquad \int_{\Omega_t} \rho^{\pm} \boldsymbol{f}(x,t) \cdot \boldsymbol{\eta}_i(x)\,\mathrm{d}x = 0, \quad i = 1,2,3, \tag{1.4}$$

a solution of problem (1.1) satisfies also the other conservation laws

$$\int\limits_{\Omega_t} \rho^{\pm} x_j\,\mathrm{d}x = \int\limits_{\Omega_0} \rho^{\pm} x_j\,\mathrm{d}x \equiv 0, \quad j = 1,2,3, \text{ (barycenter conservation)},$$

$$\int\limits_{\Omega_t} \rho^{\pm} \boldsymbol{v}(x,t)\,\mathrm{d}x = \int\limits_{\Omega_0} \rho^{\pm} \boldsymbol{v}_0(x)\,\mathrm{d}x \equiv 0 \quad \text{(momentum conservation)}, \tag{1.5}$$

$$\int\limits_{\Omega_t} \rho^{\pm} \boldsymbol{v}(x,t) \cdot \boldsymbol{\eta}_i(x)\,\mathrm{d}x = \int\limits_{\Omega_0} \rho^{\pm} \boldsymbol{v}_0(x) \cdot \boldsymbol{\eta}_i(x)\,\mathrm{d}x \equiv \omega \int\limits_{\mathcal{F}} \bar{\rho} \boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_i(x)\,\mathrm{d}x = \beta\delta_i^3,$$
$$\text{(angular momentum conservation)},$$

where $\boldsymbol{\eta}_i(x) = \boldsymbol{e}_i \times \boldsymbol{x}$, $i = 1,2,3$, $\bar{\rho}$ is the step-function of density equal to ρ^- in $\mathcal{F}^-$ and ρ^+ in $\mathcal{F}^+$, δ_i^k is the Kronecker delta; ω is the angular velocity of the rotation,

$$\beta = \omega \int\limits_{\mathcal{F}} \bar{\rho}(x)|x'|^2 \,\mathrm{d}x \equiv \omega \mathcal{I}$$

is the angular momentum of the rotating liquids. One can prove that (1.5) holds for all $t > 0$ if it is satisfied for $t = 0$ (see [9]) and conditions (1.4) hold for all nonnegative t.

If one introduces the new pressure function $p - \rho^{\pm}\varkappa U$, system (1.1) transforms into the problem

$$\begin{aligned} &\rho^{\pm}\big(\mathcal{D}_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\big) - \mu^{\pm}\nabla^2\boldsymbol{v} + \nabla p = \rho^{\pm}\boldsymbol{f}, \\ &\qquad\qquad \nabla\cdot\boldsymbol{v} = 0 \ \text{ in } \ \cup\Omega_t^{\pm},\ t>0, \\ &\qquad \boldsymbol{v}(x,0) = \boldsymbol{v}_0(x) \ \text{ in } \ \cup\Omega_0^{\pm} = \Omega_0^+ \cup \Omega_0^-, \end{aligned} \tag{1.6}$$

$$\begin{aligned} &\mathbb{T}(\boldsymbol{v},p)\boldsymbol{n}\big|_{\Gamma_t^-} = (\sigma^- H^- + \rho^-\varkappa U)\boldsymbol{n} \quad \text{on } \Gamma_t^-, \\ &[\boldsymbol{v}]\big|_{\Gamma_t^+} = 0, \qquad [\mathbb{T}(\boldsymbol{v},p)\boldsymbol{n}]\big|_{\Gamma_t^+} = (\sigma^+ H^+ + [\rho^{\pm}]\big|_{\Gamma_t^+}\varkappa U)\boldsymbol{n} \quad \text{on } \Gamma_t^+, \\ &V_{\boldsymbol{n}} = \boldsymbol{v}\cdot\boldsymbol{n} \quad \text{on } \Gamma_t = \Gamma_t^+ \cup \Gamma_t^-. \end{aligned} \tag{1.7}$$

Two-phase liquid mass uniformly rotating about the x_3-axis with constant angular velocity $\omega = \beta/I_0$ has velocity vector field

$$\boldsymbol{\mathcal{V}}(x) = \omega \boldsymbol{e}_3 \times \boldsymbol{x} \equiv \omega\boldsymbol{\eta}_3.$$

and pressure function

$$\mathcal{P}(x) = \bar{\rho}\frac{\omega^2}{2}|x'|^2 + p_0^{\pm},$$

where $|x'|^2 = x_1^2 + x_2^2$ and $\bar{\rho}$, $p_0^{\pm}$ are step-functions in $\mathcal{F}^{\pm}$. This motion is governed by the homogeneous steady Navier-Stokes equations

$$\bar{\rho}(\boldsymbol{\mathcal{V}}\cdot\nabla)\boldsymbol{\mathcal{V}} - \bar{\mu}\nabla^2\boldsymbol{\mathcal{V}} + \nabla\mathcal{P} = 0, \qquad \nabla\cdot\boldsymbol{\mathcal{V}} = 0 \quad \text{in } \mathcal{F} = \cup\mathcal{F}^{\pm}$$

with the step-function $\bar{\mu} \equiv \mu^+$ in $\mathcal{F}^+$ and $\bar{\mu} \equiv \mu^-$ in $\mathcal{F}^-$. If one substitutes $\boldsymbol{\mathcal{V}}, \mathcal{P}$ into boundary conditions (1.7), one obtains the equations for the surface $\mathcal{G}^-$ of the domain $\mathcal{F}$ and for the interface $\mathcal{G}^+$ between the fluids

$$\begin{aligned} &\sigma^-\mathcal{H}^-(x) + \rho^-\frac{\omega^2}{2}|x'|^2 + \rho^-\varkappa\mathcal{U} + p_0^- = 0, \quad x \in \mathcal{G}^-, \\ &\sigma^+\mathcal{H}^+(x) + [\bar{\rho}]\big|_{\mathcal{G}^+}\frac{\omega^2}{2}|x'|^2 + [\bar{\rho}]\big|_{\mathcal{G}^+}\varkappa\mathcal{U} + [p_0^{\pm}]\big|_{\mathcal{G}^+} = 0, \quad x \in \mathcal{G}^+, \end{aligned} \tag{1.8}$$

where $\mathcal{H}^-, \mathcal{H}^+$ are twice the mean curvatures of $\mathcal{G}^-, \mathcal{G}^+, \mathcal{U}(x) = \int\limits_{\mathcal{F}} \frac{\bar\rho\, \mathrm{d}z}{|x-z|}$. In [9] it was proved the existence of the surfaces $\mathcal{G}^-$, $\mathcal{G}^+$ satisfying equations (1.8) provided that β is small enough, and $\varkappa, [\bar\rho]\big|_{\mathcal{G}^+} > 0$ (Proposition 3.3). It was noted there that $\mathcal{G}^\pm$ are flattened spheroids.

Proposition 1.1. *Let the angular momentum β be small enough, and $\varkappa > 0, \rho^+ > \rho^-$. Then for given volumes $|\mathcal{F}^+|$ and $|\mathcal{F}|$, there exists a unique equilibrium figure which is axially symmetric about the axis x_3 and symmetric about the plane $x_3 = 0$. The surfaces $\mathcal{G}^+$ and $\mathcal{G}^-$ are close to the spheres $S_{R_+} = \{|x| = R_+\}$ and $S_{R_-} = \{|x| = R_-\}$, respectively, where R_+, R_- are such that*

$$\frac{4\pi}{3} R_+^3 = |\mathcal{F}^+|, \quad \frac{4\pi}{3} R_-^3 = |\mathcal{F}|$$

and

$$\mathcal{G}^\pm = \{y = z + \varphi^\pm \boldsymbol{z}/|z|, \quad z \in S_{R_\pm}\}.$$

So, we assume the axial symmetry of $\mathcal{F}^\pm$ and the symmetry of them about the plane $x_3 = 0$; it implies that

$$\int\limits_{\mathcal{F}} \bar\rho(x) x_i dx = 0, \quad i = 1, 2. \tag{1.9}$$

$$\int\limits_{\mathcal{F}} \bar\rho x_3 dx = 0, \quad \int\limits_{\mathcal{F}} \bar\rho x_3 x_j dx = 0, \quad j = 1, 2.$$

Condition (1.9) corresponds to the first relation in (1.5) which means that the barycenter of the liquids coincides with the origin all the time. The other conditions in (1.5), the conservation of momentum and angular one, take the form

$$\int\limits_{\Omega_t} \rho^\pm \boldsymbol{v}(x,t) dx = \int\limits_{\mathcal{F}} \bar\rho \boldsymbol{\mathcal{V}}(x) dx = 0, \tag{1.10}$$

$$\int\limits_{\Omega_t} \rho^\pm \boldsymbol{v}(x,t) \cdot \boldsymbol{\eta}_i(x) dx = \int\limits_{\mathcal{F}} \bar\rho \boldsymbol{\mathcal{V}}(x) \cdot \boldsymbol{\eta}_i(x) dx = \delta_i^3 \beta, \quad i = 1, 2, 3.$$

It is reasonable to work with the problem for the perturbations of the velocity and pressure,

$$\boldsymbol{v}_r(x,t) = \boldsymbol{v}(x,t) - \boldsymbol{\mathcal{V}}(x), \quad p_r(x,t) = p(x,t) - \mathcal{P}(x)$$

written in the coordinate system rotating about the x_3-axis with the angular velocity ω.

We introduce the new coordinates $\{y_i\}$ and the new unknown functions $(\tilde{\boldsymbol{v}}, \tilde{p})$ by the formulas

$$x = \mathcal{Z}(\omega t)y,$$
$$\tilde{\boldsymbol{v}}(y,t) = \mathcal{Z}^{-1}(\omega t)\boldsymbol{v}_r(\mathcal{Z}(\omega t)y,t), \quad \tilde{p}(y,t) = p_r(\mathcal{Z}(\omega t)y,t),$$

where

$$\mathcal{Z}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We note that

$$\mathcal{Z}^{-1}(\omega t)(\boldsymbol{\mathcal{V}}\cdot\nabla_x)\boldsymbol{v}_r = \omega\big(\boldsymbol{\eta}_3(x)\cdot\nabla_x\big)\tilde{\boldsymbol{v}}(y,t) = \omega\big(\mathcal{Z}^{-1}\boldsymbol{\eta}_3(y)\cdot\nabla_y\big)\tilde{\boldsymbol{v}}$$
$$= \omega\big(\boldsymbol{\eta}_3(y)\cdot\nabla_y\big)\tilde{\boldsymbol{v}}(y,t) = \omega\big(y_2\frac{\partial\tilde{\boldsymbol{v}}}{\partial y_1} - y_1\frac{\partial\tilde{\boldsymbol{v}}}{\partial y_2}\big)$$

and $\mathcal{D}_t\boldsymbol{v}_r|_{x=\mathcal{Z}y} = \mathcal{D}_t\boldsymbol{v}_r(\mathcal{Z}y,t) - (\boldsymbol{\mathcal{V}}\cdot\nabla)\boldsymbol{v}_r$. Substituting this in (1.6), (1.7), acting by $\mathcal{Z}^{-1}$ and taking (1.8) into account in the boundary conditions, we arrive at the free boundary problem for the perturbations of the velocity $\tilde{\boldsymbol{v}}$ and pressure $\tilde{p}$:

$$\begin{aligned}
&\rho^{\pm}\big(\mathcal{D}_t\tilde{\boldsymbol{v}} + (\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}} + 2\omega(e_3\times\tilde{\boldsymbol{v}})\big) - \mu^{\pm}\nabla^2\tilde{\boldsymbol{v}} + \nabla\tilde{p} = \rho^{\pm}\tilde{\boldsymbol{f}},\\
&\nabla\cdot\tilde{\boldsymbol{v}} = 0 \quad \text{in } \cup\tilde{\Omega}_t^{\pm} \equiv \tilde{\Omega}_t^{-}\cup\tilde{\Omega}_t^{+}, \quad t>0,\\
&\tilde{\boldsymbol{v}}(y,0) = \boldsymbol{v}_0(y) - \boldsymbol{\mathcal{V}}(y) \equiv \tilde{\boldsymbol{v}}_0(y), \quad y\in\cup\tilde{\Omega}_0^{\pm} \equiv \tilde{\Omega}_0^{-}\cup\tilde{\Omega}_0^{+},\\
&-\tilde{p}\tilde{\boldsymbol{n}} + \mu^{-}\mathbb{S}(\tilde{\boldsymbol{v}})\tilde{\boldsymbol{n}}\big|_{\tilde{\Gamma}_t^{-}} = \big\{\sigma^{-}\big(\tilde{H}^{-}(y) - \mathcal{H}^{-}(z)\big) + \rho^{-}\omega^2\big(|y'|^2 - |z'|^2\big)/2\\
&\qquad\qquad + \varkappa\rho^{-}\big(\tilde{U}(y,t) - \mathcal{U}(z)\big)\big\}\tilde{\boldsymbol{n}},\\
&[\tilde{\boldsymbol{v}}]\big|_{\tilde{\Gamma}_t^{+}} = 0, \qquad\qquad (1.11)\\
&[-\tilde{p}\tilde{\boldsymbol{n}} + \mu^{\pm}\mathbb{S}(\tilde{\boldsymbol{v}})\tilde{\boldsymbol{n}}]|_{\tilde{\Gamma}_t^{+}} = \big\{\sigma^{+}\big(\tilde{H}^{+}(y) - \mathcal{H}^{+}(z)\big) + [\rho^{\pm}]\omega^2\big(|y'|^2 - |z'|^2\big)/2\\
&\qquad\qquad + \varkappa[\rho^{\pm}]|_{\Gamma_t^{+}}\big(\tilde{U}(y,t) - \mathcal{U}(z)\big)\big\}\tilde{\boldsymbol{n}},\\
&\tilde{V}_{\tilde{\boldsymbol{n}}} = \tilde{\boldsymbol{v}}\cdot\tilde{\boldsymbol{n}} \quad \text{on } \tilde{\Gamma}_t \equiv \tilde{\Gamma}_t^{-}\cup\tilde{\Gamma}_t^{+},
\end{aligned}$$

where

$$\tilde{\Omega}_t^{\pm} = \mathcal{Z}^{-1}(\omega t)\Omega_t^{\pm}, \ \tilde{\Gamma}_t^{\pm} = \mathcal{Z}^{-1}(\omega t)\Gamma_t^{\pm}, \ \tilde{\boldsymbol{f}}(y,t) = \mathcal{Z}^{-1}(\omega t)\boldsymbol{f}(\mathcal{Z}(\omega t)y,t),$$

$\tilde{\boldsymbol{n}}$ is the outward normal to $\tilde{\Gamma}_t$, $\boldsymbol{n} = \mathcal{Z}\tilde{\boldsymbol{n}}$, $y' = (y_1, y_2, 0)$, etc. We note that the kinematic boundary condition $V_{\boldsymbol{n}} = \boldsymbol{v}\cdot\boldsymbol{n}$, conserves its form (see [14]) (is invariant with respect to these transformations).

Relations (1.3), (1.5), (1.10) go over into

$$|\tilde\Omega_t^+| = |\mathcal{F}^+|, \quad |\tilde\Omega_t| = |\mathcal{F}|, \tag{1.12}$$

$$\begin{gathered}\int\limits_{\tilde\Omega_t} \rho^\pm y_j\,\mathrm{d}y = 0, \quad j=1,2,3, \quad \text{(barycenter conservation)}\\ \int\limits_{\tilde\Omega_t} \rho^\pm \tilde{\boldsymbol{v}}(y,t)\,\mathrm{d}y = 0, \quad \text{(momentum conservation)}\\ \int\limits_{\tilde\Omega_t} \rho^\pm \tilde{\boldsymbol{v}}(y,t)\cdot\boldsymbol{\eta}_i(y)\,\mathrm{d}y + \omega\int\limits_{\tilde\Omega_t} \rho^\pm \boldsymbol{\eta}_3\cdot\boldsymbol{\eta}_i(y)\,\mathrm{d}y = \omega\int\limits_{\mathcal{F}} \bar\rho\boldsymbol{\eta}_3\cdot\boldsymbol{\eta}_i(y)\,\mathrm{d}y = \beta\delta_i^3,\end{gathered} \tag{1.13}$$

where $\boldsymbol{\eta}_i(y) = \boldsymbol{e}_i \times \boldsymbol{y}$, $i=1,2,3$.

We recall the definition of the Sobolev–Slobodetskiĭ spaces which we use in the present paper. The isotropic space $W_2^l(\Omega)$, $\Omega\subset\mathbb{R}^n$, is the space with the norm

$$\|u\|^2_{W_2^l(\Omega)} = \sum_{0\leqslant|\boldsymbol{j}|\leqslant l} \|\mathcal{D}_x^{\boldsymbol{j}}u\|^2_\Omega \equiv \sum_{0\leqslant|\boldsymbol{j}|\leqslant l}\int\limits_\Omega |\mathcal{D}_x^{\boldsymbol{j}}u(x)|^2\,\mathrm{d}x$$

if $l=[l]$, i. e., l is an integral number, and

$$\|u\|^2_{W_2^l(\Omega)} = \|u\|^2_{W_2^{[l]}(\Omega)} + \sum_{|\boldsymbol{j}|=[l]}\int\limits_\Omega\int\limits_\Omega |\mathcal{D}_x^{\boldsymbol{j}}u(x)-\mathcal{D}_y^{\boldsymbol{j}}u(y)|^2\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{n+2\lambda}}$$

if $l=[l]+\lambda$, $\lambda\in(0,1)$. As usual, $\mathcal{D}_x^{\boldsymbol{j}}u$ denotes a (generalized) partial derivative $\frac{\partial^{|\boldsymbol{j}|}u}{\partial x_1^{j_1}\ldots\partial x_n^{j_n}}$, where $\boldsymbol{j}=(j_1,j_2,\ldots j_n)$ and $|\boldsymbol{j}| = j_1+\ldots+j_n$.

We introduce the anisotropic spaces

$$W_2^{l,0}(Q_T) = L_2\big((0,T),W_2^l(\Omega)\big), \quad W_2^{0,l/2}(Q_T) = W_2^{l/2}\big((0,T),L_2(\Omega)\big);$$

$Q_T=\Omega\times(0,T)$, the squares of norms in these spaces coincide, respectively, with

$$\|u\|^2_{W_2^{l,0}(Q_T)} = \int\limits_0^T \|u(\cdot,t)\|^2_{W_2^l(\Omega)}\,\mathrm{d}t, \quad \|u\|^2_{W_2^{0,l/2}(Q_T)} = \int\limits_\Omega \|u(x,\cdot)\|^2_{W_2^{l/2}(0,T)}\,\mathrm{d}x.$$

The space $W_2^{l,l/2}(Q_T)\equiv W_2^{l,0}(Q_T)\cap W_2^{0,l/2}(Q_T)$ can be supplied with the norm

$$\|u\|_{W_2^{l,l/2}(Q_T)} \equiv \|u\|_{W_2^{l,0}(Q_T)} + \|u\|_{W_2^{0,l/2}(Q_T)}.$$

We will use also another equivalent norm in $W_2^{l,l/2}(Q_T)$.

The Sobolev–Slobodetskiĭ spaces of functions given on smooth surfaces, in particular, on $\mathcal{G}^\pm$ and on $G_T^\pm = \mathcal{G}^\pm \times (0,T)$, $T \leqslant \infty$, are introduced in the standard way, with the help of local maps and partition of unity.

Moreover, we introduce also the norm

$$|u|_{G_T^\pm}^{(s+l,l/2)} = \|u\|_{W_2^{s+l,0}(G_T^\pm)} + \|u\|_{W_2^{l/2}\left(0,T;W_2^s(\mathcal{G}^\pm)\right)}, \quad s>0.$$

Finally, we set

$$\|u\|^2_{W_2^l(\cup\mathcal{F}^\pm)} \equiv \|u\|^2_{W_2^l(\mathcal{F}^+)} + \|u\|^2_{W_2^l(\mathcal{F}^-)}, \quad \|u\|_\Omega \equiv \|u\|_{L_2(\Omega)}.$$

§2. Linearization of the problem

Let us suppose that the surfaces $\tilde{\Gamma}_t^\pm$ can be given by the relations

$$\tilde{\Gamma}_t^\pm = \{y = z + \boldsymbol{N}(z)r(z,t), \quad z \in \mathcal{G}^\pm\},$$

and we map $\tilde{\Omega}_t^\pm$ on $\mathcal{F}^\pm$ by the transformation the inverse of which is

$$y = z + \boldsymbol{N}^*(z)r^*(z,t) \equiv e_r(z,t), \tag{2.1}$$

where $\boldsymbol{N}^*$ and r^* are extensions of $\boldsymbol{N}$ and r into $\mathcal{F}$, respectively.

An analysis of nonstationary problem with free boundaries for the Navier–Stokes equations (1.11) with initial data close to the regime of rotation of a two-layer fluid as a solid (see Fig. 1) is based on the linearization of it. In order to linearize problem (1.11), we calculate the first variation with respect to $r^\pm$ of the expressions $H(y)-\mathcal{H}(z)$, $|y'|^2-|z'|^2$, $U(y,t)-\mathcal{U}(z)$, where y is connected with z by the relation (2.1). We apply the following formulas for the first and second variations of a functional $R[r]$ with respect to r

$$\delta_0 R[r] = \frac{\mathrm{d}}{\mathrm{d}s}R[sr]\big|_{s=0}, \qquad \delta_0^2 R[r] = \frac{\mathrm{d}^2}{\mathrm{d}s^2}R[sr]\big|_{s=0}. \tag{2.2}$$

It is clear that

$$\delta_0\left(|y'|^2-|z'|^2\right) = \frac{\mathrm{d}}{\mathrm{d}s}\left(|\boldsymbol{z}'+\boldsymbol{N}'sr|^2-|\boldsymbol{z}'|^2\right)\big|_{s=0} = 2\boldsymbol{z}'\cdot\boldsymbol{N}'r, \;\; \boldsymbol{N}' = (N_1,N_2,0),$$

and, according to [15],

$$\delta_0\left(H^\pm(y)-\mathcal{H}^\pm(z)\right) = \Delta^\pm r^\pm + \left(\mathcal{H}^{\pm^2}(z)-2\mathcal{K}^\pm(z)\right)r^\pm,$$

where $\Delta^\pm$ are the Laplace – Beltrami operators on $\mathcal{G}^\pm$, respectively. Moreover, as mentioned in [9],

$$\delta_0\left(U(y,t)-\mathcal{U}(z)\right) = \frac{\partial\mathcal{U}}{\partial\boldsymbol{N}}r + \mathcal{W}[r](z,t),$$

where

$$\mathcal{U}(z)=\int\limits_{\mathcal{F}}\frac{\bar{\rho}\,\mathrm{d}x}{|z-x|},\quad \mathcal{W}[r](z,t)\equiv\rho^-\int\limits_{\mathcal{G}^-}\frac{r(x,t)}{|z-x|}\,\mathrm{d}\mathcal{G}_x+[\rho^\pm]\big|_{\mathcal{G}^+}\int\limits_{\mathcal{G}^+}\frac{r(x,t)}{|z-x|}\,\mathrm{d}\mathcal{G}_x.$$

Since $V_{\boldsymbol{n}}\equiv\mathcal{D}_t\boldsymbol{y}\cdot\boldsymbol{n}|_{\mathcal{G}}$, we have the kinematic condition in the form

$$\mathcal{D}_t r\boldsymbol{N}\cdot\boldsymbol{n}=\tilde{\boldsymbol{v}}\cdot\boldsymbol{n}. \tag{2.3}$$

Applying (2.1) and using the above relations, we arrive at the linear problem corresponding to (1.11) with respect to the unknown velocity vector field $\boldsymbol{w}$ and pressure function p:

$$\begin{aligned}
&\bar{\rho}\big(\mathcal{D}_t\boldsymbol{w}+2\omega(e_3\times\boldsymbol{w})\big)-\bar{\mu}\nabla^2\boldsymbol{w}+\nabla p=\bar{\rho}\boldsymbol{f},\\
&\nabla\cdot\boldsymbol{w}=f\equiv\nabla\cdot\boldsymbol{F}\quad\text{in }\ \mathcal{F}\equiv\mathcal{F}^-\cup\mathcal{F}^+,\quad t>0,\\
&\boldsymbol{w}(z,0)=\boldsymbol{v}_0(z)-\boldsymbol{\mathcal{V}}(z)\equiv\boldsymbol{w}_0(z),\quad z\in\mathcal{F},\\
&\mathbb{T}(\boldsymbol{w},p)\boldsymbol{N}+\boldsymbol{N}\mathcal{B}_0^-(r^-)=\boldsymbol{d}\quad\text{on }\mathcal{G}^-,\\
&[\boldsymbol{w}]\big|_{\mathcal{G}^+}=0,\quad [\mathbb{T}(\boldsymbol{w},p)\boldsymbol{N}]\big|_{\mathcal{G}^+}+\boldsymbol{N}\mathcal{B}_0^+(r^+)=\boldsymbol{d}\quad\text{on }\mathcal{G}^+,\\
&\mathcal{D}_t r-\boldsymbol{w}\cdot\boldsymbol{N}=g\quad\text{on }\mathcal{G}\equiv\mathcal{G}^-\cup\mathcal{G}^+,\qquad r\big|_{t=0}=r_0\quad\text{on }\mathcal{G},
\end{aligned}\tag{2.4}$$

where the operators

$$\begin{aligned}
&\mathcal{B}_0^-(r)=-\sigma^-\Delta^-r-b^-(z)r-\rho^-\varkappa\mathcal{W}[r],\quad z\in\mathcal{G}^-,\\
&\mathcal{B}_0^+(r)=-\sigma^+\Delta^+r-b^+(z)r-[\bar{\rho}]\big|_{\mathcal{G}^+}\varkappa\mathcal{W}[r],\quad z\in\mathcal{G}^+,
\end{aligned}\tag{2.5}$$

with

$$\begin{aligned}
&b^-(z)=\sigma^-({\mathcal{H}^-}^2-2\mathcal{K}^-)+\rho^-\omega^2\boldsymbol{N}\cdot\boldsymbol{z}'+\rho^-\varkappa\partial\mathcal{U}/\partial\boldsymbol{N},\\
&b^+(z)=\sigma^+({\mathcal{H}^+}^2-2\mathcal{K}^+)+[\bar{\rho}]\big|_{\mathcal{G}^+}\omega^2\boldsymbol{N}\cdot\boldsymbol{z}'+[\bar{\rho}]\big|_{\mathcal{G}^+}\varkappa\partial\mathcal{U}/\partial\boldsymbol{N},
\end{aligned}$$

$\boldsymbol{z}'=(z_1,z_2,0)$, $\mathcal{K}^\pm$ are the Gaussian curvatures of $\mathcal{G}^\pm$, ω is the angular velocity of the rotation, $r(x,t)$ is an unknown function defining the surfaces $\Gamma_t^\pm$; $\boldsymbol{N}$ is the outward unit normal to $\mathcal{G}^-\cup\mathcal{G}^+$; $\boldsymbol{f},f,\boldsymbol{d},g$, $\boldsymbol{w}_0,r_0$ are given functions.

First, we study homogeneous problem (2.4):

$$\begin{aligned}
&\bar{\rho}(\mathcal{D}_t \boldsymbol{w} + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{w})) - \bar{\mu}\nabla^2 \boldsymbol{w} + \nabla p = 0, \qquad \nabla \cdot \boldsymbol{w} = 0 \text{ in } \mathcal{F},\ \ t > 0,\\
&\boldsymbol{w}\big|_{t=0} = \boldsymbol{w}_0 \quad \text{in } \mathcal{F},\\
&\mathbb{T}(\boldsymbol{w}, p)\boldsymbol{N}\big|_{\mathcal{G}^-} + \boldsymbol{N}\mathcal{B}_0^-(r) = 0 \text{ on } \mathcal{G}^-,\\
&[\boldsymbol{w}]\big|_{\mathcal{G}^+} = 0, \quad [\mathbb{T}(\boldsymbol{w}, p)\boldsymbol{N}]\big|_{\mathcal{G}^+} + \boldsymbol{N}\mathcal{B}_0^+(r) = 0 \text{ on } \mathcal{G}^+,\\
&\mathcal{D}_t r - \boldsymbol{w} \cdot \boldsymbol{N} = 0 \quad \text{on } \mathcal{G}, \quad r\big|_{t=0} = r_0 \quad \text{on } \mathcal{G}.
\end{aligned} \tag{2.6}$$

We assume that the domains $\mathcal{F}^{\pm}$ are symmetric with respect to z_1, z_2, z_3, as well as the initial data satisfy, in accordance with the linearization of assumptions (1.12), (1.13), orthogonality conditions

$$\begin{aligned}
&\int\limits_{\mathcal{G}^{\pm}} r_0(z)\,\mathrm{d}\mathcal{G} = 0,\\
&\rho^- \int\limits_{\mathcal{G}^-} r_0(z) z_j\,\mathrm{d}\mathcal{G} + [\bar{\rho}]\big|_{\mathcal{G}^+} \int\limits_{\mathcal{G}^+} r_0(z) z_j\,\mathrm{d}\mathcal{G} = 0, \quad j = 1, 2, 3,
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
&\int\limits_{\mathcal{F}} \bar{\rho}\boldsymbol{w}_0(z)\,\mathrm{d}z = 0,\\
&\int\limits_{\mathcal{F}} \bar{\rho}\boldsymbol{w}_0(z) \cdot \boldsymbol{\eta}_j(z)\,\mathrm{d}z + \omega\big(\rho^- \int\limits_{\mathcal{G}^-} r_0(z)\boldsymbol{\eta}_3(z) \cdot \boldsymbol{\eta}_j(z)\,\mathrm{d}\mathcal{G}\\
&\qquad\qquad + [\bar{\rho}]\big|_{\mathcal{G}^+} \int\limits_{\mathcal{G}^+} r_0(z)\boldsymbol{\eta}_3(z) \cdot \boldsymbol{\eta}_j(z)\,\mathrm{d}\mathcal{G}\big) = 0.
\end{aligned} \tag{2.8}$$

We introduce the notation $Q_T^{\pm} = \mathcal{F}^{\pm} \times (0, T)$, $G_T^{\pm} = \mathcal{G}^{\pm} \times (0, T)$, $D_T = Q_T^+ \cup Q_T^-$, $Q_T = Q_T^+ \cup \overline{Q_T^-}$, $G_T = G_T^+ \cup G_T^-$, $T \in (0, \infty]$.

Proposition 2.1. *A solution of problem* (2.6) – (2.8) *satisfies conditions* (2.7)*,* (2.8) *for all* $t > 0$.

This proposition is proved in the same way as Proposition 2.1 in [14] by virtue of Proposition 2.2.

In view of impulse conservation law, it is valid the following statement.

Corollary 2.1. *There holds the following decomposition*

$$\boldsymbol{w} = \boldsymbol{w}^{\perp} + \sum_{i=1}^{3} d_i(r)\boldsymbol{\eta}_i,$$

where $\boldsymbol{w}^\perp$ is a vector field orthogonal to all the vectors of rigid motion $\boldsymbol{\eta}$, i. e.,

$$\int_{\mathcal{F}} \bar\rho \boldsymbol{w}^\perp \cdot \boldsymbol{\eta}\,\mathrm{d}z = 0, \qquad \boldsymbol{\eta}(z) = \boldsymbol{e}_i \quad or \quad \boldsymbol{\eta}(z) = \boldsymbol{\eta}_i(z), \quad i = 1,2,3,$$

and

$$d_i(r) = -\frac{\omega}{S_i}\big(\rho^- \int_{\mathcal{G}^-} r\boldsymbol{\eta}_3 \cdot \boldsymbol{\eta}_i\,\mathrm{d}\mathcal{G} + [\bar\rho]|_{\mathcal{G}^+} \int_{\mathcal{G}^+} r\boldsymbol{\eta}_3 \cdot \boldsymbol{\eta}_i\,\mathrm{d}\mathcal{G}\big), \quad S_i = \int_{\mathcal{F}} \bar\rho |\boldsymbol{\eta}_i|^2\,\mathrm{d}z. \tag{2.9}$$

Proposition 2.2. *The following relations hold:*

$$\begin{aligned} &\mathcal{B}_0^-(\boldsymbol{\eta}\cdot\boldsymbol{N}) = -\omega^2\rho^-\boldsymbol{\eta}\cdot\boldsymbol{z}', \quad z\in\mathcal{G}^-,\\ &\mathcal{B}_0^+(\boldsymbol{\eta}\cdot\boldsymbol{N}) = -\omega^2[\bar\rho]|_{\mathcal{G}_i^+}\boldsymbol{\eta}\cdot\boldsymbol{z}', \quad z\in\mathcal{G}^+, \end{aligned} \tag{2.10}$$

where $\boldsymbol{\eta}$ is an arbitrary vector of rigid motion.

Proof. Let Ω_ε be a bounded domain with the boundary Γ_ε, and $\boldsymbol{n}_\varepsilon$ be the external normal to Γ_ε. The equality

$$\begin{aligned} &\int_{\Gamma_\varepsilon} \Big(\sigma H_\varepsilon(x) + \rho\frac{\omega^2}{2}|x'|^2 + \rho\varkappa U + p_0\Big)\boldsymbol{n}_\varepsilon(x)\cdot\boldsymbol{\eta}_i(x)\,\mathrm{d}\Gamma_\varepsilon\\ &\qquad = \rho\omega^2 \int_{\Omega_\varepsilon} \boldsymbol{\eta}_i(x)\cdot\boldsymbol{x}'\,\mathrm{d}x, \quad i=1,2,3, \end{aligned} \tag{2.11}$$

follows from

$$\begin{aligned} \int_{\Gamma_\varepsilon} H_\varepsilon(x)\boldsymbol{n}_\varepsilon\cdot\boldsymbol{\eta}_i\,\mathrm{d}\Gamma_\varepsilon &= \int_{\Gamma_\varepsilon} \Delta_{\Gamma_\varepsilon}\boldsymbol{x}\cdot\boldsymbol{\eta}_i\,\mathrm{d}\Gamma_\varepsilon = \int_{\Gamma_\varepsilon} \boldsymbol{x}\cdot\Delta_{\Gamma_\varepsilon}\boldsymbol{\eta}_i\,\mathrm{d}\Gamma_\varepsilon\\ &= -\int_{\Gamma_\varepsilon} \boldsymbol{\eta}_i\cdot\Delta_{\Gamma_\varepsilon}\boldsymbol{x}\,\mathrm{d}\Gamma_\varepsilon = 0, \end{aligned}$$

which is a consequence of the well-known Weierstrass formula

$$H_\varepsilon(x)\boldsymbol{n}_\varepsilon = \Delta_{\Gamma_\varepsilon}\boldsymbol{x},$$

and from

$$\int_{\Gamma_\varepsilon} |x'|^2\boldsymbol{n}_\varepsilon\cdot\boldsymbol{\eta}_i(x)\,\mathrm{d}\Gamma_\varepsilon = \int_{\Omega_\varepsilon} \nabla\cdot|x'|^2\boldsymbol{\eta}_i\,\mathrm{d}x = 2\int_{\Omega_\varepsilon} \boldsymbol{\eta}_i(x)\cdot\boldsymbol{x}'\,\mathrm{d}x,$$

$$\int\limits_{\Gamma_\varepsilon} \rho\varkappa U \boldsymbol{n}_\varepsilon(x)\cdot\boldsymbol{\eta}_i(x)\,\mathrm{d}\Gamma_\varepsilon = \rho\int\limits_{\Omega_\varepsilon} \rho\varkappa\nabla\cdot\big(U\boldsymbol{\eta}_i(x)\big)\,\mathrm{d}x$$
$$= \rho\varkappa\int\limits_{\Omega_\varepsilon}\int\limits_{\Omega_\varepsilon}\rho\frac{\boldsymbol{x}\cdot\boldsymbol{\eta}_i(x)}{|x-y|^3}\,\mathrm{d}y\,\mathrm{d}x = 0,$$

$$\int\limits_{\Gamma_\varepsilon} \boldsymbol{n}_\varepsilon\cdot\boldsymbol{\eta}_i\,\mathrm{d}S = \int\limits_{\Omega_\varepsilon}\nabla\cdot\boldsymbol{\eta}_i(x)\,\mathrm{d}x = 0.$$

Next, by $\Gamma_\varepsilon^\pm$ we denote the surfaces given by $x = z + \varepsilon\boldsymbol{N}^\pm r^\pm$, $z\in\mathcal{G}^\pm$, and Ω_ε^+, Ω_ε^- mean the domains bounded by the surfaces Γ_ε^+, $\Gamma_\varepsilon^+\cup\Gamma_\varepsilon^-$ and close to $\mathcal{F}^\pm$, respectively; $\Omega_\varepsilon \equiv \overline{\Omega_\varepsilon^+}\cup\Omega_\varepsilon^-$. Finally, let $\boldsymbol{N}^*$ and r^* be the extensions of $\boldsymbol{N}^\pm$ and $r^\pm$ into $\mathcal{F}$.

We generalize (2.11) on the surfaces $\Gamma_\varepsilon^\pm$:

$$\int\limits_{\Gamma_\varepsilon^-}\Big(\sigma^- H_\varepsilon^-(x) + \rho^-\frac{\omega^2}{2}|x'|^2 + \rho^-\varkappa U + p_0^-\Big)\boldsymbol{n}_\varepsilon^-(x)\cdot\boldsymbol{\eta}_i(x)\,\mathrm{d}\Gamma_\varepsilon$$
$$+\int\limits_{\Gamma_\varepsilon^+}\Big(\sigma^+ H_\varepsilon^+(x) + [\rho^\pm]\big|_{\Gamma_\varepsilon^+}\frac{\omega^2}{2}|x'|^2 + [\rho^\pm]\big|_{\Gamma_\varepsilon^+}\varkappa U + p_0^+ - p_0^-\Big)\boldsymbol{n}_\varepsilon^+(x)\cdot\boldsymbol{\eta}_i(x)\,\mathrm{d}\Gamma_\varepsilon$$
$$= \omega^2\Big(\int\limits_{\Omega_\varepsilon}\rho^-\boldsymbol{\eta}_i\cdot\boldsymbol{x}'\,\mathrm{d}x + \int\limits_{\Omega_\varepsilon^+}[\rho^\pm]\big|_{\Gamma_\varepsilon^+}\boldsymbol{\eta}_i\cdot\boldsymbol{x}'\,\mathrm{d}x\Big)$$
$$= \omega^2\Big(\int\limits_{\Omega_\varepsilon^-}\rho^-\boldsymbol{\eta}_i\cdot\boldsymbol{x}'\,\mathrm{d}x + \int\limits_{\Omega_\varepsilon^+}\rho^+\boldsymbol{\eta}_i\cdot\boldsymbol{x}'\,\mathrm{d}x\Big) = \omega^2\int\limits_{\Omega_\varepsilon}\rho^\pm\boldsymbol{\eta}_i\cdot\boldsymbol{x}'\,\mathrm{d}x.$$

By using equations (1.8) for $\mathcal{G}^\pm$, we obtain

$$\varepsilon^{-1}\Big\{\int\limits_{\mathcal{G}^-}\Big(\sigma^-\big(H_\varepsilon^-(x) - \mathcal{H}^-(z)\big) + \rho^-\frac{\omega^2}{2}\big(|x'|^2 - |z'|^2\big)$$
$$+ \rho^-\varkappa(U-\mathcal{U})\Big)\boldsymbol{n}_\varepsilon(x)\cdot\boldsymbol{\eta}_i(x)\big|_{x=z+\varepsilon\boldsymbol{N}^- r^-}|\widehat{\mathbb{L}}_\varepsilon^T(z)\boldsymbol{N}(z)|\,\mathrm{d}\mathcal{G}$$
$$+\int\limits_{\mathcal{G}^+}\Big(\sigma^+\big(H_\varepsilon^+(x) - \mathcal{H}^+(z)\big) + [\rho^\pm]\big|_{\Gamma_\varepsilon^+}\frac{\omega^2}{2}\big(|x'|^2 - |z'|^2\big) \qquad (2.12)$$
$$+ [\rho^\pm]\big|_{\Gamma_\varepsilon^+}\varkappa(U-\mathcal{U})\Big)\boldsymbol{n}_\varepsilon(x)\cdot\boldsymbol{\eta}_i(x)\big|_{x=z+\varepsilon\boldsymbol{N}^+ r^+}|\widehat{\mathbb{L}}_\varepsilon^T(z)\boldsymbol{N}(z)|\,\mathrm{d}\mathcal{G}\Big\}$$

$$= \frac{\omega^2}{\varepsilon}\Big\{\rho^-\Big(\int\limits_{\Omega_\varepsilon^-}\boldsymbol{\eta}_i(x)\cdot\boldsymbol{x}'\,\mathrm{d}x - \int\limits_{\mathcal{F}^-}\boldsymbol{\eta}_i(z)\cdot\boldsymbol{z}'\,\mathrm{d}z\Big)$$

$$+\,\rho^+\Big(\int\limits_{\Omega_\varepsilon^+}\boldsymbol{\eta}_i(x)\cdot\boldsymbol{x}'\,\mathrm{d}x - \int\limits_{\mathcal{F}^+}\boldsymbol{\eta}_i(z)\cdot\boldsymbol{z}'\,\mathrm{d}z\Big)\Big\},$$

where $\mathbb{L}_\varepsilon$ is the Jacobi matrix of the (invertible) transformation

$$x = z + \varepsilon\boldsymbol{N}^* r^* : \quad \mathcal{F} \to \Omega_\varepsilon,$$

$\widehat{\mathbb{L}}_\varepsilon$ is its co-factor matrix.

The first variation of (2.12) leads to

$$\int\limits_{\mathcal{G}^-}\mathcal{B}_0^-(r^-)\boldsymbol{N}\cdot\boldsymbol{\eta}_i(z)\,\mathrm{d}\mathcal{G} + \int\limits_{\mathcal{G}^+}\mathcal{B}_0^+(r^+)\boldsymbol{N}\cdot\boldsymbol{\eta}_i(z)\,\mathrm{d}\mathcal{G}$$

$$= -\lim_{\varepsilon\to 0}\varepsilon^{-1}\omega^2\rho^-\Big(\int\limits_{\Omega_\varepsilon^-}\boldsymbol{\eta}_i(x)\cdot\boldsymbol{x}'\,\mathrm{d}x - \int\limits_{\mathcal{F}^-}\boldsymbol{\eta}_i(z)\cdot\boldsymbol{z}'\,\mathrm{d}z\Big)$$

$$-\lim_{\varepsilon\to 0}\varepsilon^{-1}\omega^2\rho^+\Big(\int\limits_{\Omega_\varepsilon^+}\boldsymbol{\eta}_i(x)\cdot\boldsymbol{x}'\,\mathrm{d}x - \int\limits_{\mathcal{F}^+}\boldsymbol{\eta}_i(z)\cdot\boldsymbol{z}'\,\mathrm{d}z\Big)$$

$$= -\omega^2\rho^-\Big(\int\limits_{\mathcal{G}^-}\boldsymbol{\eta}_i(z)\cdot\boldsymbol{z}'r^-\,\mathrm{d}\mathcal{G} - \int\limits_{\mathcal{G}^+}\boldsymbol{\eta}_i(z)\cdot\boldsymbol{z}'r^+\,\mathrm{d}\mathcal{G}\Big) - \omega^2\rho^+\int\limits_{\mathcal{G}^+}\boldsymbol{\eta}_i(z)\cdot\boldsymbol{z}'r^+\,\mathrm{d}\mathcal{G}$$

$$= -\omega^2\rho^-\int\limits_{\mathcal{G}^-}\boldsymbol{\eta}_i(z)\cdot\boldsymbol{z}'r^-\,\mathrm{d}\mathcal{G} - \omega^2[\bar\rho]\big|_{\mathcal{G}^+}\int\limits_{\mathcal{G}^+}\boldsymbol{\eta}_i(z)\cdot\boldsymbol{z}'r^+\,\mathrm{d}\mathcal{G}$$

which implies

$$\int\limits_{\mathcal{G}^-}\mathcal{B}_0^-(r^-)\boldsymbol{N}\cdot\boldsymbol{\eta}_i(z)\,\mathrm{d}\mathcal{G} = -\omega^2\rho^-\int\limits_{\mathcal{G}^-}\boldsymbol{\eta}_i(z)\cdot\boldsymbol{z}'r^-\,\mathrm{d}\mathcal{G}$$

and

$$\int\limits_{\mathcal{G}^+}\mathcal{B}_0^+(r^+)\boldsymbol{N}\cdot\boldsymbol{\eta}_i(z)\,\mathrm{d}\mathcal{G} = -\omega^2[\bar\rho]\big|_{\mathcal{G}^+}\int\limits_{\mathcal{G}^+}\boldsymbol{\eta}_i(z)\cdot\boldsymbol{z}'r^+\,\mathrm{d}\mathcal{G}.$$

It is true the same for $\boldsymbol{e}_i$ instead of $\boldsymbol{\eta}_i$. In view of the self-adjointness of the operators $\mathcal{B}_0^\pm(r)$, equalities (2.10) are proved. □

Theorem 2.1 (**Local Solvability of the Linear Problem**). *Let* $\mathcal{G} \in W_2^{3/2+l}$ *and* $r_0 \in W_2^{2+l}(\mathcal{G})$ *with* $l \in (1/2, 1)$. *For arbitrary* $\boldsymbol{f} \in \boldsymbol{W}_2^{l,l/2}(D_T)$,

$f \in W_2^{1+l,0}(D_T)$, $f = \nabla \cdot \boldsymbol{F}$, $\boldsymbol{F} \in \boldsymbol{W}_2^{0,1+\frac{l}{2}}(D_T)$, $[\boldsymbol{F} \cdot \boldsymbol{N}]|_{\mathcal{G}} = 0$, $\boldsymbol{w}_0 \in W_2^{1+l}(\mathcal{F})$, $\boldsymbol{d} = \boldsymbol{d}_\tau + d\boldsymbol{N}$, $\boldsymbol{d}_\tau \in \boldsymbol{W}_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(G_T)$, $\boldsymbol{N} \cdot \boldsymbol{d}_\tau = 0$, $d \in W_2^{l+\frac{1}{2},0}(G_T) \cap W_2^{l/2}\big(0,T; W_2^{1/2}(\mathcal{G})\big)$, $g \in W_2^{3/2+l,3/4+l/2}(G_T)$, $T < \infty$, *satisfying compatibility conditions*

$$\nabla \cdot \boldsymbol{w}_0 = f|_{t=0},$$
$$[\boldsymbol{w}_0]|_{\mathcal{G}^+} = 0,\ \ [\mu^{\pm}\Pi_{\mathcal{G}}\mathbb{S}(\boldsymbol{w}_0)\boldsymbol{N}]|_{\mathcal{G}^+} = \boldsymbol{d}_\tau|_{t=0},\ \ \mu^{-}\Pi_{\mathcal{G}}\mathbb{S}(\boldsymbol{w}_0)\boldsymbol{N}|_{\mathcal{G}^-} = \boldsymbol{d}_\tau|_{t=0},$$

where $\Pi_{\mathcal{G}}\boldsymbol{b} = \boldsymbol{b} - (\boldsymbol{N} \cdot \boldsymbol{b})\boldsymbol{N}$, *problem* (2.4) *has a unique solution* $(\boldsymbol{w}, p, r)$ *such that* $\boldsymbol{w} \in \boldsymbol{W}_2^{2+l,1+\frac{l}{2}}(D_T)$, $p \in \boldsymbol{W}_2^{l,\frac{l}{2}}(D_T)$, $\nabla p \in \boldsymbol{W}_2^{l,\frac{l}{2}}(D_T)$, $r(\cdot,t) \in W_2^{2+l}(\mathcal{G})$ *for any* $t \in (0,T)$ *and*

$$\begin{aligned}
&\|\boldsymbol{w}\|_{\boldsymbol{W}_2^{2+l,1+l/2}(D_T)} + \|\nabla p\|_{\boldsymbol{W}_2^{l,l/2}(D_T)} + \|p\|_{W_2^{l,l/2}(D_T)} + \|r\|_{W_2^{5/2+l,5/4+l/2}(G_T)} \\
&\quad + \|\mathcal{D}_t r\|_{W_2^{3/2+l,3/4+l/2}(G_T)} \leqslant c(T)\Big\{\|\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(D_T)} + \|f\|_{W_2^{1+l,0}(D_T)} \\
&\qquad + \|\boldsymbol{F}\|_{W_2^{0,1+l/2}(D_T)} + \|\boldsymbol{d}_\tau\|_{\boldsymbol{W}_2^{l+1/2,l/2+1/4}(G_T)} + \sigma|d|_{G_T}^{(l+1/2,l/2)} \\
&\qquad + \|g\|_{W_2^{3/2+l,3/4+l/2}(G_T)} + \|\boldsymbol{w}_0\|_{\boldsymbol{W}_2^{1+l}(\mathcal{F})} + \|r_0\|_{W_2^{2+l}(\mathcal{G})}\Big\}. \quad (2.13)
\end{aligned}$$

Remark 2.1. *From trace theorem for* $\rho \in W_2^{1,1}(G_T)$, *it follows that*

$$\|\rho(\cdot,t)\|_{W_2^{1/2}(\mathcal{G})} \leqslant c\Big\{\|\rho\|_{W_2^{1,0}(G_T)} + \|\mathcal{D}_t\rho\|_{G_T}\Big\},\ t \in [0,T],$$

which implies the inequality

$$\|r(\cdot,t)\|_{W_2^{2+l}(\mathcal{G})} \leqslant c\Big\{\|r\|_{W_2^{5/2+l,0}(G_T)} + \|\mathcal{D}_t r\|_{W_2^{3/2+l,0}(G_T)}\Big\}.$$

This means that $\Gamma_t^{\pm} \in W_2^{2+l}$ *for all* $t \in [0,T]$.

Proof. Let r_1 be a function satisfying the conditions

$$r_1(y,0) = r_0(y),$$
$$\mathcal{D}_t r_1(y,0) = g(y,0) + \boldsymbol{w}_0(y) \cdot \boldsymbol{N}(y) \equiv r_0'(y)$$

and the estimates

$$\begin{aligned}
&|r_1|_{G_T}^{(\frac{5}{2}+l,\frac{l}{2})} + \|\mathcal{D}_t r_1\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(G_T)} \\
&\qquad \leqslant c\big\{\|r_1\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(G_T)} + \|\mathcal{D}_t r_1\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(G_T)}\big\} \\
&\qquad\qquad \leqslant c\big\{\|r_0\|_{W_2^{2+l}(\mathcal{G})} + \|r_0'\|_{W_2^{l+1/2}(\mathcal{G})}\big\}. \quad (2.14)
\end{aligned}$$

Such r_1 exists due to Proposition 4.1 in [19] and equivalent normalizations of the Sobolev–Slobodetskiĭ spaces.

We can write

$$\mathcal{B}_0^{\pm} r(y,t) = \mathcal{B}_0^{\pm} r_1(y,t) + \int\limits_0^t \mathcal{B}_0^{\pm} \mathcal{D}_t\big(r(y,\tau) - r_1(y,\tau)\big)\,\mathrm{d}\tau$$
$$= \mathcal{B}_0^{\pm} r_1(y,t) + \int\limits_0^t \mathcal{B}_0^{\pm}\Big(g(y,\tau) + \boldsymbol{w}(y,\tau)\cdot\boldsymbol{N}(y) - \mathcal{D}_t r_1(y,\tau)\Big)\,\mathrm{d}\tau.$$

Consequently, system (2.4) can be transformed to the form:

$$\begin{aligned}
&\bar\rho\big(\mathcal{D}_t\boldsymbol{w} + 2\omega(\boldsymbol{e}_3\times\boldsymbol{w})\big) - \bar\mu\nabla^2\boldsymbol{w} + \nabla p = \bar\rho\boldsymbol{f}, \quad \nabla\cdot\boldsymbol{w} = f \text{ in } \mathcal{F},\ t>0,\\
&\quad \boldsymbol{w}(y,0) = \boldsymbol{w}_0(y) \quad \text{ in } \mathcal{F},\\
&\mu^-\Pi_{\mathcal{G}}\mathbb{S}(\boldsymbol{w})\boldsymbol{N}\big|_{\mathcal{G}^-} = \boldsymbol{d}_\tau, \quad [\boldsymbol{w}]|_{\mathcal{G}^+} = 0, \quad [\mu^{\pm}\Pi_{\mathcal{G}}\mathbb{S}(\boldsymbol{w})\boldsymbol{N}]\big|_{\mathcal{G}^+} = \boldsymbol{d}_\tau,\\
&\boldsymbol{N}\cdot\mathbb{T}(\boldsymbol{w},p)\boldsymbol{N}\big|_{\mathcal{G}^-} - \sigma^-\boldsymbol{N}\cdot\Delta^-\int\limits_0^t \boldsymbol{w}|_{\mathcal{G}^-}\,\mathrm{d}\tau = d' + \sigma^-\int\limits_0^t B'\,\mathrm{d}\tau\\
&+ \rho^-\varkappa\int\limits_0^t\Big(\mathcal{W}[\boldsymbol{N}\cdot\boldsymbol{w}] + \frac{\partial\mathcal{U}}{\partial\boldsymbol{N}}\boldsymbol{N}\cdot\boldsymbol{w}\Big)\,\mathrm{d}\tau + \sigma^-\nabla_{\mathcal{G}}\mathcal{H}\cdot\int\limits_0^t\boldsymbol{w}\,\mathrm{d}\tau\\
&\qquad - \sigma^-\omega^2\rho^-\boldsymbol{N}\cdot\boldsymbol{y}'\int\limits_0^t\boldsymbol{w}\cdot\boldsymbol{N}\,\mathrm{d}\tau + 2\sigma^-\int\limits_0^t\nabla_{\mathcal{G}}\boldsymbol{w}:\nabla_{\mathcal{G}}\boldsymbol{N}\,\mathrm{d}\tau \quad \text{on } \mathcal{G}^-,\\
&[\boldsymbol{N}\cdot\mathbb{T}(\boldsymbol{w},p)\boldsymbol{N}]\big|_{\mathcal{G}^+} - \sigma^+\boldsymbol{N}\cdot\Delta^+\int\limits_0^t\boldsymbol{w}|_{\mathcal{G}^+}\,\mathrm{d}\tau = d' + \sigma^+\int\limits_0^t B'\,\mathrm{d}\tau\\
&+ [\bar\rho]\big|_{\mathcal{G}}\varkappa\int\limits_0^t\Big(\mathcal{W}[\boldsymbol{N}\cdot\boldsymbol{w}] + \frac{\partial\mathcal{U}}{\partial\boldsymbol{N}}\boldsymbol{N}\cdot\boldsymbol{w}\Big)\,\mathrm{d}\tau + \sigma^+\nabla_{\mathcal{G}}\mathcal{H}\cdot\int\limits_0^t\boldsymbol{w}\,\mathrm{d}\tau\\
&\qquad - \sigma^+\omega^2[\bar\rho]\big|_{\mathcal{G}^+}\boldsymbol{N}\cdot\boldsymbol{y}'\int\limits_0^t\boldsymbol{w}\cdot\boldsymbol{N}\,\mathrm{d}\tau
\end{aligned} \tag{2.15}$$

$$+\,2\sigma^{+}\int\limits_0^t \nabla_{\mathcal G}\boldsymbol w:\nabla_{\mathcal G}\boldsymbol N\,\mathrm d\tau \quad \text{on } \mathcal G^{+},$$

where $d' = d - \sigma\mathcal B_0^{\pm} r_1$, $B' = -\mathcal B_0^{\pm}(g - \mathcal D_t r_1)$, $\nabla_{\mathcal G} = \Pi_{\mathcal G}\nabla$ is the surface gradient on $\mathcal G^{+}$; $\mathbb S:\mathbb T \equiv S_{ij}T_{ij}$. Here we have used that

$$\Delta^{\pm}\boldsymbol N = \nabla_{\mathcal G}\mathcal H^{\pm} - ({\mathcal H^{\pm}}^2 - 2\mathcal K^{\pm})\boldsymbol N$$

(Lemma 10.7 in [16]). Such problems were investigated in [12,21,22], where, in particular, the solvability of (2.15) without the terms $2\omega(\boldsymbol e_3\times\boldsymbol w)$ and

$$[\bar\rho]\big|_{\mathcal G}\varkappa\int\limits_0^t\Big(\mathcal W[\boldsymbol N\cdot\boldsymbol w] + \frac{\partial\mathcal U}{\partial\boldsymbol N}\boldsymbol N\cdot\boldsymbol w\Big)\,\mathrm d\tau + \sigma^{\pm}\nabla_{\mathcal G}\mathcal H\cdot\int\limits_0^t\boldsymbol w\,\mathrm d\tau$$

$$-\,\sigma^{\pm}\omega^2[\bar\rho]|_{\mathcal G^{\pm}}\boldsymbol N\cdot\boldsymbol y'\int\limits_0^t \boldsymbol w\cdot\boldsymbol N|_{\mathcal G^{\pm}}\,\mathrm d\tau + 2\sigma^{\pm}\int\limits_0^t \nabla_{\mathcal G}\boldsymbol w(y,t):\nabla_{\mathcal G}\boldsymbol N(y)|_{\mathcal G^{\pm}}\,\mathrm d\tau,$$

as well as the estimate of its solution

$$\begin{aligned}\|\boldsymbol w\|_{W_2^{2+l,1+l/2}(D_T)} + \|\nabla p\|_{\boldsymbol W_2^{l,l/2}(D_T)} + \|p\|_{W_2^{0,l/2}(D_T)}&\\ \leqslant c(T)\Big\{\|\boldsymbol f\|_{\boldsymbol W_2^{l,l/2}(D_T)}&\\ + \|f\|_{W_2^{1+l,0}(D_T)} + \|\boldsymbol F\|_{\boldsymbol W_2^{0,1+l/2}(D_T)} + \|\boldsymbol d_\tau\|_{\boldsymbol W_2^{l+1/2,l/2+1/4}(G_T)}&\\ + |d'|_{G_T}^{(l+1/2,l/2)} + \|B'\|_{W_2^{l-1/2,l/2-1/4}(G_T)} + \|\boldsymbol w_0\|_{\boldsymbol W_2^{1+l}(\mathcal F)}\Big\}&\end{aligned} \tag{2.16}$$

were established. We now need to evaluate the remaining terms of $\mathcal B_0^{\pm}[\boldsymbol w\cdot\boldsymbol N]$. For instance, the terms connected with self-gravity are estimated by Lemma 2.1 in [20] which gives the inequality for the norm of the product of two functions in the Sobolev–Slobodetskiĭ spaces as follows:

$$\begin{aligned}\|\mathcal W[\boldsymbol w\cdot\boldsymbol N]\|_{W_2^{l-1/2,l/2-1/4}(G_T)} &\leqslant c\|\boldsymbol w\cdot\boldsymbol N\|_{W_2^{l-1/2,l/2-1/4}(G_T)}\\ &\leqslant c(T)\|\boldsymbol w\|_{W_2^{l+1/2,l/2+1/4}(G_T)}\|\boldsymbol N\|_{W_2^{l-1/2}(\mathcal G)},\end{aligned} \tag{2.17}$$

$$\Big\|\frac{\partial\mathcal{U}}{\partial\boldsymbol{N}}\boldsymbol{w}\cdot\boldsymbol{N}\Big\|_{W_2^{l-1/2,l/2-1/4}(G_T)} \leqslant c(T)\Big\|\frac{\partial\mathcal{U}}{\partial\boldsymbol{N}}\Big\|_{W_2^{l-1/2}(\mathcal{G})}\|\boldsymbol{w}\cdot\boldsymbol{N}\|_{W_2^{l+1/2,l/2+1/4}(G_T)} \leqslant c(T)\|\boldsymbol{w}\|_{W_2^{l+1/2,l/2+1/4}(G_T)}\|\boldsymbol{N}\|_{W_2^{l+1/2}(\mathcal{G})}. \quad (2.18)$$

The others terms can be treated in a similar way. So, inequality (2.16) together with (2.14), (2.17), (2.18) implies estimate (2.13) because the additional terms are of lower order and have no essential influence on the final result. In addition, in [21, 22], we considered the whole space with a closed interface. We note that the results for bounded domains are similar [12]. Near the outer boundary, one should apply the estimates obtained in [23] for a single liquid of finite volume. □

Now we consider homogeneous problem (2.6) with $\boldsymbol{w}_0$ and r_0 satisfying orthogonality conditions (2.7), (2.8). At first, exponentially weighted L_2-estimates of $\boldsymbol{w}$ and r will be obtained.

Proposition 2.3. *Assume that the form*

$$R_0(r) = \int\limits_{\mathcal{G}} r\mathcal{B}_0^{\pm} r\,\mathrm{d}\mathcal{G} \quad (2.19)$$

is positive definite, i. e.,

$$c^{-1}\|r\|^2_{W_2^1(\mathcal{G})} \leqslant R_0(r) \leqslant c\|r\|^2_{W_2^1(\mathcal{G})} \quad (2.20)$$

for arbitrary $r(x)$ satisfying (2.7). *Then a solution of* (2.6) – (2.8) *satisfies the inequality*

$$\|e^{\beta_1 t}\boldsymbol{w}(\cdot,t)\|^2_{\mathcal{F}} + \|e^{\beta_1 t}r(\cdot,t)\|^2_{W_2^1(\mathcal{G})} \leqslant c\big\{\|\boldsymbol{w}_0\|^2_{\mathcal{F}} + \|r_0\|^2_{W_2^1(\mathcal{G})}\big\}, \quad t>0, \quad (2.21)$$

where β_1, $c>0$ are independent of t.

Proof. In order to prove (2.21), we multiply the first equation in problem (2.6) by $\boldsymbol{w}$ and integrate by parts. As a result, using the boundary conditions and the self-adjointness of $\mathcal{B}_0^{\pm}(r)$, we have energy relations

$$\begin{aligned}
0 &= \int_{\mathcal{F}} \big(\overline{\rho}\mathcal{D}_t \boldsymbol{w}\cdot\boldsymbol{w} - \nabla\cdot\mathbb{T}(\boldsymbol{w},q)\cdot\boldsymbol{w}\big)\,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}}\overline{\rho}|\boldsymbol{w}|^2\,\mathrm{d}x + \int_{\mathcal{F}}\overline{\mu}|\mathbb{S}(\boldsymbol{w})|^2\,\mathrm{d}x \\
&- \int_{\mathcal{G}^-}\mathbb{T}(\boldsymbol{w},p)\boldsymbol{N}\cdot\boldsymbol{w}\,\mathrm{d}\mathcal{G} - \int_{\mathcal{G}^+}[\mathbb{T}(\boldsymbol{w},p)\boldsymbol{N}]\big|_{\mathcal{G}^+}\cdot\boldsymbol{w}\,\mathrm{d}\mathcal{G} \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}}\overline{\rho}|\boldsymbol{w}|^2\,\mathrm{d}x + \int_{\mathcal{F}}\overline{\mu}|\mathbb{S}(\boldsymbol{w})|^2\,\mathrm{d}x + \int_{\mathcal{G}^+}\mathcal{B}_0^+(r)\boldsymbol{w}\cdot\boldsymbol{N}\,\mathrm{d}\mathcal{G} + \int_{\mathcal{G}^-}\mathcal{B}_0^-(r)\boldsymbol{w}\cdot\boldsymbol{N}\,\mathrm{d}\mathcal{G} \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\int_{\mathcal{F}}\overline{\rho}|\boldsymbol{w}|^2\,\mathrm{d}x + \int_{\mathcal{G}^+} r\mathcal{B}_0^+(r)\,\mathrm{d}\mathcal{G} + \int_{\mathcal{G}^-} r\mathcal{B}_0^-(r)\,\mathrm{d}\mathcal{G}\Big\} + \int_{\mathcal{F}}\overline{\mu}|\mathbb{S}(\boldsymbol{w})|^2\,\mathrm{d}x.
\end{aligned} \tag{2.22}$$

Making the same but with $\boldsymbol{W}\in W_2^1(\mathcal{F})$ such that

$$\nabla\cdot\boldsymbol{W} = 0 \quad \text{in} \quad \mathcal{F}, \quad \boldsymbol{W}\cdot\boldsymbol{N}|_{\mathcal{G}^{\pm}} = r^{\pm},$$

$$\begin{aligned}
&\|\boldsymbol{W}\|_{W_2^1(\mathcal{F})} \leqslant c\|r\|_{W_2^{1/2}(\mathcal{G})}, \\
&\|\mathcal{D}_t\boldsymbol{W}\|_{\mathcal{F}} \leqslant c\|\mathcal{D}_t r\|_{\mathcal{G}} \leqslant c\|\boldsymbol{w}\cdot\boldsymbol{N}\|_{\mathcal{G}},
\end{aligned}$$

we obtain

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}}\overline{\rho}\boldsymbol{w}\cdot\boldsymbol{W}\,\mathrm{d}x - \int_{\mathcal{F}}\overline{\rho}\boldsymbol{w}\cdot\mathcal{D}_t\boldsymbol{W}\,\mathrm{d}x + 2\omega\int_{\mathcal{F}}(\boldsymbol{e}_3\times\boldsymbol{w})\cdot\boldsymbol{W}\,\mathrm{d}x \\
&+ \int_{\mathcal{G}^-}\boldsymbol{W}\cdot\boldsymbol{N}\mathcal{B}_0^-(r)\,\mathrm{d}\mathcal{G} + \int_{\mathcal{G}^+}\mathcal{B}_0^+(r)\boldsymbol{W}\cdot\boldsymbol{N}\,\mathrm{d}\mathcal{G} + \int_{\mathcal{F}}\overline{\mu}\mathbb{S}(\boldsymbol{w}):\mathbb{S}(\boldsymbol{W})\,\mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}}\overline{\rho}\boldsymbol{w}\cdot\boldsymbol{W}\,\mathrm{d}x - \int_{\mathcal{F}}\overline{\rho}\boldsymbol{w}\cdot\mathcal{D}_t\boldsymbol{W}\,\mathrm{d}x + 2\omega\int_{\mathcal{F}}(\boldsymbol{e}_3\times\boldsymbol{w})\cdot\boldsymbol{W}\,\mathrm{d}x \\
&+ \int_{\mathcal{F}}\overline{\mu}\mathbb{S}(\boldsymbol{w}):\mathbb{S}(\boldsymbol{W})\,\mathrm{d}x + \int_{\mathcal{G}^-} r^-\mathcal{B}_0^-(r)\,\mathrm{d}\mathcal{G} + \int_{\mathcal{G}^+} r^+\mathcal{B}_0^+(r)\,\mathrm{d}\mathcal{G}.
\end{aligned} \tag{2.23}$$

Since $\int_{\mathcal{G}^{\pm}} r^{\pm}\,\mathrm{d}\mathcal{G} = 0$ due to Proposition 2.1, such $\boldsymbol{W}$ exists.

Now we estimate generalized energy. We multiply (2.23) by small $\gamma > 0$ and add to (2.22), which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + \mathcal{E}_1(t) = 0,$$

where $\mathcal{E}$ is the function of generalized energy,

$$\mathcal{E}=\frac{1}{2}\Big(\int\limits_{\mathcal{F}}\bar{\rho}|\boldsymbol{w}|^2\,\mathrm{d}x+\int\limits_{\mathcal{G}^+}r\mathcal{B}_0^+(r)\,\mathrm{d}\mathcal{G}+\int\limits_{\mathcal{G}^-}r\mathcal{B}_0^-(r)\,\mathrm{d}\mathcal{G}+\gamma\int\limits_{\mathcal{F}}\bar{\rho}\boldsymbol{w}\cdot\boldsymbol{W}\,\mathrm{d}x\Big),$$

$$\mathcal{E}_1=\int\limits_{\mathcal{F}}\bar{\mu}|\mathbb{S}(\boldsymbol{w})|^2\,\mathrm{d}x+\gamma\Big(2\omega\int\limits_{\mathcal{F}}(\boldsymbol{e}_3\times\boldsymbol{w})\cdot\boldsymbol{W}\,\mathrm{d}x-\int\limits_{\mathcal{F}}\bar{\rho}\boldsymbol{w}\cdot\mathcal{D}_t\boldsymbol{W}\,\mathrm{d}x$$
$$+\int\limits_{\mathcal{F}}\bar{\mu}\mathbb{S}(\boldsymbol{w}):\mathbb{S}(\boldsymbol{W})\,\mathrm{d}x+\int\limits_{\mathcal{G}^-}r^-\mathcal{B}_0^-(r)\,\mathrm{d}\mathcal{G}+\int\limits_{\mathcal{G}^+}r^+\mathcal{B}_0^+(r)\,\mathrm{d}\mathcal{G}\Big).$$

By virtue (2.20), we have

$$c_3\big\{\|\boldsymbol{w}\|^2_{\mathcal{F}}+\|r\|^2_{W_2^1(\mathcal{G})}\big\}\leqslant\mathcal{E}\leqslant c_4\big\{\|\boldsymbol{w}\|^2_{\mathcal{F}}+\|r\|^2_{W_2^1(\mathcal{G})}\big\}.$$

In view of Corollary 2.1, $\boldsymbol{w}=\boldsymbol{w}^\perp+\sum\limits_{i=1}^{3}d_i(r)\boldsymbol{\eta}_i(x)\equiv\boldsymbol{w}^\perp+\boldsymbol{w}'$ and, hence,

$$\|\sqrt{\bar{\rho}}\boldsymbol{w}\|^2_{\mathcal{F}}=\|\sqrt{\bar{\rho}}\boldsymbol{w}^\perp\|^2_{\mathcal{F}}+\|\sqrt{\bar{\rho}}\boldsymbol{w}'\|^2_{\mathcal{F}},$$

where $\|\sqrt{\bar{\rho}}\boldsymbol{w}'\|^2_{\mathcal{F}}=\sum\limits_{k,j=1}^{3}d_kd_jS_{kj}=\sum\limits_{j=1}^{3}S_jd_j^2$, $S_{kj}=\int\limits_{\mathcal{F}}\bar{\rho}\boldsymbol{\eta}_k\cdot\boldsymbol{\eta}_j\,\mathrm{d}x$, $S_j\equiv S_{jj}$, and d_j, $j=1,2,3$, are defined by (2.9). It is easily seen that $\|\sqrt{\bar{\rho}}\boldsymbol{w}'\|^2_{\mathcal{F}}$ is a positive quadratic form with respect to r. Consequently,

$$c_5\big\{\|\boldsymbol{w}^\perp\|^2_{\mathcal{F}}+\|r\|^2_{W_2^1(\mathcal{G})}\big\}\leqslant\mathcal{E}\leqslant c_6\big\{\|\boldsymbol{w}^\perp\|^2_{\mathcal{F}}+\|r\|^2_{W_2^1(\mathcal{G})}\big\}.$$

Next, we apply the Korn inequality, valid for the functions orthogonal to all rigid displacement vectors [24],

$$c_7\|\nabla\boldsymbol{w}^\perp\|^2_{\mathcal{F}}\leqslant c_8\|\sqrt{\bar{\mu}}\mathbb{S}(\boldsymbol{w}^\perp)\|^2_{\mathcal{F}}=c_8\|\sqrt{\bar{\mu}}\mathbb{S}(\boldsymbol{w})\|^2_{\mathcal{F}}.$$

Then we can use the Poincaré inequality

$$c_9\|\boldsymbol{w}^\perp\|^2_{\mathcal{F}}\leqslant c_{10}\|\bar{\rho}\boldsymbol{w}^\perp\|^2_{\mathcal{F}}\leqslant c_{11}\|\bar{\rho}\nabla\boldsymbol{w}^\perp\|^2_{\mathcal{F}}$$

since

$$0=\int\limits_{\mathcal{F}}\bar{\rho}\boldsymbol{w}\,\mathrm{d}x=\int\limits_{\mathcal{F}}\bar{\rho}\boldsymbol{w}^\perp\,\mathrm{d}x$$

due to (2.8) and (1.9).

Hence, by the Hölder inequality, for small enough γ, we have

$$\mathcal{E}_1\geqslant 2\beta_1\mathcal{E}$$

with some $\beta_1 > 0$. Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + 2\beta_1\mathcal{E}(t) \leqslant 0$$

which implies

$$\mathcal{E} \leqslant e^{-2\beta_1 t}\mathcal{E}(0)$$

and inequality (2.21). □

Remark 2.2. *We observe that condition* (2.20) *coincides with the positiveness of the second variation of the potential energy*

$$G(r) = \sigma^+|\Gamma_t^+| + \sigma^-|\Gamma_t^-| - \frac{\omega^2}{2}\int\limits_{\Omega_t}\rho^\pm|x'|^2\,\mathrm{d}x - \frac{\varkappa}{2}\int\limits_{\Omega_t}\rho^\pm U\,\mathrm{d}x - p_0^+|\Omega_t^+| - p_0^-|\Omega_t^-|$$

for given volumes of $\Omega_t^\pm$. One can calculate it by (2.2)*:*

$$\delta_0^2 G(r) = \int\limits_{\mathcal{G}^-}\Big\{\sigma^-(|\nabla_{\mathcal{G}} r|^2 + 2\mathcal{K}r^2) - \rho^-\frac{\omega^2}{2}\Big(\frac{\partial}{\partial \boldsymbol{N}}|x'|^2 - |x'|^2\mathcal{H}\Big)r^2 + p_0^-\mathcal{H}r^2\Big\}\mathrm{d}\mathcal{G}$$

$$+\int\limits_{\mathcal{G}^+}\Big\{\sigma^+(|\nabla_{\mathcal{G}} r|^2 + 2\mathcal{K}r^2) - [\bar\rho]\big|_{\mathcal{G}^+}\frac{\omega^2}{2}\Big(\frac{\partial}{\partial \boldsymbol{N}}|x'|^2 - |x'|^2\mathcal{H}\Big)r^2 + (p_0^+ - p_0^-)\mathcal{H}r^2\Big\}\mathrm{d}\mathcal{G}$$

$$-\varkappa\rho^-\int\limits_{\mathcal{G}^-}\Big(\frac{\partial\mathcal{U}}{\partial\boldsymbol{N}} - \mathcal{H}\mathcal{U}\Big)r^2\,\mathrm{d}\mathcal{G} - \varkappa[\bar\rho]\big|_{\mathcal{G}^+}\int\limits_{\mathcal{G}^+}\Big(\frac{\partial\mathcal{U}}{\partial\boldsymbol{N}} - \mathcal{H}\mathcal{U}\Big)r^2\,\mathrm{d}\mathcal{G}$$

$$-\varkappa\rho^-\int\limits_{\mathcal{G}^-}r(x)\mathcal{W}[r](x.t)\,\mathrm{d}\mathcal{G}_x - \varkappa[\bar\rho]\big|_{\mathcal{G}^+}\int\limits_{\mathcal{G}^+}r(x)\mathcal{W}[r](x.t)\,\mathrm{d}\mathcal{G}_x$$

(see [7, 9]). Due to equations (1.8)*, this yields*

$$\begin{aligned}\delta_0^2 G(r) = &\int\limits_{\mathcal{G}^-}\Big\{\sigma^-|\nabla_{\mathcal{G}} r|^2 + \Big(\sigma^-(2\mathcal{K} - \mathcal{H}^2) - \rho^-\omega^2\boldsymbol{N}\cdot\boldsymbol{x}'\Big)r^2\Big\}\,\mathrm{d}\mathcal{G}\\ &+\int\limits_{\mathcal{G}^+}\Big\{\sigma^+|\nabla_{\mathcal{G}} r|^2 + \Big(\sigma^+(2\mathcal{K} - \mathcal{H}^2) - [\bar\rho]\big|_{\mathcal{G}^+}\omega^2\boldsymbol{N}\cdot\boldsymbol{x}'\Big)r^2\Big\}\,\mathrm{d}\mathcal{G}\\ &-\varkappa\rho^-\int\limits_{\mathcal{G}^-}\frac{\partial\mathcal{U}}{\partial\boldsymbol{N}}r^2\,\mathrm{d}\mathcal{G} - \varkappa[\bar\rho]\big|_{\mathcal{G}^+}\int\limits_{\mathcal{G}^+}\frac{\partial\mathcal{U}}{\partial\boldsymbol{N}}r^2\,\mathrm{d}\mathcal{G}\\ &-\varkappa\rho^-\int\limits_{\mathcal{G}^-}r(x)\mathcal{W}[r](x.t)\,\mathrm{d}\mathcal{G}_x - \varkappa[\bar\rho]\big|_{\mathcal{G}^+}\int\limits_{\mathcal{G}^+}r(x)\mathcal{W}[r](x.t)\,\mathrm{d}\mathcal{G}_x.\end{aligned}$$

The nonnegativity of the second variation of the potential $G(r)$ on the subspace of r satisfying orthogonality conditions (2.7) guarantees weak lower semicontinuity of it whence together with the coerciveness of the potential it follows the existence of a minimum. It is clear that the minimum realizes at $r=0$ which implies the stability of equilibrium figures $\mathcal{F}$ and $\mathcal{F}^+$ given by (1.8) that are the Euler equations for the potential $G(r)$.

This approach corresponds to the variational setting for stability problem of the boundaries $\mathcal{G}^{\pm}$.

Theorem 2.2 (Global Solvability of the Linear Homogeneous Problem). *If estimate (2.20) is valid for the functional $R_0(r)$ defined by (2.19) then problem (2.6) with $\boldsymbol{w}_0 \in W_2^{1+l}(\mathcal{F})$, $r_0 \in W_2^{2+l}(\mathcal{G})$, $l\in(1/2,1)$, satisfying compatibility conditions*

$$\begin{gathered}\nabla\cdot\boldsymbol{w}_0=0,\quad [\boldsymbol{w}_0]|_{\mathcal{G}^+}=0,\quad [\mu^{\pm}\Pi_{\mathcal{G}}\mathbb{S}(\boldsymbol{w}_0)\boldsymbol{N}]|_{\mathcal{G}^+}=0,\\ \mu^-\Pi_{\mathcal{G}}\mathbb{S}(\boldsymbol{w}_0)\boldsymbol{N}|_{\mathcal{G}^-}=0,\end{gathered}\tag{2.24}$$

and orthogonality conditions (2.7), (2.8), has a unique solution $(\boldsymbol{w},p,r)$ such that $\boldsymbol{w}\in\boldsymbol{W}_2^{2+l,1+l/2}(D_\infty)$, $p\in\boldsymbol{W}_2^{l,l/2}(D_\infty)$, $\nabla p\in\boldsymbol{W}_2^{l,l/2}(D_\infty)$, $r(\cdot,t)\in W_2^{2+l}(\mathcal{G})$ for any $t\in(0,\infty)$. This solution is subjected to the inequality

$$\begin{aligned}&\|e^{\beta t}\boldsymbol{w}\|_{\boldsymbol{W}_2^{2+l,1+\frac{l}{2}}(D_\infty)}+\|e^{\beta t}\nabla p\|_{\boldsymbol{W}_2^{l,\frac{l}{2}}(D_\infty)}\\ &\quad+\|e^{\beta t}p\|_{W_2^{l,\frac{l}{2}}(D_\infty)}+\|e^{\beta t}r\|^2_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(G_\infty)}+\|e^{\beta t}\mathcal{D}_t r\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(G_\infty)}\\ &\qquad\leqslant c\big\{\|\boldsymbol{w}_0\|_{\boldsymbol{W}_2^{1+l}(\mathcal{F})}+\|r_0\|_{W_2^{2+l}(\mathcal{G})}\big\}\end{aligned}\tag{2.25}$$

with certain $\beta>0$ and the constant c independent of T.

For obtaining bounds for higher order norms of the solution similar to (2.21), we invoke a local-in-time estimate of the solution.

Proposition 2.4. *Let $T>2$. The solution of problem (2.6), (2.7), (2.8) is subject to the inequality*

$$\begin{aligned}&\|\boldsymbol{w}\|_{\boldsymbol{W}_2^{2+l,1+\frac{l}{2}}(D_{t_0-1,t_0})}+\|\nabla p\|_{\boldsymbol{W}_2^{l,\frac{l}{2}}(D_{t_0-1,t_0})}+\|p\|_{W_2^{l,\frac{l}{2}}(D_{t_0-1,t_0})}\\ &\quad+\|r\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(G_{t_0-1,t_0})}+\|\mathcal{D}_t r\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(G_{t_0-1,t_0})}\\ &\qquad\leqslant c\big\{\|\boldsymbol{w}\|_{Q_{t_0-2,t_0}}+\|r\|_{G_{t_0-2,t_0}}\big\},\end{aligned}$$

where $2<t_0\leqslant T$, $D_{t_1,t_2}=\mathcal{F}\times(t_1,t_2)$, $Q_{t_1,t_2}=\Omega\times(t_1,t_2)$, $\Omega=\overline{\mathcal{F}^+}\cup\mathcal{F}^-$, $G_{t_1,t_2}=\mathcal{G}\times(t_1,t_2)$.

The proof of this statement is similar to that of Prop. 2.4 in [14].

Proof of Theorem 2.2. By Theorem 2.1 and Proposition 2.4, one has

$$e^{2\beta(T-j)}\Big\{\|\boldsymbol{w}\|^2_{\boldsymbol{W}_2^{2+l,1+l/2}(D_{T-j-1,T-j})} + \|\nabla p\|^2_{\boldsymbol{W}_2^{l,l/2}(D_{T-j-1,T-j})} + \|p\|^2_{W_2^{l,l/2}(D_{T-j-1,T-j})} + \|r\|^2_{W_2^{5/2+l,5/4+l/2}(G_{T-j-1,T-j})} + \|\mathcal{D}_t r\|^2_{W_2^{3/2+l,3/4+l/2}(G_{T-j-1,T-j})}\Big\} \leqslant ce^{2\beta(T-j)}\Big\{\|\boldsymbol{w}\|^2_{Q_{T-j-2,T-j}} + \|r\|^2_{G_{T-j-2,T-j}}\Big\},$$
$$j = 0, 1, \dots, [T]-2. \quad (2.26)$$

Taking the sum of (2.26) from $j = 0$ to $j = [T]–2$, we obtain the inequality which implies

$$Y^2_{T-[T]+1,T}(e^{\beta t}\boldsymbol{w}, e^{\beta t}p, e^{\beta t}r) \leqslant c \int\limits_{T-[T]}^{T} e^{2\beta t}\Big(\|\boldsymbol{w}(\cdot,t)\|^2_\Omega + \|r(\cdot,t)\|^2_{\mathcal{G}}\Big)\,\mathrm{d}t, \quad (2.27)$$

where

$$Y_{t_1,t_2}(\boldsymbol{u}, q, \rho) = \|\boldsymbol{u}\|_{\boldsymbol{W}_2^{2+l,1+l/2}(D_{t_1,t_2})} + \|\nabla q\|_{\boldsymbol{W}_2^{l,l/2}(D_{t_1,t_2})} + \|q\|_{W_2^{l,l/2}(D_{t_1,t_2})} + \|\rho\|_{W_2^{5/2+l,5/4+l/2}(G_{t_1,t_2})} + \|\mathcal{D}_t\rho\|_{W_2^{3/2+l,3/4+l/2}(G_{t_1,t_2})}.$$

By adding the estimate

$$Y^2_{0,2}(\boldsymbol{w}, p, r) \leqslant c\Big\{\|\boldsymbol{w}_0\|^2_{\boldsymbol{W}_2^{1+l}(\mathcal{F})} + \|r_0\|^2_{W_2^{2+l}(\mathcal{G})}\Big\}$$

to (2.27), choosing $\beta < \beta_1$ and making use of (2.21), we arrive at an inequality equivalent to (2.25). □

§3. The nonlinear problem

After transformation (2.1) and separating the normal and tangent parts in the boundary conditions, in view of (2.3) and (2.5), problem (1.11) can be written in the form [11]:

$$
\begin{aligned}
&\bar\rho\big(\mathcal{D}_t\boldsymbol{u}+2\omega(\boldsymbol{e}_3\times\boldsymbol{u})\big)-\bar\mu\nabla^2\boldsymbol{u}+\nabla q=\bar\rho\hat{\boldsymbol{f}}+\boldsymbol{l}_1(\boldsymbol{u},q,r),\\
&\qquad\qquad\nabla\cdot\boldsymbol{u}=l_2(\boldsymbol{u},r)\equiv\nabla\cdot\boldsymbol{L}(\boldsymbol{u},r)\qquad\text{in }\mathcal{F},\ \ t>0,\\
&\mu^-\Pi_{\mathcal{G}}\mathbb{S}(\boldsymbol{u})\boldsymbol{N}=\boldsymbol{l}_3^-(\boldsymbol{u},r)\quad\text{on }\mathcal{G}^-,\\
&[\bar\mu\Pi_{\mathcal{G}}\mathbb{S}(\boldsymbol{u})\boldsymbol{N}]\big|_{\mathcal{G}^+}=\boldsymbol{l}_3^+(\boldsymbol{u},r)\quad\text{on }\mathcal{G}^+,\\
&-q+\mu^-\boldsymbol{N}\cdot\mathbb{S}(\boldsymbol{u})\boldsymbol{N}+\mathcal{B}_0^-r=l_4^-(\boldsymbol{u},r)+l_5^-(r)\quad\text{on }\mathcal{G}^-,\\
&[\boldsymbol{u}]\big|_{\mathcal{G}^+}=0,\quad[-q+\bar\mu\boldsymbol{N}\cdot\mathbb{S}(\boldsymbol{u})\boldsymbol{N}]\big|_{\mathcal{G}^+}+\mathcal{B}_0^+r=l_4^+(\boldsymbol{u},r)+l_5^+(r)\quad\text{on }\mathcal{G}^+,\\
&\mathcal{D}_tr-\boldsymbol{u}\cdot\boldsymbol{N}=l_6(\boldsymbol{u},r)\quad\text{on }\mathcal{G},\\
&\boldsymbol{u}\big|_{t=0}=\boldsymbol{u}_0\quad\text{in }\mathcal{F},\quad r\big|_{t=0}=r_0\quad\text{on }\mathcal{G},
\end{aligned}
\tag{3.1}
$$

where

$$
\boldsymbol{u}(z,t)=\tilde{\boldsymbol{v}}\big(e_r(z,t),t\big),\ \ \boldsymbol{u}_0(z)=\tilde{\boldsymbol{v}}\big(e_{r_0}(z,0),0\big),\ \ q(z,t)=\tilde p\big(e_r(z,t),t\big),
$$

$$
\hat{\boldsymbol{f}}(z,t)=\tilde{\boldsymbol{f}}\big(e_r(z,t),t\big),
$$

$$
\begin{aligned}
&\boldsymbol{l}_1(\boldsymbol{u},q,r)=\bar\mu(\widetilde\nabla^2-\nabla^2)\boldsymbol{u}+(\nabla-\widetilde\nabla)q+\bar\rho\mathcal{D}_tr^*(\mathcal{L}^{-1}\boldsymbol{N}^*\cdot\nabla)\boldsymbol{u}-\bar\rho(\mathcal{L}^{-1}\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\\
&l_2(\boldsymbol{u},r)=(\mathcal{I}-\widehat{\mathcal{L}}^T)\nabla\cdot\boldsymbol{u}=\nabla\cdot\boldsymbol{L}(\boldsymbol{u},r),\quad\boldsymbol{L}(\boldsymbol{u},r)=(\mathcal{I}-\widehat{\mathcal{L}})\boldsymbol{u},\\
&\boldsymbol{l}_3^-(\boldsymbol{u},r)=\mu^-\Pi_{\mathcal{G}}\big(\Pi_{\mathcal{G}}\mathbb{S}(\boldsymbol{u})\boldsymbol{N}-\tilde\Pi\widetilde{\mathbb{S}}(\boldsymbol{u})\tilde{\boldsymbol{n}}(e_r)\big),\\
&\boldsymbol{l}_3^+(\boldsymbol{u},r)=\big[\bar\mu\Pi_{\mathcal{G}}\big(\Pi_{\mathcal{G}}\mathbb{S}(\boldsymbol{u})\boldsymbol{N}-\tilde\Pi\widetilde{\mathbb{S}}(\boldsymbol{u})\tilde{\boldsymbol{n}}(e_r)\big)\big]\big|_{\mathcal{G}^+},\\
&l_4^-(\boldsymbol{u},r)=\mu^-\big(\boldsymbol{N}\cdot\mathbb{S}(\boldsymbol{u})\boldsymbol{N}-\tilde{\boldsymbol{n}}(e_r)\cdot\widetilde{\mathbb{S}}(\boldsymbol{u})\tilde{\boldsymbol{n}}(e_r)\big),\\
&l_4^+(\boldsymbol{u},r)=\big[\bar\mu\big(\boldsymbol{N}\cdot\mathbb{S}(\boldsymbol{u})\boldsymbol{N}-\tilde{\boldsymbol{n}}(e_r)\cdot\widetilde{\mathbb{S}}(\boldsymbol{u})\tilde{\boldsymbol{n}}(e_r)\big)\big]\big|_{\mathcal{G}^+},\\
&l_5^-(r)=\sigma^-\int\limits_0^1(1-s)\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big(\mathcal{L}^{-T}(z,sr)\nabla_{\mathcal{G}}\cdot\frac{\widehat{\mathcal{L}}^T(z,sr)\boldsymbol{N}}{|\widehat{\mathcal{L}}^T(z,sr)\boldsymbol{N}|}\Big)\,\mathrm{d}s+\frac{\omega^2}{2}\rho^-|\boldsymbol{N}'|^2r^2\\
&\qquad+\varkappa\rho^-\int\limits_0^1(1-s)\frac{\mathrm{d}^2}{\mathrm{d}s^2}\widetilde U(e_{sr}(z),t)\,\mathrm{d}s,\\
&l_5^+(r)=\sigma^+\int\limits_0^1(1-s)\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big(\mathcal{L}^{-T}(z,sr)\nabla_{\mathcal{G}}\cdot\frac{\widehat{\mathcal{L}}^T(z,sr)\boldsymbol{N}}{|\widehat{\mathcal{L}}^T(z,sr)\boldsymbol{N}|}\Big)\,\mathrm{d}s+\frac{\omega^2}{2}[\bar\rho]\big|_{\mathcal{G}^+}|\boldsymbol{N}'|^2r^2
\end{aligned}
\tag{3.2}
$$

$$+\varkappa[\bar\rho]\big|_{\mathcal{G}^+}\int\limits_0^1 (1-s)\frac{\mathrm{d}^2}{\mathrm{d}s^2}\widetilde{U}(e_{sr}(z),t)\,\mathrm{d}s,$$

$$l_6(\boldsymbol{u},r)=\Big(\frac{\widehat{\mathcal{L}}^T\boldsymbol{N}}{\boldsymbol{N}\cdot\widehat{\mathcal{L}}^T\boldsymbol{N}}-\boldsymbol{N}\Big)\cdot\boldsymbol{u},$$

$\mathcal{I}$ is the identity matrix, $\mathcal{L}$ is the Jacobi matrix of transformation (2.1), $L\equiv\det\mathcal{L}$, $\widehat{\mathcal{L}}\equiv L\mathcal{L}^{-1}$. Clearly,

$$\mathcal{L}(z,r)=\Big\{\delta_j^i+\frac{\partial\big(r(z,t)N_i(z)\big)}{\partial z_j}\Big\}_{i,j=1}^3 \qquad \tilde{\boldsymbol{n}}=\frac{\widehat{\mathcal{L}}^T(z,r)\boldsymbol{N}}{|\widehat{\mathcal{L}}^T(z,r)\boldsymbol{N}|};$$

$\widetilde{\nabla}=\mathcal{L}^{-T}\nabla$ is the transformed gradient ∇_x ("T" means transposition), $\widetilde{\mathbb{S}}(\boldsymbol{u})=\widetilde{\nabla}\boldsymbol{u}+(\widetilde{\nabla}\boldsymbol{u})^T$ is the transformed doubled rate-of-strain tensor; $\widetilde{\Pi}\boldsymbol{b}=\boldsymbol{b}-\tilde{\boldsymbol{n}}\cdot\boldsymbol{b}\tilde{\boldsymbol{n}}$ is the projection of a vector $\boldsymbol{b}$ on the tangent plane to $\widetilde{\Gamma}_t$, $\nabla_{\mathcal{G}}=\Pi_{\mathcal{G}}\nabla$.

We assume the fulfillment of restrictions (1.4). Conditions (1.12), (1.13) can be expressed in terms of r as follows (see [17])

$$\begin{aligned}
&\int\limits_{\mathcal{G}^\pm}\varphi^\pm(z,r)\,\mathrm{d}\mathcal{G}=0,\ \ \rho^-\int\limits_{\mathcal{G}^-}\boldsymbol{\psi}^-(z,r)\,\mathrm{d}\mathcal{G}+[\bar\rho]\big|_{\mathcal{G}^+}\int\limits_{\mathcal{G}^+}\boldsymbol{\psi}^+(z,r)\,\mathrm{d}\mathcal{G}=0,\\
&\int\limits_{\mathcal{F}}\bar\rho\boldsymbol{u}(z,t)L(z,r)\,\mathrm{d}z=0,\\
&\int\limits_{\mathcal{F}}\bar\rho\boldsymbol{u}(z,t)\cdot\boldsymbol{\eta}_j(e_r)L(z,r)\,\mathrm{d}z+\omega\int\limits_{\mathcal{F}}\bar\rho\boldsymbol{\eta}_3(e_r)\cdot\boldsymbol{\eta}_j(e_r)L(z,r)\,\mathrm{d}z\\
&\qquad\qquad=\int\limits_{\mathcal{F}}\bar\rho\boldsymbol{\eta}_3(z)\cdot\boldsymbol{\eta}_j(z)\,\mathrm{d}z,\quad j=1,2,3,
\end{aligned}\tag{3.3}$$

where

$$\varphi^\pm(z,r)=r-\frac{r^2}{2}\mathcal{H}^\pm(z)+\frac{r^3}{3}\mathcal{K}^\pm(z),$$
$$\boldsymbol{\psi}^\pm(z,r)=\varphi^\pm(z,r)\boldsymbol{z}+\boldsymbol{N}(z)\Big(\frac{r^2}{2}-\frac{r^3}{3}\mathcal{H}^\pm(z)+\frac{r^4}{4}\mathcal{K}^\pm(z)\Big).$$

Proposition 3.1. *For arbitrary numbers $l^\pm$, vectors $\boldsymbol{l},\boldsymbol{m},\boldsymbol{M}=(M_1,M_2,M_3)$, a function $f_0\in W_2^l(\mathcal{F})$ and a vector field $\boldsymbol{b}_0\in W_2^{l+1/2}(\mathcal{G})$, there exist*

$r \in W_2^{2+l}(\mathcal{G})$ *and* $\boldsymbol{u} \in \boldsymbol{W}_2^{1+l}(\mathcal{F})$ *satisfying the conditions*

$$\int\limits_{\mathcal{G}^-} r(z)\,\mathrm{d}\mathcal{G} = l^-, \qquad \int\limits_{\mathcal{G}^+} r(z)\,\mathrm{d}\mathcal{G} = l^+,$$

$$\rho^- \int\limits_{\mathcal{G}^-} r(z)\boldsymbol{z}\,\mathrm{d}\mathcal{G} + [\bar{\rho}]|_{\mathcal{G}^+} \int\limits_{\mathcal{G}^+} r(z)\boldsymbol{z}\,\mathrm{d}\mathcal{G} = \boldsymbol{l},$$

$$\int\limits_{\mathcal{F}} \bar{\rho}\boldsymbol{u}(z)\,\mathrm{d}z = \boldsymbol{m},$$

$$\int\limits_{\mathcal{F}} \bar{\rho}\boldsymbol{u}(z)\cdot\boldsymbol{\eta}_j(z)\,\mathrm{d}z + \omega\Big(\rho^- \int\limits_{\mathcal{G}^-} r(z)\boldsymbol{\eta}_3(z)\cdot\boldsymbol{\eta}_j(z)\,\mathrm{d}\mathcal{G}$$

$$+ [\bar{\rho}]|_{\mathcal{G}^+} \int\limits_{\mathcal{G}^+} r(z)\boldsymbol{\eta}_3(z)\cdot\boldsymbol{\eta}_j(z)\,\mathrm{d}\mathcal{G}\Big) = M_j, \quad j = 1,2,3,$$

$$\nabla\cdot\boldsymbol{u} = f_0 \quad \text{in } \mathcal{F}, \quad \boldsymbol{b}_0\cdot\boldsymbol{n}_0 = 0 \ \text{on } \mathcal{G}^\pm,$$

$$\mu^-\Pi_{\mathcal{G}}\mathbb{S}(\boldsymbol{u})\boldsymbol{N} = \boldsymbol{b}_0 \quad \text{on } \mathcal{G}^-, \ [\boldsymbol{u}]|_{\mathcal{G}^+} = 0, \quad [\bar{\mu}\Pi_{\mathcal{G}}\mathbb{S}(\boldsymbol{u})\boldsymbol{N}]|_{\mathcal{G}^+} = \boldsymbol{b}_0 \ \text{on } \mathcal{G}^+,$$

and the inequality

$$\|r\|_{W_2^{2+l}(\mathcal{G})} + \|\boldsymbol{u}\|_{W_2^{1+l}(\mathcal{F})}$$
$$c\Big(|l^+| + |l^-| + |\boldsymbol{l}| + |\boldsymbol{m}| + |\boldsymbol{M}| + \|f_0\|_{W_2^l(\mathcal{F})} + \|\boldsymbol{b}_0\|_{W_2^{l+1/2}(\mathcal{G})}\Big).$$

This proposition was proved in [14].

Theorem 3.1 (**Local Solvability of the Nonlinear Problem**). *Let* $T_0 < \infty$ *and let compatibility conditions*

$$\begin{aligned} &\nabla\cdot\boldsymbol{u}_0 = l_2(\boldsymbol{u}_0, r_0) \ \text{in } \mathcal{F}, \ \mu^-\Pi_{\mathcal{G}}\mathbb{S}(\boldsymbol{u}_0)\boldsymbol{N} = \boldsymbol{l}_3^-(\boldsymbol{u}_0, r_0) \ \text{on } \mathcal{G}^-, \\ &[\boldsymbol{u}_0]\big|_{\mathcal{G}^+} = 0, \ [\bar{\mu}\Pi_{\mathcal{G}}\mathbb{S}(\boldsymbol{u}_0)\boldsymbol{N}]\big|_{\mathcal{G}^+} = \boldsymbol{l}_3^+(\boldsymbol{u}_0, r_0) \ \text{on } \mathcal{G}^+ \end{aligned} \tag{3.4}$$

be satisfied. Then there exists a value $\varepsilon(T_0) \ll 1$ *such that problem* (3.1) *with small data:*

$$\|\boldsymbol{u}_0\|_{\boldsymbol{W}_2^{1+l}(\mathcal{F})} + \|r_0\|_{W_2^{2+l}(\mathcal{G})} + \|\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(Q_{T_0})} + \|\nabla\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(Q_{T_0})} \leqslant \varepsilon, \tag{3.5}$$

has a unique solution $(\boldsymbol{u}, q, r)$ *on the interval* $(0, T_0]$ *and the inequalities*

$$Y_{0,T_0}(\boldsymbol{u}, q, r) \leqslant c(\varepsilon)\Big\{N(\boldsymbol{u}_0, r_0) + \|\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(Q_{T_0})}\Big\}, \tag{3.6}$$

$$N(\boldsymbol{u}(\cdot,T_0),r(\cdot,T_0)) \leqslant \vartheta N(\boldsymbol{u}_0,r_0) + c\|\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(Q_{T_0})}, \tag{3.7}$$

hold, where $\vartheta < 1/2$,

$$\begin{aligned} Y_{0,T}(\boldsymbol{u},q,r) \equiv &\|\boldsymbol{u}\|_{W_2^{2+l,1+l/2}(D_T)} + \|\nabla q\|_{W_2^{l,l/2}(D_T)} + \|q\|_{W_2^{l,l/2}(D_T)} \\ &+ \|r\|_{W_2^{5/2+l,5/4+l/2}(G_T)} + \|\mathcal{D}_t r\|_{W_2^{3/2+l,3/4+l/2}(G_T)}. \end{aligned}$$

and

$$N(\boldsymbol{w},\rho) \equiv \|\boldsymbol{w}\|_{\boldsymbol{W}_2^{1+l}(\mathcal{F})} + \|\rho\|_{W_2^{2+l}(\mathcal{G})}.$$

The proof of Theorem 3.1 relies on Theorem 2.1 and on the following estimates of the nonlinear terms.

Proposition 3.2. *If*

$$\|r(\cdot,t)\|_{W_2^{3/2+l}(\mathcal{G})} + \|\mathcal{D}_t r(\cdot,t)\|_{W_2^{1/2+l}(\mathcal{G})} + \|\boldsymbol{u}(\cdot,t)\|_{\mathcal{F}} \leqslant \delta, \quad t \leqslant T, \tag{3.8}$$

where δ *is a certain small positive number, then nonlinear terms* (3.2) *and* $\hat{\boldsymbol{f}}(z,t) \equiv \tilde{\boldsymbol{f}}(e_r(z,t),t)$ *are subject to the inequalities*

$$\begin{aligned} Z_{0,T}(\boldsymbol{u},q,r) &\equiv \|\boldsymbol{l}_1(\boldsymbol{u},r)\|_{\boldsymbol{W}_2^{l,\frac{l}{2}}(D_T)} + \|l_2(\boldsymbol{u},r)\|_{W_2^{1+l,0}(D_T)} \\ &+ \|\boldsymbol{L}(\boldsymbol{u},r)\|_{W_2^{0,1+\frac{l}{2}}(D_T)} + \|\boldsymbol{l}_3(\boldsymbol{u},r)\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(G_T)} + |l_4(\boldsymbol{u},r)|_{G_T}^{(\frac{1}{2}+l,\frac{l}{2})} \\ &+ |l_5(r)|_{G_T}^{(\frac{1}{2}+l,\frac{l}{2})} + \|l_6(\boldsymbol{u},r)\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(G_T)} \leqslant cY_{0,T}^2(\boldsymbol{u},q,r), \end{aligned} \tag{3.9}$$

$$\begin{aligned} &\|\hat{\boldsymbol{f}}\|_{\boldsymbol{W}_2^{l,\frac{l}{2}}(Q_T)} \\ &\leqslant c\Big\{\|\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,\frac{l}{2}}(Q_T)} + \sup_{t<T}\Big(\|\mathcal{D}_t r(\cdot,t)\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})} + \|\boldsymbol{u}(\cdot,t)\|_{\Omega}\Big)\|\nabla \boldsymbol{f}\|_{Q_T}\Big\}. \end{aligned}$$

If $(\boldsymbol{u},r)$ *and* $(\boldsymbol{u}',r')$ *satisfy* (3.8), *then*

$$\begin{aligned} Z_{0,T}(\boldsymbol{u}-\boldsymbol{u}',q-q',r-r') &\leqslant c\delta Y_{0,T}(\boldsymbol{u}-\boldsymbol{u}',q-q',r-r'), \\ \|\widetilde{\boldsymbol{f}}-\widetilde{\boldsymbol{f}}'\|_{\boldsymbol{W}_2^{l,l/2}(Q_T)} &\leqslant c\delta Y_{0,T}(\boldsymbol{u}-\boldsymbol{u}',q-q',r-r'), \end{aligned}$$

where $\hat{\boldsymbol{f}}' = \tilde{\boldsymbol{f}}(e_{r'}(z,t),t)$.

A similar statement was proved in [11] (Proposition 3.2). See also [18] and [9].

The main result of the paper is as follows.

Theorem 3.2 (**Global Solvability of the Nonlinear Problem**). *Let* $\varkappa \geqslant 0, [\bar\rho]\big|_{\mathcal{G}^+} > 0$ *and, in addition, let* $\boldsymbol{f} \in W_2^{l,l/2}(Q_\infty)$, $\boldsymbol{u}_0 \in W_2^{1+l}(\mathcal{F})$, $r_0 \in W_2^{2+l}(\mathcal{G})$, $l \in (1/2, 1)$. *We assume that compatibility conditions* (3.4) *and smallness one*

$$\|\boldsymbol{u}_0\|_{W_2^{1+l}(\mathcal{F})} + \|r_0\|_{W_2^{2+l}(\mathcal{G})} \leqslant \varepsilon \ll 1, \tag{3.10}$$

as well as inequality (2.20), *restrictions* (3.3) *at* $t = 0$ *and* (1.4) *hold. Moreover, we assume that* $\boldsymbol{f}$ *has small finite norms*:

$$\|e^{\beta t}\boldsymbol{f}\|_{W_2^{l,l/2}(Q_\infty)} \leqslant \varepsilon, \quad \beta > 0, \quad \sup_{\tau>0}\|\mathcal{D}_x^{\boldsymbol{i}}\boldsymbol{f}\|_{Q_{\tau,\tau+T_0}} \leqslant \varepsilon, \quad |\boldsymbol{i}| = 1, 2, \tag{3.11}$$

where $Q_\infty = \Omega \times (0,\infty)$, $T_0 > 2$ *is an appropriate fixed number.*

Then problem (3.1) *has a unique solution defined in the infinite time interval* $t > 0$ *and*

$$\begin{aligned} &\|\mathrm{e}^{\alpha t}\boldsymbol{u}\|_{W_2^{2+l,1+l/2}(D_\infty)} + \|\mathrm{e}^{\alpha t}\nabla q\|_{W_2^{l,l/2}(D_\infty)} + \|\mathrm{e}^{\alpha t} q\|_{W_2^{l/2}(0,\infty;W_2^{1/2}(\mathcal{G}))} \\ &+ \|\mathrm{e}^{\alpha t} q\|_{W_2^{l,l/2}(D_\infty)} + \|\mathrm{e}^{\alpha t} r\|_{W_2^{5/2+l,5/4+l/2}(G_\infty)} + \|\mathrm{e}^{\alpha t}\mathcal{D}_t r\|^2_{W_2^{3/2+l,3/4+l/2}(G_\infty)} \\ &\qquad \leqslant c_1(\varepsilon)\Big\{\|\mathrm{e}^{at}\boldsymbol{f}\|_{W_2^{l,l/2}(Q_\infty)} + \|\boldsymbol{u}_0\|_{W_2^{1+l}(\mathcal{F})} + \|r_0\|_{W_2^{2+l}(\mathcal{G})}\Big\} \end{aligned} \tag{3.12}$$

with a certain $0 < \alpha < \beta$; $c_1(\varepsilon)$ *is a bounded function of* ε.

We note that a similar theorem in the case of $\varkappa = 0$ was proved in [14], the restriction $[\bar\rho]\big|_{\mathcal{G}^+} > 0$ being not necessary.

Proof. We outline the main ideas of the proof.

A solution to (3.1) is sought in the form of the sum

$$\boldsymbol{u} = \boldsymbol{u}' + \boldsymbol{u}'', \quad q = q' + q'', \quad r = r' + r''.$$

We write conditions (3.3) in the form

$$\begin{aligned}
&\int\limits_{\mathcal{G}^\pm} r^\pm \,\mathrm{d}\mathcal{G} = \int\limits_{\mathcal{G}^\pm} (r^\pm - \varphi(z,r^\pm))\,\mathrm{d}\mathcal{G}, \quad \varphi(z,r^\pm) = \varphi^\pm(z,r^\pm) \quad \text{on } \mathcal{G}^\pm,\\
&\rho^- \int\limits_{\mathcal{G}^-} r^- \boldsymbol{z}\,\mathrm{d}\mathcal{G} + [\bar\rho]|_{\mathcal{G}^+} \int\limits_{\mathcal{G}^+} r^+ \boldsymbol{z}\,\mathrm{d}\mathcal{G} = \rho^- \int\limits_{\mathcal{G}^-} \big(r^- \boldsymbol{z} - \boldsymbol{\psi}^-(z,r^-)\big)\,\mathrm{d}\mathcal{G}\\
&\qquad + [\bar\rho]|_{\mathcal{G}^+} \int\limits_{\mathcal{G}^+} \big(r^+ \boldsymbol{z} - \boldsymbol{\psi}^+(z,r^+)\big)\,\mathrm{d}\mathcal{G}, \qquad\qquad (3.13)\\
&\int\limits_{\mathcal{F}} \bar\rho \boldsymbol{u}\,\mathrm{d}z = \int\limits_{\mathcal{F}} \bar\rho \boldsymbol{u}\big(1 - L(z,r)\big)\,\mathrm{d}z,\\
&\int\limits_{\mathcal{F}} \bar\rho \boldsymbol{u}\cdot\boldsymbol{\eta}_j(z)\,\mathrm{d}z + \omega\Big(\rho^- \int\limits_{\mathcal{G}_i} r\boldsymbol{\eta}_3\cdot\boldsymbol{\eta}_j\,\mathrm{d}\mathcal{G} + [\bar\rho]|_{\mathcal{G}^+} \int\limits_{\mathcal{G}_i} r\boldsymbol{\eta}_3\cdot\boldsymbol{\eta}_j\,\mathrm{d}\mathcal{G}\Big)\\
&\quad = \omega\Big(\rho^- \int\limits_{\mathcal{G}_i} r\boldsymbol{\eta}_3\cdot\boldsymbol{\eta}_j\,\mathrm{d}\mathcal{G} + [\bar\rho]|_{\mathcal{G}^+} \int\limits_{\mathcal{G}_i} r\boldsymbol{\eta}_3\cdot\boldsymbol{\eta}_j\,\mathrm{d}\mathcal{G} - \int\limits_{\widetilde\Omega_t} \rho^\pm \boldsymbol{\eta}_3(y)\cdot\boldsymbol{\eta}_j(y)\,\mathrm{d}y\Big)\\
&\qquad + \int\limits_{\mathcal{F}} \bar\rho\boldsymbol{u}\cdot\boldsymbol{\eta}_j(z)\big(1 - L(z,r)\big)\,\mathrm{d}z + \int\limits_{\mathcal{F}} \bar\rho\boldsymbol{\eta}_3(z)\cdot\boldsymbol{\eta}_j(z)\,\mathrm{d}z, \quad j = 1,2,3,
\end{aligned}$$

and construct the functions $\boldsymbol{u}_0'', r_0''$ satisfying the relations (see Proposition 3.1)

$$\begin{aligned}
&\int\limits_{\mathcal{G}^\pm} r_0^{\pm''}\,\mathrm{d}\mathcal{G} = \int\limits_{\mathcal{G}^\pm} \big(r_0^\pm - \varphi(z,r_0^\pm)\big)\,\mathrm{d}\mathcal{G},\\
&\rho^- \int\limits_{\mathcal{G}^-} r_0^{-''} \boldsymbol{z}\,\mathrm{d}\mathcal{G} + [\bar\rho]|_{\mathcal{G}^+} \int\limits_{\mathcal{G}^+} r_0^{+''} \boldsymbol{z}\,\mathrm{d}\mathcal{G} = \rho^- \int\limits_{\mathcal{G}^-} \big(r_0^- \boldsymbol{z} - \boldsymbol{\psi}^-(z,r_0^-)\big)\,\mathrm{d}\mathcal{G}\\
&\qquad + [\bar\rho]\big|_{\mathcal{G}^+} \int\limits_{\mathcal{G}^+} \big(r_0^+ \boldsymbol{z} - \boldsymbol{\psi}^+(z,r_0^+)\big)\,\mathrm{d}\mathcal{G},\\
&\int\limits_{\mathcal{F}} \bar\rho \boldsymbol{u}_0''\,\mathrm{d}z = \int\limits_{\mathcal{F}} \bar\rho \boldsymbol{u}_0\big(1 - L(z,r_0)\big)\,\mathrm{d}z,\\
&\int\limits_{\mathcal{F}} \bar\rho \boldsymbol{u}_0''\cdot\boldsymbol{\eta}_j(z)\,\mathrm{d}z + \omega\Big(\rho^- \int\limits_{\mathcal{G}^-} r_0''\boldsymbol{\eta}_3\cdot\boldsymbol{\eta}_j\,\mathrm{d}\mathcal{G} + [\bar\rho]\big|_{\mathcal{G}^+} \int\limits_{\mathcal{G}^+} r_0''\boldsymbol{\eta}_3\cdot\boldsymbol{\eta}_j\,\mathrm{d}\mathcal{G}\Big)
\end{aligned}$$

$$= \omega\Big(\rho^- \int\limits_{\mathcal{G}^-} r_0 \boldsymbol{\eta}_3 \cdot \boldsymbol{\eta}_j \, \mathrm{d}\mathcal{G} + [\bar\rho]\big|_{\mathcal{G}^+} \int\limits_{\mathcal{G}_i} r_0 \boldsymbol{\eta}_3 \cdot \boldsymbol{\eta}_j \, \mathrm{d}\mathcal{G} - \int\limits_{\widetilde{\Omega}_0} \rho^{\pm} \boldsymbol{\eta}_3(y) \cdot \boldsymbol{\eta}_j(y) \, \mathrm{d}y\Big)$$

$$+ \int\limits_{\mathcal{F}} \bar\rho \boldsymbol{u}_0 \cdot \boldsymbol{\eta}_j(z)\big(1 - L(z,r)\big) \, \mathrm{d}z + \int\limits_{\mathcal{F}} \bar\rho \boldsymbol{\eta}_3(z) \cdot \boldsymbol{\eta}_j(z) \, \mathrm{d}z, \quad j = 1,2,3,$$

$$\begin{aligned} &\nabla \cdot \boldsymbol{u}_0'' = l_2(\boldsymbol{u}_0, r_0) \quad \text{in } \mathcal{F}, \\ &\mu^- \Pi_{\mathcal{G}} \mathbb{S}(\boldsymbol{u}_0'') \boldsymbol{N}\big|_{\mathcal{G}^-} = \boldsymbol{l}_3^-(\boldsymbol{u}_0, r_0), \\ &[\boldsymbol{u}_0'']\big|_{\mathcal{G}^+} = 0, \quad [\bar\mu \Pi_{\mathcal{G}} \mathbb{S}(\boldsymbol{u}_0'') \boldsymbol{N}]\big|_{\mathcal{G}^+} = \boldsymbol{l}_3^+(\boldsymbol{u}_0, r_0). \end{aligned} \tag{3.14}$$

Then we set $\boldsymbol{u}_0' \equiv \boldsymbol{u}_0 - \boldsymbol{u}_0''$, $r_0' \equiv r_0 - r_0''$ and define $(\boldsymbol{u}', q', r')$ as a solution to the problem

$$\begin{aligned} &\bar\rho\big(\mathcal{D}_t \boldsymbol{u}'(z,t) + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{u}')\big) - \bar\mu \nabla^2 \boldsymbol{u}' + \nabla q' = 0, \ \ \nabla \cdot \boldsymbol{u}' = 0 \text{ in } \mathcal{F}, \\ &\mu^- \Pi_{\mathcal{G}} \mathbb{S}(\boldsymbol{u}') \boldsymbol{N}\big|_{\mathcal{G}^-} = 0, \ \ -q' + \mu^- \boldsymbol{N} \cdot \mathbb{S}(\boldsymbol{u}') \boldsymbol{N} + \mathcal{B}_0^- r' = 0 \text{ on } \mathcal{G}^-, \\ &[\boldsymbol{u}']\big|_{\mathcal{G}^+} = 0, \ \ \big[\bar\mu \Pi_{\mathcal{G}} \mathbb{S}(\boldsymbol{u}') \boldsymbol{N}(z)\big]\big|_{\mathcal{G}^+} = 0, \\ &\big[-q' + \mu^- \boldsymbol{N} \cdot \mathbb{S}(\boldsymbol{u}') \boldsymbol{N}\big]\big|_{\mathcal{G}^+} + \mathcal{B}_0^+ r' = 0 \text{ on } \mathcal{G}^+, \\ &\mathcal{D}_t r' - \boldsymbol{u}' \cdot \boldsymbol{N} = 0 \text{ on } \mathcal{G}, \\ &\boldsymbol{u}'(z,0) = \boldsymbol{u}_0'(z), \ \ z \in \mathcal{F}, \ \ r'(z,0) = r_0'(z), \ \ z \in \mathcal{G}. \end{aligned} \tag{3.15}$$

We note that the initial data $\boldsymbol{u}_0'$, r_0' satisfy (2.7), (2.8) and homogeneous compatibility conditions (2.24).

Finally, we find $(\boldsymbol{u}'', q'', r'')$ as a solution to the system

$$\begin{aligned} &\bar\rho\big(\mathcal{D}_t \boldsymbol{u}'' + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{u}'')\big) - \bar\mu \nabla^2 \boldsymbol{u}'' + \nabla q'' = \bar\rho \hat{\boldsymbol{f}} + \boldsymbol{l}_1(\boldsymbol{u}' + \boldsymbol{u}'', q' + q'', r' + r''), \\ &\nabla \cdot \boldsymbol{u}'' = l_2(\boldsymbol{u}' + \boldsymbol{u}'', r' + r'') \quad \text{in } \mathcal{F}, \ \ t > 0, \\ &\mu^- \Pi_{\mathcal{G}} \mathbb{S}(\boldsymbol{u}'') \boldsymbol{N} = \boldsymbol{l}_3^-(\boldsymbol{u}' + \boldsymbol{u}'', r' + r'') \quad \text{on } \mathcal{G}^-, \\ &[\boldsymbol{u}'']\big|_{\mathcal{G}^+} = 0, \quad \big[\bar\mu \Pi_{\mathcal{G}} \mathbb{S}(\boldsymbol{u}'') \boldsymbol{N}\big]\big|_{\mathcal{G}^+} = \boldsymbol{l}_3^+(\boldsymbol{u}' + \boldsymbol{u}'', r' + r'') \quad \text{on } \mathcal{G}^+, \\ &-q'' + \mu^- \boldsymbol{N} \cdot \mathbb{S}(\boldsymbol{u}'') \boldsymbol{N} + \mathcal{B}_0^- r'' = l_4^-(\boldsymbol{u}' + \boldsymbol{u}'', r' + r'') + l_5^-(r' + r'') \text{ on } \mathcal{G}^-, \\ &[-q'' + \bar\mu \boldsymbol{N} \cdot \mathbb{S}(\boldsymbol{u}'') \boldsymbol{N}]|_{\mathcal{G}^+} + \mathcal{B}_0^+ r'' = l_4^+(\boldsymbol{u}' + \boldsymbol{u}'', r' + r'') + l_5^+(r' + r'') \text{ on } \mathcal{G}^+, \\ &\mathcal{D}_t r'' - \boldsymbol{u}'' \cdot \boldsymbol{N} = l_6(\boldsymbol{u}' + \boldsymbol{u}'', r' + r'') \text{ on } \mathcal{G}, \\ &\boldsymbol{u}''\big|_{t=0} = \boldsymbol{u}_0'' \quad \text{in } \mathcal{F}, \quad r''\big|_{t=0} = r_0'' \quad \text{on } \mathcal{G}. \end{aligned} \tag{3.16}$$

We consider restrictions (3.14). If (3.10) holds, then the expressions

$$l^{\pm}=\int\limits_{\mathcal{G}^{\pm}}\big(r_0^{\pm}-\varphi(z,r_0^{\pm})\big)\,\mathrm{d}\mathcal{G},$$

$$\boldsymbol{l}=\rho^{-}\int\limits_{\mathcal{G}^{-}}\big(r_0^{-}\boldsymbol{z}-\boldsymbol{\psi}^{-}(z,r_0^{-})\big)\,\mathrm{d}\mathcal{G}+[\bar\rho]|_{\mathcal{G}^{+}}\int\limits_{\mathcal{G}^{+}}(r_0^{+}\boldsymbol{z}-\boldsymbol{\psi}^{+}(z,r_0^{+}))\,\mathrm{d}\mathcal{G},$$

$$\boldsymbol{m}=\int\limits_{\mathcal{F}}\rho\boldsymbol{u}_0\big(1-L(z,r_0)\big)\,\mathrm{d}z,$$

$$M_j=\int\limits_{\mathcal{F}}\rho\boldsymbol{u}_0\cdot\boldsymbol{\eta}_j\big(1-L(z,r_0)\big)\,\mathrm{d}z,\quad j=1,2,3,$$

and the functions $f_0=l_2(\boldsymbol{u}_0,r_0)$, $\boldsymbol{b}_0(z)=\boldsymbol{l}_3^{\pm}(\boldsymbol{u}_0,r_0),\ z\in\mathcal{G}^{\pm}$, satisfy the inequality

$$|l^{+}|+|l^{-}|+|\boldsymbol{l}|+|\boldsymbol{m}|+|\boldsymbol{M}|+\|f_0\|_{W_2^{l}(\cup\mathcal{F}^{\pm})}+\|\boldsymbol{b}_0\|_{W_2^{l+1/2}(\mathcal{G})}\leqslant c\varepsilon\big(\|\boldsymbol{u}_0\|_{W_2^{1+l}(\mathcal{F})}+\|r_0\|_{W_2^{2+l}(\mathcal{G})}\big).$$

Hence

$$\|\boldsymbol{u}_0''\|_{W_2^{1+l}(\mathcal{F})}+\|r_0''\|_{W_2^{2+l}(\mathcal{G})}\leqslant c\varepsilon\big(\|\boldsymbol{u}_0\|_{W_2^{1+l}(\mathcal{F})}+\|r_0\|_{W_2^{2+l}(\mathcal{G})}\big),$$
$$\|\boldsymbol{u}_0'\|_{W_2^{1+l}(\mathcal{F})}+\|r_0'\|_{W_2^{2+l}(\mathcal{G})}\leqslant c\big(\|\boldsymbol{u}_0\|_{W_2^{1+l}(\mathcal{F})}+\|r_0\|_{W_2^{2+l}(\mathcal{G})}\big).$$

Moreover, in view of (3.13), (3.14), $\boldsymbol{u}_0',r_0'$ is subject to the necessary conditions

$$\int\limits_{\mathcal{G}^{\pm}}r_0^{'\pm}\,\mathrm{d}\mathcal{G}=\int\limits_{\mathcal{G}^{\pm}}(r_0-r_0^{''\pm})\,\mathrm{d}\mathcal{G}=\int\limits_{\mathcal{G}^{\pm}}\varphi(z,r_0)\,\mathrm{d}S=0,$$

$$\rho^{-}\int\limits_{\mathcal{G}^{-}}r_0^{'-}z_j\,\mathrm{d}\mathcal{G}+[\bar\rho]\big|_{\mathcal{G}^{+}}\int\limits_{\mathcal{G}^{+}}r_0^{'+}z_j\,\mathrm{d}\mathcal{G}=\rho^{-}\int\limits_{\mathcal{G}^{-}}\psi_j^{-}(z,r_0)\,\mathrm{d}\mathcal{G}+[\bar\rho]\big|_{\mathcal{G}^{+}}\int\limits_{\mathcal{G}^{+}}\psi_j^{'+}\,\mathrm{d}\mathcal{G}=0,$$

$$\int\limits_{\mathcal{F}}\bar\rho\boldsymbol{u}_0'\,\mathrm{d}\mathcal{G}=0,$$

$$\int\limits_{\mathcal{F}}\bar\rho\boldsymbol{u}_0'\cdot\boldsymbol{\eta}_j(z)\,\mathrm{d}z+\omega\Big(\rho^{-}\int\limits_{\mathcal{G}^{-}}r_0^{'-}\boldsymbol{\eta}_3\cdot\boldsymbol{\eta}_j\,\mathrm{d}\mathcal{G}+[\bar\rho]|_{\mathcal{G}^{+}}\int\limits_{\mathcal{G}^{+}}r_0^{'+}\boldsymbol{\eta}_3\cdot\boldsymbol{\eta}_j\,\mathrm{d}S\Big)=0.$$

By Theorem 2.2, the solution $(\boldsymbol{u}', q', r')$ of problem (3.15) satisfies the inequality

$$N\big(\boldsymbol{u}'(\cdot,T), r'(\cdot,T)\big) \equiv \|\boldsymbol{u}'(\cdot,T)\|_{W_2^{1+l}(\mathcal{F})} + \|r'(\cdot,T)\|_{W_2^{2+l}(\mathcal{G})} \leqslant c_1 \mathrm{e}^{-\beta T}\big\{\|\boldsymbol{u}_0\|_{W_2^{1+l}(\mathcal{F})} + \|r_0\|_{W_2^{2+l}(\mathcal{G})}\big\}.$$

We fix $T = T_0$ so large that

$$c_1 e^{-\beta T_0} \leqslant \theta/2 \ll 1/2, \qquad \beta > 0.$$

As for the problem (3.16), it is solved by iterations, as in [11], on the basis of Theorem 2.1 and estimate (3.9) of nonlinear terms (3.2):

$$Z_{0,T}(\boldsymbol{u}, q, r) \leqslant c Y_{0,T}^2(\boldsymbol{u}, q, r).$$

Thus, if ε is small enough, by inequalities (3.6) and (3.7), we obtain

$$Y_{0,T_0}(\boldsymbol{u}'', q'', r'') \leqslant c_2(\varepsilon)\big(\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_{T_0})} + \|\boldsymbol{u}_0\|_{W_2^{1+l}(\mathcal{F})} + \|r_0\|_{W_2^{2+l}(\mathcal{G})}\big),$$

$$N\big(\boldsymbol{u}(\cdot,T_0), r(\cdot,T_0)\big) \leqslant N\big(\boldsymbol{u}'(\cdot,T_0), r'(\cdot,T_0)\big) + N\big(\boldsymbol{u}''(\cdot,T_0), r''(\cdot,T_0)\big) \leqslant (\theta/2 + \vartheta)\big(\|\boldsymbol{u}_0\|_{W_2^{1+l}(\mathcal{F})} + \|r_0\|_{W_2^{2+l}(\mathcal{G})}\big) + c\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_{T_0})}.$$

We set $\lambda \equiv \theta/2 + \vartheta < 1$, due to (3.10), this implies

$$\begin{aligned} &Y_{0,T_0}(\boldsymbol{u}, q, r) \leqslant c\Big(\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_{T_0})} + \|\boldsymbol{u}_0\|_{W_2^{1+l}(\mathcal{F})} + \|r_0\|_{W_2^{2+l}(\mathcal{G})}\Big) \leqslant c\varepsilon, \\ &N\big(\boldsymbol{u}(\cdot,T_0), r(\cdot,T_0)\big) \leqslant \lambda\big(\|\boldsymbol{u}_0\|_{W_2^{1+l}(\mathcal{F})} + \|r_0\|_{W_2^{2+l}(\mathcal{G})}\big) \\ &\qquad + c\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_{T_0})} \leqslant C\varepsilon. \end{aligned} \tag{3.17}$$

Inequalities (3.17) allow us to extend the solution $(\boldsymbol{u}, q, r)$ to the intervals $(T_0, 2T_0)$, ..., $(kT_0, (k+1)T_0)$, ... up to the infinite interval $t > 0$ by the repeated applications of the obtained local result and to complete the proof of Theorem 3.2, as in [11].

Let us suppose the solution is already found for $t \leqslant kT_0$. Then it can be defined for $t \in (kT_0, (k+1)T_0]$ as a solution to the problem with the initial conditions $\boldsymbol{u}(z, kT_0) = \boldsymbol{u}(z, kT_0 - 0) \equiv \boldsymbol{u}_k(z)$, $r(z, kT_0) = r(z, kT_0 - 0) \equiv r_k(z)$.

Let us consider the case $k = 1$. From (3.5) and (3.6) it follows that

$$N_1 \equiv N(\boldsymbol{u}_1, r_1) \leqslant C\varepsilon,$$

hence, by replacing ε with $C^{-1}\varepsilon$, we see that this problem is solvable in the time interval $(T_0, 2T_0]$ and, by (3.17), the estimates

$$Y_1(\boldsymbol{u}, q, r) \leqslant c\Big\{N_1 + \|\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(Q_{T_0,2T_0})}\Big\},$$
$$N_2 \leqslant \lambda N_1 + c\|\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(Q_{T_0,2T_0})} \leqslant C\varepsilon,$$

are satisfied, where

$$N_k \equiv N_{T_0}(\boldsymbol{u}_k, r_k), \qquad Y_k(\boldsymbol{u}, q, r) \equiv Y_{kT_0,(k+1)T_0}(\boldsymbol{u}, q, r).$$

If the solution is found for $t \leqslant kT_0$ and the inequalities

$$N_j^2 \leqslant \lambda^2 N_{j-1}^2 + c\|\boldsymbol{f}\|^2_{W_2^{l,l/2}(Q_{(j-1)T_0,jT_0})}, \quad \lambda < 1, \tag{3.18}$$
$$Y_j^2 \leqslant c\Big\{N_j^2 + \|\boldsymbol{f}\|^2_{W_2^{l,l/2}(Q_{(j-1)T_0,jT_0})}\Big\}, \quad j = 1, ..., k-1,$$

are proved, then

$$N_j^2 \leqslant ... \leqslant \lambda^{2j} N_0^2 + c\sum_{i=0}^{j-1} \lambda^{2(j-1-i)}\|\boldsymbol{f}\|^2_{W_2^{l,l/2}(Q_{iT_0,(i+1)T_0})} \leqslant c\lambda^{2(j-1)}\varepsilon^2 \tag{3.19}$$

with the constants c independent of j (we have used inequalities (3.11) for $\boldsymbol{f}$). Since $\lambda^j \to 0$ as $j \to \infty$, the right-hand side of (3.19) is less than ε^2 for $j \geqslant j_0$, and the replacement of ε with $C^{-1}\varepsilon$ can be made only finite number of times.

Let $\lambda_1 > \lambda$ ($\lambda_1 = \mathrm{e}^{-aT_0}$, $a < b$). We take the sum of (3.18) multiplied by λ_1^{-2j}. This leads to

$$\sum_{j=0}^{k} \lambda_1^{-2j} N_j^2 \leqslant N_0^2 + \frac{\lambda^2}{\lambda_1^2}\sum_{j=1}^{k} \lambda_1^{-2j+2} N_{j-1}^2 + c\sum_{j=1}^{k} \lambda_1^{-2j}\|\boldsymbol{f}\|^2_{W_2^{l,l/2}(Q_{(j-1)T_0,jT_0})}$$
$$\leqslant \frac{\lambda_1^2}{\lambda_1^2 - \lambda^2} N_0^2 + \frac{c\lambda_1^2}{\lambda_1^2 - \lambda^2}\sum_{j=1}^{k} \lambda_1^{-2j}\|\boldsymbol{f}\|^2_{W_2^{l,l/2}(Q_{(j-1)T_0,jT_0})}.$$

Finally, by passing to the limit as $k \to \infty$ in

$$\sum_{j=0}^{k} \lambda_1^{-2j} Y_j^2(\boldsymbol{u}, q, r) \leqslant c\Big\{N_0^2 + \sum_{j=0}^{k} \lambda_1^{-2j}\|\boldsymbol{f}\|^2_{W_2^{l,l/2}(Q_{jT_0,(j+1)T_0})}\Big\},$$

we arrive at an inequality equivalent to (3.12).

□

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