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SMALL WEIGHTS IN CACCIOPPOLI'S INEQUALITY AND APPLICATIONS TO LIOUVILLE-TYPE THEOREMS FOR NON-STANDARD PROBLEMS

Abstract. Using a variant of Caccioppoli's inequality involving small weights, i.e. weights of the form $(1 + |\nabla u|^2)^{-\alpha/2}$ for some $\alpha > 0$, we establish several Liouville-type theorems under general non-standard growth conditions.

Dedicated to the 70th birthday of Gregory Seregin

§1. Introduction

Throughout this manuscript we always suppose that $u\colon \mathbb{R}^2 \to \mathbb{R}$, $u \in C^2(\mathbb{R}^2)$, is a solution of the nonlinear equation

$$\operatorname{div}\big[\nabla f(\nabla u)\big] = 0 \quad \text{on } \mathbb{R}^2 . \tag{1.1}$$

Our main goal is the discussion of Liouville-type results under rather general hypotheses on the convex density $f\colon \mathbb{R}^2 \to \mathbb{R}$ including non-standard growth conditions such as the case of linear growth or even allowing a certain degree of anisotropy in the superlinear case. For technical simplicity we restrict ourselves to the twodimensional case.

It is out of reach to give a complete overview on all the recent contributions on Liouville-type results. We refer to the beautiful survey of Farina [1] in the case of general elliptic problems including a lot of historical references. We also refer to Seregin's discussion [2] of Liouville-type theorems in the case of the Navier-Stokes equations.

Contributions in the case of linear or anisotropic growth are quite rarely found. We just mention the papers [3–6] and the references quoted therein.

Before going into details we fix our main assumption which will be supplemented with appropriate hypotheses adapted to the applications of Section 4 and of Section 5.

Key words and phrases: non-standard growth problems, Liouville-type theorems, Caccioppoli's inequality.

Assumption 1.1. *The convex energy density $f\colon \mathbb{R}^2 \to \mathbb{R}$ is of class $C^2(\mathbb{R}^2)$ and satisfies the non-uniform ellipticity condition*

$$c_1\big(1+|Z|^2\big)^{-\frac{\mu}{2}}|Y|^2 \leqslant D^2 f(Z)\big(Y,Y\big) \leqslant c_2\big(1+|Z|^2\big)^{-\frac{\overline{\mu}}{2}}|Y|^2 \qquad (1.2)$$

for all Z, $Y \in \mathbb{R}^2$ with exponents $\mu > 1$, $\overline{\mu} \leqslant 1$ and constants c_1, $c_2 > 0$. We note that $\overline{\mu}$ may be negative, which will be of particular interest in the case of superlinear growth.

Condition (1.2) also serves as one main assumption in the recent paper [6] on Liouville-type results in two dimensions for functionals satisfying a linear growth condition. These results are restricted to the case $\overline{\mu} = 1$, for instance we have:

Theorem 1.1 ([6, Theorem 1,1, c, the case $N = 1$])**.** *Let $u \in C^2(\mathbb{R}^2)$ denote a solution of* (1.1) *with density f such that for some real number $M > 0$ we have*

$$|\nabla f(Z)| \leqslant M \qquad \text{for all} \quad Z \in \mathbb{R}^2$$

and such (1.2) *holds with the choice $\overline{\mu} = 1$.*

If we have

$$\limsup_{|x|\to\infty} \frac{|u(x)|}{|x|} < \infty\,, \qquad (1.3)$$

then u is affine.

Let us give some further explanations concerning the hypotheses of Theorem 1.1: from (1.2) and the boundedness of ∇f it follows that f is of linear growth, and this actually holds for arbitrary exponents $\overline{\mu} \leqslant 1 < \mu$. We refer to [6, Lemma 2.1].

At the same time, if f is of linear growth satisfying inequality (1.2), then according to [7, Lemma 1.1] we necessarily get the restriction $\overline{\mu} \leqslant 1$, whereas the bound $\mu > 1$ is an immediate consequence of the linear growth of f. To sum up, Theorem 1.1 addresses the case of energy densities with linear growth, but just covers the "limit case" for which $\overline{\mu} = 1$. So the first question arises, whether the result of Theorem 1.1 keeps valid, if we allow exponents $\overline{\mu} < 1$.

Closely related is the following setting in the case of superlinear growth. Suppose that we have (1.2) with exponent $\overline{\mu} = 2 - q$, $q > 1$, on the right-hand side. Then we are interested in the anisotropic case, which means that we do not narrow our discussion by assuming q-power growth of f. We just impose the inequality

$$a|Z|^s - b \leqslant f(Z) \qquad \text{for all} \quad Z \in \mathbb{R}^2\,,$$

with exponent $1 < s \leqslant q$ and with constants a, $A > 0$, a, $b \geqslant 0$ as lower bound for the density f (see Section 5 for some further comments on the assumptions). A Liouville-type result in this setting is established in Section 5. We emphasize that we are not aware of similar Liouville-type theorems w.r.t. this general kind of anisotropic hypotheses.

Let us fix the notation.

Notation. We always abbreviate

$$\Gamma := 1 + |\nabla u|^2,\ \ \Gamma_Q := 1 + |\nabla u - Q|^2 \ \ \text{for vectors}\, Q \in \mathbb{R}^2\,.$$

For fixed radii r, $R > 0$ we consider open disks B_r and define

$$T_R := B_{2R} - \overline{B_R}\,, \quad \hat{T}_R := B_{5R/2} - \overline{B_{R/2}}\,,$$

where the center x_0 is not indicated.

Moreover, for $r > 0$ we let

$$(\nabla u) = (\nabla u)_r = \fint_{B_r} \nabla u \,\mathrm{d}x\,.$$

Then we have

Theorem 1.2. *Assume that Assumption* 1.1 *holds. Moreover, suppose that there are real numbers γ, $\gamma_Q \geqslant 0$ such that*

$$\gamma + \gamma_Q < \frac{1}{2} \tag{1.4}$$

and that there exist $Q = Q(R) \in \mathbb{R}^2$ such that

$$\liminf_{R\to\infty} \frac{1}{R^2}\Xi(R) := \liminf_{R\to\infty} \frac{1}{R^2} \int_{T_R} \Gamma^{-\frac{\gamma+\mu}{2}} \Gamma_Q^{-\frac{\gamma_Q}{2}} |\nabla u - Q|^2 \,\mathrm{d}x < \infty\,. \tag{1.5}$$

Then u is an affine function.

We note that Theorem 1.2 just relies on condition (1.2) and it does not matter, whether the energy density f is of linear growth or even shows a completely anisotropic behaviour. Moreover, the conclusion of the theorem is independent of the value of the exponent μ.

Let us shortly comment on the main idea for the proof of Theorem 1.2 recalling the approach towards Theorem 1.1. This theorem is proved by first using a Caccioppoli-type inequality for the differentiated Euler equation. Since we have $\overline{\mu} = 1$ as a hypothesis of Theorem 1.1, the right-hand side of this inequality can be measured in terms of the quantity $\nabla f(\nabla u) \cdot \nabla u$ which in turn occurs on the left-hand side of the weak form of (1.1) applied to a suitable testfunction.

In the case $\overline{\mu} < 1$ a serious gap arises which cannot be closed by obvious arguments. Here, as the main new feature, we introduce small weights in Caccioppoli's inequality such that both sides again fit together. The same arguments bridge the gap in the superlinear anisotropic case.

Theorem 1.2 immediately gives the following elementary corollary.

Corollary 1.1. *Suppose that we have Assumption* 1.1*. Then u is an affine function if one of the following conditions holds.*

i) $\sup\limits_{x\in\mathbb{R}^2} |\nabla u(x)| < \infty$.

ii) *There exists some $\varepsilon > 0$ such that*

$$\fint_{T_R} \Gamma^{\frac{1}{2}\left(\frac{3}{2}-\overline{\mu}\right)+\varepsilon}\, \mathrm{d}x \leqslant c$$

with a finite constant $c > 0$ not depending on R.

Before formulating more refined corollaries, we establish our Caccioppoli-type estimate in the next section as the main tool for proving Theorem 1.2 in Section 3.

Section 4 is devoted to applications in the linear growth setting while Section 5 concentrates on the main corollary in the superlinear case.

We finally note that the generality of Theorem 1.2 may be used to discuss a series of other applications which is left to the particular interest of the reader.

§2. A Caccioppoli-type inequality

We start with a Caccioppoli-type inequality weighted with negative powers of Γ and Γ_Q.

Lemma 2.1. *Given Assumption* 1.1 *we fix $Q \in \mathbb{R}^2$, consider real numbers $s_Q > -1/4$, $s_1 > -1/4$ and let*

$$c_Q := \begin{cases} 4|s_Q| & \text{if } s_Q \leqslant 0\,, \\ 0 & \text{if } s_Q > 0, \end{cases} \qquad c_1 := \begin{cases} 4|s_1| & \text{if } s_1 \leqslant 0\,, \\ 0 & \text{if } s_1 > 0\,, \end{cases}$$

and suppose that $\eta \in C_0^\infty\big(\mathbb{R}^2\big)$, $0 \leqslant \eta \leqslant 1$. *If* $c_Q + c_1 < 1$, *then we have* (*summation w.r.t.* $\alpha = 1, 2$)

$$\begin{aligned}\big[1 - c_Q - c_1\big] &\int_{\mathbb{R}^2} D^2 f(\nabla u)\big(\nabla\partial_\alpha u, \nabla\partial_\alpha u\big)\Gamma_Q^{s_Q}\Gamma^{s_1}\eta^2 \,\mathrm{d}x \\ &\leqslant c\Bigg[\int_{\operatorname{spt}\nabla\eta} D^2 f(\nabla u)\big(\nabla\partial_\alpha u, \nabla\partial_\alpha u\big)\Gamma_Q^{s_Q}\Gamma^{s_1}\eta^2 \,\mathrm{d}x\Bigg]^{\frac{1}{2}} \\ &\cdot\Bigg[\int_{\operatorname{spt}\nabla\eta} D^2 f(\nabla u)\big(\nabla\eta, \nabla\eta\big)\big|\nabla u - Q\big|^2\Gamma_Q^{s_Q}\Gamma^{s_1} \,\mathrm{d}x\Bigg]^{\frac{1}{2}}, \end{aligned} \tag{2.1}$$

where the constant c *is not sdepending on* η. *In particular we have*

$$\begin{aligned}&\int_{\mathbb{R}^2} D^2 f(\nabla u)\big(\nabla\partial_\alpha u, \nabla\partial_\alpha u\big)\Gamma_Q^{s_Q}\Gamma^{s_1}\eta^2 \,\mathrm{d}x \\ &\quad\leqslant c\int_{\operatorname{spt}\nabla\eta} D^2 f(\nabla u)\big(\nabla\eta, \nabla\eta\big)\big|\nabla u - Q\big|^2\Gamma_Q^{s_Q}\Gamma^{s_1} \,\mathrm{d}x\,. \end{aligned} \tag{2.2}$$

Proof. We first consider the case that $-1/4 < s_Q \leqslant 0$. For $\alpha = 1, 2$ and for all $\psi \in C_0^\infty(\mathbb{R}^2)$ equation (1.1) yields

$$0 = \int_{\mathbb{R}^2} D^2 f(\nabla u)\big(\nabla\partial_\alpha u, \nabla\psi\big)\,\mathrm{d}x\,. \tag{2.3}$$

Inserting $\psi := \eta^2\big(\partial_\alpha u - Q_\alpha\big)\Gamma_Q^{s_Q}\Gamma^{s_1}$ in (2.3) we obtain for $\alpha = 1, 2$ and any testfunction η

$$\begin{aligned}&\int_{\mathbb{R}^2} D^2 f(\nabla u)\big(\nabla\partial_\alpha u, \nabla\partial_\alpha u\big)\Gamma_Q^{s_Q}\Gamma^{s_1}\eta^2 \,\mathrm{d}x \\ &= -\int_{\mathbb{R}^2} D^2 f(\nabla u)\big(\nabla\partial_\alpha u, \big(\partial_\alpha u - Q_\alpha\big)\nabla\Gamma_Q^{s_Q}\big)\Gamma^{s_1}\eta^2 \,\mathrm{d}x \\ &- \int_{\mathbb{R}^2} D^2 f(\nabla u)\big(\nabla\partial_\alpha u, \big(\partial_\alpha u - Q_\alpha\big)\nabla\Gamma^{s_1}\big)\Gamma_Q^{s_Q}\eta^2 \,\mathrm{d}x \\ &- 2\int_{\mathbb{R}^2} D^2 f(\nabla u)\big(\nabla\partial_\alpha u, \nabla\eta\big)\big(\partial_\alpha u - Q_\alpha\big)\Gamma_Q^{s_Q}\Gamma^{s_1}\eta \,\mathrm{d}x\,. \end{aligned} \tag{2.4}$$

We denote the bilinear form $D^2f(\cdot,\cdot)$ by $\langle\cdot,\cdot\rangle$ and discuss the left-hand side of (2.4) by observing that

$$\begin{aligned}&\sum_{\alpha=1}^{2}\Big\langle\partial_\alpha\nabla u,\partial_\alpha\nabla u\Big\rangle\Gamma_Q^{s_Q}\\&\geqslant\sum_{\alpha=1}^{2}\Big\langle\partial_\alpha\nabla u,\partial_\alpha\nabla u\Big\rangle\Big[\big(\partial_1u-Q_1\big)^2+\big(\partial_2u-Q_2\big)^2\Big]\Gamma_Q^{s_Q-1}\\&\geqslant\sum_{\alpha=1}^{2}\Big\langle\big(\partial_\alpha u-Q_\alpha\big)\partial_\alpha\nabla u,\big(\partial_\alpha u-Q_\alpha\big)\partial_\alpha\nabla u\Big\rangle\Gamma_Q^{s_Q-1}\,.\end{aligned}\tag{2.5}$$

Moreover, for discussing the first integral on the right-hand side of (2.4) we write

$$\begin{aligned}&\sum_{\alpha=1}^{2}\Big\langle\partial_\alpha\nabla u,\big(\partial_\alpha u-Q_\alpha\big)\nabla\Gamma_Q^{s_Q}\Big\rangle\\&=s_Q\sum_{\alpha=1}^{2}\Big\langle\big(\partial_\alpha u-Q_\alpha\big)\partial_\alpha\nabla u,\nabla\sum_{i=1}^{2}\big(\partial_iu-Q_i\big)^2\Big\rangle\Gamma_Q^{s_Q-1}\\&=2s_Q\sum_{\alpha=1}^{2}\Big\langle\big(\partial_\alpha u-Q_\alpha\big)\partial_\alpha\nabla u,\big(\partial_\alpha u-Q_\alpha\big)\partial_\alpha\nabla u\Big\rangle\Gamma_Q^{s_Q-1}\\&+4s_Q\Big\langle\big(\partial_1u-Q_1\big)\partial_1\nabla u,\big(\partial_2u-Q_2\big)\partial_2\nabla u\Big\rangle\Gamma_Q^{s_Q-1}\,.\end{aligned}\tag{2.6}$$

On account of

$$\begin{aligned}&\Big|\Big\langle\big(\partial_1u-Q_1\big)\partial_1\nabla u,\big(\partial_2u-Q_2\big)\partial_2\nabla u\Big\rangle\Big|\\&\leqslant\frac{1}{2}\sum_{\alpha=1}^{2}\Big\langle\big(\partial_\alpha u-Q_\alpha\big)\partial_\alpha\nabla u,\big(\partial_\alpha u-Q_\alpha\big)\partial_\alpha\nabla u\Big\rangle\end{aligned}$$

we obtain from (2.6)

$$\begin{aligned}&\Big|\sum_{\alpha=1}^{2}\Big\langle\partial_\alpha\nabla u,\big(\partial_\alpha u-Q_\alpha\big)\nabla\Gamma_Q^{s_Q}\Big\rangle\Big|\\&\leqslant4|s_Q|\sum_{\alpha=1}^{2}\Big\langle\big(\partial_\alpha u-Q_\alpha\big)\partial_\alpha\nabla u,\big(\partial_\alpha u-Q_\alpha\big)\partial_\alpha\nabla u\Big\rangle\Gamma_Q^{s_Q-1}\,.\end{aligned}\tag{2.7}$$

Combining (2.7) and (2.5) we get (where from now on we take the sum w.r.t. $\alpha = 1, 2$)

$$-\int_{\mathbb{R}^2} D^2 f(\nabla u)\big(\partial_\alpha \nabla u, \big(\partial_\alpha u - Q_\alpha\big)\nabla \Gamma_Q^{s_Q}\big)\Gamma^{s_1}\eta^2 \,\mathrm{d}x$$
$$\leqslant 4|s_Q| \int_{\mathbb{R}^2} D^2 f(\nabla u)\big(\partial_\alpha \nabla u, \partial_\alpha \nabla u\big)\Gamma_Q^{s_Q}\Gamma^{s_1}\eta^2 \,\mathrm{d}x\,. \tag{2.8}$$

In the case $0 < s_Q$ we just use the positive sign of

$$\Big\langle \partial_\alpha \nabla u\big(\partial_\alpha u - Q_\alpha\big), \nabla \Gamma_Q^{s_Q}\Big\rangle = \frac{1}{2}\Big\langle \nabla\big(\partial_\alpha u - Q_\alpha\big)^2, \nabla \Gamma_Q^{s_Q}\Big\rangle\,. \tag{2.9}$$

Having established (2.8) and (2.9) we recall $c_Q := 4|s_Q|$ if $-1/4 < s_Q \leqslant 0$ and $c_Q = 0$ if $s_Q > 0$. Then we summarize (2.8) and (2.9) by writing

$$-\int_{\mathbb{R}^2} D^2 f(\nabla u)\big(\partial_\alpha \nabla u, \big(\partial_\alpha u - Q_\alpha\big)\nabla \Gamma_Q^{s_Q}\big)\Gamma^{s_1}\eta^2 \,\mathrm{d}x$$
$$\leqslant c_Q \int_{\mathbb{R}^2} D^2 f(\nabla u)\big(\partial_\alpha \nabla u, \partial\alpha \nabla u\big)\Gamma_Q^{s_Q}\Gamma^{s_1}\eta^2 \,\mathrm{d}x\,. \tag{2.10}$$

In the same way we recall $c_1 := 4|s_1|$ if $-1/4 < s_1 \leqslant 0$ and $c_1 = 0$ if $s_1 > 0$. With exactly the same reasoning as above we additionally obtain

$$-\int_{\mathbb{R}^2} D^2 f(\nabla u)\big(\partial_\alpha \nabla u, \big(\partial_\alpha u - Q_\alpha\big)\nabla \Gamma^{s_1}\big)\Gamma_Q^{s_Q}\eta^2 \,\mathrm{d}x$$
$$\leqslant c_1 \int_{\mathbb{R}^2} D^2 f(\nabla u)\big(\partial_\alpha \nabla u, \partial\alpha \nabla u\big)\Gamma_Q^{s_Q}\Gamma^{s_1}\eta^2 \,\mathrm{d}x\,. \tag{2.11}$$

Returning to (2.4) and using (2.10) and (2.11) we get

$$\big[1 - c_\varepsilon - c_1\big] \int_{\mathbb{R}^n} D^2 f(\nabla u)\big(\nabla \partial_\alpha u, \nabla \partial_\alpha u\big)\Gamma_Q^{s}\eta^2 \,\mathrm{d}x$$
$$\leqslant -2 \int_{\mathbb{R}^n} D^2 f(\nabla u)\big(\eta \nabla \partial_\alpha u, \big(\partial_\alpha u - Q_\alpha\big)\nabla \eta\big)\Gamma_Q^{s_Q}\Gamma^{s_1} \,\mathrm{d}x\,. \tag{2.12}$$

On the right-hand side of (2.12) we observe that the integration is performed w.r.t. the domain spt $\nabla\eta$ and an application of the Cauchy-Schwarz inequality completes the proof of Lemma 2.1. □

§3. Proof of Theorem 1.2

For the proof of Theorem 1.2 we fix a disk $B_r \subset \mathbb{R}^2$. We apply the Sobolev-Poincaré inequality to the solution $u \in C^2(\mathbb{R}^2)$ under consideration and get the inequality

$$\begin{aligned}
\int_{B_r} \big|\nabla u - (\nabla u)\big|^2 \,\mathrm{d}x &\leqslant c\Bigg[\int_{B_r} \big|\nabla^2 u\big| \,\mathrm{d}x\Bigg]^2 \\
&\leqslant c\Bigg[\int_{B_r} \Gamma^{-\frac{\mu}{4}}|\nabla^2 u|\Gamma^{\frac{\mu}{4}}\Gamma^{-\frac{\gamma}{4}}\Gamma_Q^{-\frac{\gamma_Q}{4}}\Gamma^{\frac{\gamma}{4}}\Gamma_Q^{\frac{\gamma_Q}{4}} \,\mathrm{d}x\Bigg]^2 \\
&\leqslant c\Bigg[\int_{B_r} \Gamma^{-\frac{\mu}{2}}|\nabla^2 u|^2\Gamma^{-\frac{\gamma}{2}}\Gamma_Q^{-\frac{\gamma_Q}{2}} \,\mathrm{d}x\Bigg]\Bigg[\int_{B_r} \Gamma^{\frac{\mu+\gamma}{2}}\Gamma_Q^{\frac{\gamma_Q}{2}} \,\mathrm{d}x\Bigg] \\
&\leqslant c\int_{B_r} \Gamma^{-\frac{\mu}{2}}|\nabla^2 u|^2\Gamma^{-\frac{\gamma}{2}}\Gamma_Q^{-\frac{\gamma_Q}{2}} \,\mathrm{d}x\,, \qquad (3.1)
\end{aligned}$$

where we used the fact that $|\nabla u|$ is bounded on the fixed ball B_r, however c may depend on the radius r.

Now we choose $R \gg r$ and let $\eta \in C_0^\infty\big(B_{2R}\big)$, $0 \leqslant \eta \leqslant 1$, such that $\eta \equiv 1$ on B_R, $|\nabla\eta| \leqslant c/R$. Then (2.1) of Lemma 2.1 gives recalling (1.4)

$$\begin{aligned}
&\int_{B_r} \Gamma^{-\frac{\mu}{2}}|\nabla^2 u|^2\Gamma^{-\frac{\gamma}{2}}\Gamma_Q^{-\frac{\gamma_Q}{2}} \,\mathrm{d}x \\
&\leqslant \int_{B_{2R}} \Gamma^{-\frac{\mu}{2}}|\nabla^2 u|^2\Gamma^{-\frac{\gamma}{2}}\Gamma_Q^{-\frac{\gamma_Q}{2}}\eta^2 \,\mathrm{d}x \\
&\leqslant c\Bigg[\int_{\operatorname{spt}\eta} D^2 f\big(\nabla\partial_\alpha u, \nabla\partial_\alpha u\big)\Gamma^{-\frac{\gamma}{2}}\Gamma_Q^{-\frac{\gamma_Q}{2}}\eta^2 \,\mathrm{d}x\Bigg]^{\frac{1}{2}} \\
&\cdot\Bigg[\frac{c}{R^2}\int_{T_R} \Gamma^{-\frac{\gamma+\overline{\mu}}{2}}\Gamma_Q^{-\frac{\gamma_Q}{2}}\big|\nabla u - Q\big|^2 \,\mathrm{d}x\Bigg]^{\frac{1}{2}} \\
&=: cI(R)\cdot\Bigg[\frac{1}{R^2}\,\Xi(R)\Bigg]^{\frac{1}{2}}. \qquad (3.2)
\end{aligned}$$

We observe that (2.2) implies (again recalling (1.4))

$$\int\limits_{B_R} D^2 f(\nabla u)\big(\nabla\partial_\alpha u, \nabla\partial_\alpha u\big)\Gamma^{-\frac{\gamma}{2}}\Gamma_Q^{-\frac{\gamma_Q}{2}}\,\mathrm{d}x \leqslant \frac{c}{R^2}\int\limits_{T_R}\Gamma^{-\frac{\gamma+\overline{\mu}}{2}}\Gamma_Q^{-\frac{\gamma_Q}{2}}|\nabla u - Q|^2\,\mathrm{d}x\,, \tag{3.3}$$

hence we can make use of our assumption (1.5) by choosing a suitable subsequence $R \to \infty$ and obtain

$$\int\limits_{\mathbb{R}^2} D^2 f(\nabla u)\big(\nabla\partial_\alpha u, \nabla\partial_\alpha u\big)\Gamma^{-\frac{\gamma}{2}}\Gamma_Q^{-\frac{\gamma_Q}{2}}\,\mathrm{d}x < \infty\,,$$

thus $I(R) \to 0$ as $R \to \infty$.

With this information we return to (3.1), (3.2) and obtain

$$\int\limits_{B_r}\big|\nabla u - (\nabla u)\big|^2\,\mathrm{d}x \leqslant I(R)\cdot\left[\frac{1}{R^2}\,\Xi(R)\right]^{\frac{1}{2}}$$

with $I(R) \to 0$ as $R \to \infty$. This proves the theorem with the help of hypothesis (1.5). □

§4. Applications to the linear growth case

Throughout this section we replace Assumption 1.1 by a suitable stronger variant specifying the linear growth condition. More precisely, we require

Assumption 4.1. *The convex energy density* $f\colon \mathbb{R}^2 \to \mathbb{R}$ *is of class* $C^2(\mathbb{R}^2)$ *and satisfies the non-uniform ellipticity condition* (1.2) *with exponents* $\mu > 1$, $\overline{\mu} \leqslant 1$. *Moreover we assume that there exists a constant* $M > 0$ *such that for all* $Z \in \mathbb{R}^2$

$$|\nabla f(Z)| \leqslant M\,. \tag{4.1}$$

As outlined in [6] (compare the discussion after inequality (1.3) in this reference), Assumption 4.1 implies with constants a, $A > 0$, b, $B \geqslant 0$ and for all $Z \in \mathbb{R}^2$ the linear growth condition

$$a|Z| - b \leqslant f(Z) \leqslant A|Z| + B\,. \tag{4.2}$$

Before summarizing some corollaries of Theorem 1.2 in this particular setting, we will show the following proposition which follows the line of the proof of Theorem 1.1 of [6].

Proposition 4.1. *Suppose that we have Assumption* 4.1*. Then*

$$\limsup_{|x|\to\infty} \frac{|u(x)|}{|x|} < \infty \tag{4.3}$$

implies

$$\int_{T_R} \Gamma^{\frac{1}{2}}\, \mathrm{d}x \leqslant c\big[1+R^2\big]\,. \tag{4.4}$$

Proof of Proposition 4.1. By the convexity of f we have for all $Z \in \mathbb{R}^2$

$$f(Z) \leqslant f(0) + \nabla f(Z)\cdot Z\,, \tag{4.5}$$

hence for any $\eta \in C_0^1(\mathbb{R}^2)$, $\eta \equiv 1$ on T_R, $\operatorname{spt}\eta \subset \hat{T}_R$, $0 \leqslant \eta \leqslant 1$, $|\nabla\eta| \leqslant c/R$, we obtain using (4.1), (4.2) and (4.5)

$$\begin{aligned} \int_{T_R} \Gamma^{\frac{1}{2}}\, \mathrm{d}x &\leqslant c \int_{\hat{T}_R} \big[1+f(\nabla u)\big]\eta^2\, \mathrm{d}x \\ &\leqslant cR^2 + c\int_{\hat{T}_R} \nabla f(\nabla u)\cdot\nabla u \eta^2\, \mathrm{d}x\,. \end{aligned} \tag{4.6}$$

Now we use the weak form of equation (1.1) with testfunction $\psi = u\eta^2$, i.e.

$$\begin{aligned} 0 &= \int_{\mathbb{R}^2} \nabla f(\nabla u)\cdot\nabla\big[u\eta^2\big]\, \mathrm{d}x \\ &= \int_{\hat{T}_R} \nabla f(\nabla u)\cdot\nabla u\eta^2\, \mathrm{d}x + \int_{\hat{T}_R} \nabla f(\nabla u)\cdot\nabla\eta 2\eta u\, \mathrm{d}x\,, \end{aligned}$$

hence we have

$$\int_{\mathbb{R}^2} \nabla f(\nabla u)\cdot\nabla u\eta^2\, \mathrm{d}x \leqslant \frac{c}{R}\sup_{\hat{T}_R}|u|R^2\,. \tag{4.7}$$

Recalling our assumption (4.3) and combing (4.6) and (4.7) we find the claim (4.4) of the proposition. □

The first corollary to Theorem 1.2 immediately yields an extension of Theorem 1.1.

Corollary 4.1. *Theorem* 1.1 *remains valid for* $1/2 < \overline{\mu} \leqslant 1$.

Proof of Corollary 4.1. We choose $\gamma_Q = 0$, γ sufficiently close to $-1/2$ and let $Q = 0$, i.e.

$$\Xi(R) = \int\limits_{T_R} \Gamma^{-\frac{1}{2}} |\nabla u|^2 \,\mathrm{d}x \leqslant \int\limits_{T_R} |\nabla u| \,\mathrm{d}x \,,$$

and apply Proposition 4.1 to obtain hypothesis (1.5) of Theorem 1.2. □

The next corollary gives a refinement of Corollary 4.1 by taking a measure for the relative oscillation into account (compare (4.8) and (4.9)).

Corollary 4.2. *Suppose that we have Assumption* 4.1 *with* $1/2 < \overline{\mu} < 1$. *For given* γ, $\gamma_Q > 0$, $1 - \overline{\mu} < \gamma + \gamma_Q < 1/2$, *we let*

$$p = \frac{1}{2 - \gamma_Q - \gamma - \overline{\mu}} > 1 \,, \quad q = \frac{1}{\gamma_Q + \gamma + \overline{\mu} - 1}$$

and define

$$\Theta_Q(x) := \left[\frac{\Gamma_Q}{\Gamma} \right]^{\frac{2-\gamma_Q}{2} q} \,, \quad \Theta(R) := \inf_Q \frac{1}{|T_R|} \int\limits_{T_R} \Theta_Q(x) \,\mathrm{d}x \leqslant 1 \,. \tag{4.8}$$

If we suppose that for all R *sufficiently large*

$$\sup_{\hat{T}_R} |u| \leqslant cR\Theta(R)^{-\frac{p}{q}} \tag{4.9}$$

with a constant $c > 0$ *not depending on* R, *then* u *is an affine function.*

Proof of Corollary 4.2. We estimate

$$\int\limits_{T_R} \Gamma^{-\frac{\gamma + \overline{\mu}}{2}} \Gamma_Q^{-\frac{\gamma_Q}{2}} |\nabla u - Q|^2 \,\mathrm{d}x \leqslant \int\limits_{T_R} \Theta_Q^{\frac{1}{q}} \Gamma^{\frac{2-\gamma_Q-\gamma-\overline{\mu}}{2}} \,\mathrm{d}x$$

and obtain

$$\begin{aligned} \Xi(R) &\leqslant \int\limits_{T_R} \Theta_Q^{\frac{1}{q}} \Gamma^{\frac{2-\gamma_Q-\gamma-\overline{\mu}}{2}} \,\mathrm{d}x \\ &\leqslant \left[\int\limits_{T_R} \Theta_Q \,\mathrm{d}x \right]^{\frac{1}{q}} \left[\int\limits_{T_R} \Gamma^{\frac{1}{2}} \,\mathrm{d}x \right]^{\frac{1}{p}} . \end{aligned} \tag{4.10}$$

We choose $Q = Q(R)$ such that

$$\frac{1}{|T_R|}\int\limits_{T_R} \Theta_Q \,\mathrm{d}x \leqslant 2\Theta(R)\,.$$

Then (4.10) implies

$$\Xi(R) \leqslant c\big[\Theta(R)R^2\big]^{\frac{1}{q}} \left[\int\limits_{T_R} \Gamma^{\frac{1}{2}}\,\mathrm{d}x\right]^{\frac{1}{p}}. \tag{4.11}$$

Discussing the right-hand side of (4.11) we exactly follow the proof of Proposition 4.1 and just insert hypothesis (4.9) in (4.7). This shows

$$\Xi(R) \leqslant c\big[\Theta(R)R^2\big]^{\frac{1}{q}} \cdot \big[1 + \Theta(R)^{-\frac{p}{q}}R^2\big]^{\frac{1}{p}}\,,$$

hence we obtain Corollary 4.2 by recalling $\frac{1}{p} + \frac{1}{q} = 1$. □

A rather important class of energy densities with linear growth is of splitting-type, i.e. of the form

$$f(Z) = f_1(Z_1) + f_2(Z_2)$$

with functions f_1, f_2 of linear growth. The particular features of splitting type energy densities with linear growth are discussed in [7] (compare also [8]). If f_1, f_2 satisfy (1.2) with exponents $\overline{\mu}_1$, $\overline{\mu}_2 \leqslant 0$, respectively, then we have $\overline{\mu} = 0$ for the energy density f.

Nevertheless we still can derive Liouville-type theorems from Theorem 1.2. In Corollary 4.3 we present an application, where in addition we make explicit use of the flexibility of the vector $Q \in \mathbb{R}^2$ by choosing Q as a mean value. Then a smallness condition imposed on $|\nabla^2 u|$ provides the vanishing of the second derivatives.

Corollary 4.3. *Suppose that we are given Assumption* 4.1 *with* $\overline{\mu} > -1/2$. *If we have for a finite constant c that*

$$\sup_{T_R} |\nabla^2 u| \leqslant cR^{-1}\,, \tag{4.12}$$

then u is an affine function.

Proof of Corollary (4.3). We choose $\gamma = -\overline{\mu} < 1/2$, $\gamma_Q = 0$ such that (1.4) is satisfied. We have to show that (4.12) implies (1.5), i.e. we claim that in this case

$$\Xi(R) = \int_{T_R} |\nabla u - Q|^2 \, \mathrm{d}x \leqslant c\big[1 + R^2\big] \,, \tag{4.13}$$

where we choose $Q = (\nabla u)_R$. In fact, we have by the Poincaré inequality (see [9], Theorem A.10 , as the appropriate variant)

$$\int_{T_R} |\nabla u - Q|^2 \, \mathrm{d}x \leqslant cR^2 \int_{T_R} |\nabla^2 u|^2 \, \mathrm{d}x \leqslant c\Big[\sup_{T_R} |\nabla^2 u|\Big]^2 R^4 \,,$$

which proves the corollary on account of the hypothesis (4.12). □

§5. Applications to the superlinear case

We adapt Assumption 1.1 to the case of superlinear growth, i.e. we now require

Assumption 5.1. *Suppose that we are given numbers $\mu \in \mathbb{R}$, $q > 1$ such that $-\mu \leqslant q - 2$ and let $\overline{\mu} := 2 - q$ be the exponent on the right-hand side of* (1.2)*, i.e. the convex energy density $f\colon \mathbb{R}^2 \to \mathbb{R}$ is of class $C^2(\mathbb{R}^2)$ and satisfies the non-uniform ellipticity condition*

$$c_1\big(1 + |Z|^2\big)^{-\frac{\mu}{2}} |Y|^2 \leqslant D^2 f(Z)\big(Y, Y\big) \leqslant c_2\big(1 + |Z|^2\big)^{\frac{q-2}{2}} |Y|^2 \tag{5.1}$$

for all Z, $Y \in \mathbb{R}^2$ with exponents $\mu > 1$, $q > 1$ and constants c_1, $c_2 > 0$.

Suppose that we have in addition

$$a|Z|^s - b \leqslant f(Z) \quad \textit{for some } 1 < s \leqslant q \tag{5.2}$$

and with constants $a > 0$, $b \geqslant 0$.

Note that, as in the linear growth case, the auxiliary parameter μ needs no further specification in our hypotheses.

Conditions (5.1) and (5.2) are introduced in [10] describing energy densities of (s, μ, q)-growth, we refer to [11], Section 3.2, for a more detailed discussion. In particular (5.1) implies with some constant $M > 0$ and for all $Z \in \mathbb{R}^2$

$$|\nabla f(Z)| \leqslant M\big(1 + |Z|^2\big)^{\frac{q-1}{2}} \,. \tag{5.3}$$

Examples are given, for instance, by

$$f_1(Z) = \big(1 + |Z_1|^2\big)^{\frac{s}{2}} + |Z_2|^2 \quad 1 < s \leqslant 2 \,,$$

$$f_2(Z) = \big(1 + |Z|^2\big)^{\frac{s}{2}} + \big(1 + |Z_2|\big)^{\frac{q}{2}} \quad 1 < s \leqslant q \,.$$

The main result of this section is

Corollary 5.1. *Given Assumption* 5.1 *we suppose in addition that*

$$s > q - \frac{1}{2}\,. \tag{5.4}$$

If we have

$$\limsup_{|x|\to\infty} \frac{|u(x)|}{|x|} < \infty\,, \tag{5.5}$$

then u is an affine function.

Proof of Corollary 5.1. In order to apply Theorem 1.2 we let $Q = 0$, $\gamma_Q = 0$ and since we have (5.4) we can choose $0 < \gamma < 1/2$ such that

$$s > q - \gamma\,. \tag{5.6}$$

Then, in our main Theorem 1.2 we observe

$$\Xi(R) \leqslant \int_{\hat{T}_R} \Gamma^{\frac{q-\gamma}{2}}\,\mathrm{d}x\,. \tag{5.7}$$

We follow the proof of Proposition 4.1, where now (4.6) and (4.7) are replaced by (recalling (5.2), (5.3) (5.5) and choosing $l \in \mathbb{N}$ sufficiently large)

$$\begin{aligned}\int_{\hat{T}_R} \Gamma^{\frac{s}{2}} \eta^{2l}\,\mathrm{d}x &\leqslant c \int_{\hat{T}_R} \big[1 + f(\nabla u)\big] \eta^{2l}\,\mathrm{d}x \\ &\leqslant cR^2 + \frac{c}{R} \sup_{\hat{T}_R} |u| \int_{\hat{T}_R} |\nabla f(\nabla u)| \eta^{2l-1}\,\mathrm{d}x \\ &\leqslant cR^2 + c \int_{\hat{T}_R} \Gamma^{\frac{q-1}{2}} \eta^{2l-1}\,\mathrm{d}x\,. \end{aligned} \tag{5.8}$$

Since $q - 1 < q - \gamma$ we find real numbers $\hat{q}$, $\hat{p} > 1$, $\frac{1}{\hat{q}} + \frac{1}{\hat{p}} = 1$ such that

$$\int_{\hat{T}_R} \Gamma^{\frac{q-1}{2}} \eta^{2l-1}\,\mathrm{d}x \leqslant c\left[\int_{\hat{T}_R} \Gamma^{\frac{q-\gamma}{2}} \eta^{2l}\,\mathrm{d}x\right]^{\frac{1}{\hat{q}}} R^{\frac{2}{\hat{p}}}\,. \tag{5.9}$$

From (5.6), (5.8) and (5.9) we obtain

$$\int_{\hat{T}_R} \Gamma^{\frac{q-\gamma}{2}} \eta^{2l}\,\mathrm{d}x \leqslant c \int_{\hat{T}_R} \Gamma^{\frac{s}{2}} \eta^{2l}\,\mathrm{d}x \leqslant cR^2 + c\left[\int_{\hat{T}_R} \Gamma^{\frac{q-\gamma}{2}} \eta^{2l}\,\mathrm{d}x\right]^{\frac{1}{\hat{q}}} R^{\frac{2}{\hat{p}}}\,. \tag{5.10}$$

W.l.o.g. we suppose that

$$R^2 \leqslant c\left[\int_{\hat{T}_R} \Gamma^{\frac{q-\gamma}{2}} \eta^{2l}\,\mathrm{d}x\right]^{\frac{1}{\hat{q}}} R^{\frac{2}{\hat{p}}}$$

and (5.10) yields

$$\left[\int_{\hat{T}_R} \Gamma^{\frac{q-\gamma}{2}} \eta^{2l}\,\mathrm{d}x\right]^{1-\frac{1}{\hat{q}}} \leqslant cR^{\frac{2}{\hat{p}}}\,. \tag{5.11}$$

Recalling (5.7) and $1 - \frac{1}{\hat{q}} = \frac{1}{\hat{p}}$ we have proved (1.5) and this finally implies Corollary 5.1. □

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