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A RIEMANN HYPOTHESIS ANALOG FOR THE KRAWTCHOUK AND DISCRETE CHEBYSHEV POLYNOMIALS

Abstract. As an analog to the Riemann hypothesis, we prove that the real parts of all complex zeros of the Krawtchouk polynomials, as well as of the discrete Chebyshev polynomials, of order $N = -1$ are equal to $-\frac{1}{2}$. For these polynomials, we also derive a functional equation analogous to that for the Riemann zeta function.

§1. Introduction

We recommend to consult [10] as a general reference on the Riemann zeta function, which is defined as a series or as an Euler product

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\in\mathbb{P}} \frac{1}{1-\frac{1}{p^s}}, \tag{1.1}$$

which both converge in the half-plane $\mathrm{Re}(s) > 1$. There are many ways to extend the domain of (1.1), one way of doing so is the *functional equation*

$$\zeta(s) = 2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s), \tag{1.2}$$

which extends ζ to the complex plane, except at $s = 1$ where ζ has a simple pole with residue 1.

The famous *Riemann hypothesis* is closely associated to the distribution of prime numbers, and asserts that all the nontrivial zeros of the Riemann ζ-function have real part $\frac{1}{2}$. The hypothesis being true would have a great impact on understanding the distribution of prime numbers.

The Riemann hypothesis is generally considered as one of the most important open mathematical problems. It was highlighted by D. Hilbert at the 1900 International Congress of Mathematicians, and one century

[1]A preliminary version of the article was presented at PCA 2021, St. Petersburg.

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later, the Clay Mathematics Institute offered a million dollar prize for the proof of the Riemann hypothesis [12].

Consequently, there have been many attempts to prove the Riemann hypothesis true. One interesting approach originates from the *Hilbert–Pólya conjecture*, which states that the nontrivial zeros of the Riemann ζ-function are of the form $\rho_k = \frac{1}{2} + it_k$ where t_k are the eigenvalues of a Hermitian operator, and hence real [11]. Berry and Keating suggested explicitly that such a self-adjoint operator could be a Hamiltonian xp. See [1] for the references and further (still quite speculative) progress.

In this article, we study not the Riemann hypothesis, but families of orthogonal polynomials which have a property similar to the Riemann hypothesis: all the zeros lie on a line with a fixed real part. Our approach is parallel to the Hilbert–Pólya conjecture, only with a technical difference that we associate the zeros under study to the eigenvalues of *skew-Hermitian operators*.

§2. Preliminaries

2.1. On Hermitian operators. If H is a complex inner product space and A is a linear mapping $A : H \to H$, then the *adjoint mapping* A^* is defined as $(\boldsymbol{x}, A\boldsymbol{y}) = (A^*\boldsymbol{x}, \boldsymbol{y})$. An operator is called *self-adjoint*, or *Hermitian*, if $A^* = A$. In the matrix presentation, being Hermitian means that the transpose of the complex conjugate of A coincides with A. Self-adjoint mappings play a very special role in quantum mechanics, since the states of quantum systems are described by unit-trace, self-adjoint mappings having nonnegative spectrum. Also, (sharp) observables are described as self-adjoint mappings [6].

An operator A is called *skew-Hermitian* if $A^* = -A$. It is a well-known fact that the eigenvalues of Hermitian operators are real [6], and it easily follows that the eigenvalues of skew-Hermitian operators are purely imaginary, that is, of the form $i\lambda$ where $\lambda \in \mathbb{R}$.

2.2. On discrete orthogonal polynomials. Discrete orthogonal polynomials are widely used in mathematics. Applications include combinatorics, information theory, and numerical analysis. For a general treatise on orthogonal polynomials, one can see [9].

Let N be a fixed integer. The set of real polynomials of degree $\leqslant N$ is denoted by $\mathbb{R}_{\leqslant N}[x]$, and it clearly is a vector space with respect to the obvious addition and scalar multiplication. Then, for a weight sequence

$(w_0, w_1, \ldots, w_N)$, where $w_k > 0$, the formula

$$(p, q) = \sum_{k=0}^{N} w_k p(k) q(k) \tag{2.1}$$

defines an inner product on $\mathbb{R}_{\leqslant N}[x]$.[1] It is a well-known fact that the polynomials $\{1, x, x^2, \ldots, x^N\}$ form a basis of $\mathbb{R}_{\leqslant N}[x]$, and then the Gram–Schmidt ortogonalization process guarantees that there is a sequence of polynomials p_0, p_1, …, p_N orthogonal with respect to (2.1) such that $\deg(p_i) = i$ [9].

However, it is good to note that an orthogonal sequence of polynomials is not unique. Uniqueness cannot be reached by requiring the orthonormality, meaning that the norms are all equal to 1. Uniqueness, if desired, can be obtained, for example, by requiring the polynomials to be monic. However, for many purposes there is no such need.

For any fixed $N \geqslant 0$, formula (2.1) defines some sequence of orthogonal polynomials, which we call polynomials of *order* N. On the other hand, formula (2.1) is meaningless for $N < 0$, but there is a way of defining orthogonal polynomials for $N < 0$ as well. In fact, the orthogonality property implies that the orthogonal polynomials satisfy a depth two recurrence relation:

$$p_n(x) = (\alpha_n x + \beta_n) p_{n-1}(x) - \gamma_n p_{n-2}(x), \tag{2.2}$$

where α_n, $\gamma_n > 0$. Descriptions of α_n and γ_n are given in [9], and we will see in the sequel how to use the recurrence to extend the definition to values $N < 0$.

2.3. On tridiagonal matrices.

For a more general treatise on tridiagonal matrices, we advise to consult [7, 13]. Here we mention their basic properties.

Definition 1. An $n \times n$ matrix is called tridiagonal if it is of the form

[1] It is not necessary for the evaluation points to be $\{0, 1, 2, \ldots, N\}$, any set of $N + 1$ numbers will do.

$$\mathbf{A}_n = \begin{pmatrix} m_1 & u_1 & 0 & 0 & \cdots & 0 \\ l_1 & m_2 & u_2 & 0 & \cdots & 0 \\ 0 & l_2 & m_3 & u_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & m_{n-1} & u_{n-1} \\ 0 & 0 & 0 & \cdots & l_{n-1} & m_n \end{pmatrix}. \tag{2.3}$$

A tridiagonal matrix can be more compactly written as

$$\mathbf{A}_n = \begin{pmatrix} * & u_1 & u_2 & \ldots & u_{n-1} \\ m_1 & m_2 & \ldots & m_{n-1} & m_n \\ l_1 & l_2 & \ldots & l_{n-1} & * \end{pmatrix}, \tag{2.4}$$

or even more compactly as $\mathbf{A}_n = [\mathbf{u}, \mathbf{m}, \mathbf{l}]$, where

$$\mathbf{m}=(m_1, m_2, \ldots, m_{n-1}, m_n), \quad \mathbf{u}=(u_1, u_2, \ldots, u_{n-1}), \quad \mathbf{l}=(l_1, l_2, \ldots, l_{n-1})$$

represent the main diagonal and the upper and lower subdiagonals in $\mathbf{A}_n$.

It is easy to see that the determinants $\det(\mathbf{A}_n) = |\mathbf{A}_n|$ of tridiagonal matrices obey the recurrence

$$|\mathbf{A}_n| = m_n|\mathbf{A}_{n-1}| - u_{n-1}l_{n-1}|\mathbf{A}_{n-2}| \tag{2.5}$$

if we define $|\mathbf{A}_{-1}| = 0, |\mathbf{A}_0| = 1$ [7, 13].

The recurrence (2.5) is a direct link to orthogonal polynomials: letting $p_{-1}(x) = 0$, $p_0(x) = 1$, $m_n = \alpha_n x + \beta_n$, $u_n = 1$, $l_n = \gamma_{n+1}$, and $p_n(x) = |\mathbf{A}_n|$, we have

$$p_n(x) = (\alpha_n x + \beta_n)p_{n-1}(x) - \gamma_n p_{n-2}(x). \tag{2.6}$$

This consideration is summarized in the following theorem.

Theorem 1. *Let $p_{-1} = 0$, $p_0 = 1$, and*

$$p_n(x) = (\alpha_n x + \beta_n)p_{n-1}(x) - \gamma_n p_{n-2}(x).$$

Then $p_n(x)$ can be represented as

$$p_n(x) = \det \begin{pmatrix} \alpha_1 x + \beta_1 & 1 & 0 & \cdots & 0 \\ \gamma_2 & \alpha_2 x + \beta_2 & 1 & \cdots & 0 \\ 0 & \gamma_3 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \gamma_n & \alpha_n x + \beta_n \end{pmatrix}. \tag{2.7}$$

Remark 1. As the tridiagonal matrix in (2.7) was chosen to satisfy the relation $u_{n-1}l_{n-1} = \gamma_n$, it is clear that there are many other choices as well. This can be understood in the sense that (infinitely many) similar matrices share the same determinant and characteristic polynomial.

One very important property of tridiagonal matrices, used in the main results of this study, is given by the Jacobi theorem.

Theorem 2 (Jacobi)**.** *If a (real) tridiagonal matrix* $\mathbf{A} = [\mathbf{u}, \mathbf{m}, \mathbf{l}]$ *has* $\mathbf{m} = \mathbf{0}$ *and* $u_k l_k < 0$ *for* $1 \leqslant k \leqslant n-1$*, then* $\mathbf{A}$ *is diagonally similar to a skew-Hermitian matrix* $\mathbf{S} = \mathbf{D}^{-1}\mathbf{A}\mathbf{D}$*, where*

$$\begin{aligned} &\mathbf{D} = \mathbf{diag}(d_k), \quad d_1 = 1, \\ &d_k = \operatorname{sgn}(u_k)\sqrt{\prod_{i=1}^{i=k-1} \operatorname{abs}\Big(\frac{l_i}{u_i}\Big)}, \quad 2 \leqslant k \leqslant n-1. \end{aligned} \tag{2.8}$$

§3. On the Krawtchouk polynomials

The Krawtchouk polynomials have their combinatorial meanings only for $N \geqslant 0$, but they can be as well defined for values $N < 0$. For general information about the Krawtchouk polynomials, we refer to [8, 3], but here we emphasize only three facts important for our considerations:

(1) The inner product giving rise to the Krawtchouk polynomials is

$$(p, q) = \sum_{i=0}^{N} \binom{N}{i} p(i)q(i).$$

(2) The recurrence relation for the Krawtchouk polynomials is

$$(r+1)K_{r+1}^{(N)}(z) = (N-2z)K_r^{(N)}(z) - (N-r+1)K_{r-1}^{(N)}(z), \tag{3.1}$$

with initial terms $K_0^{(N)}(z) = 1$, $K_1^{(N)}(z) = N - 2z$. [2]

(3) The generating function of the Krawtchouk polynomials is

$$T_{N,z}(t) = (1+t)^{N-z}(1-t)^z = \sum_{r=0}^{\infty} K_r^{(N)}(z)t^r, \qquad z \in \mathbb{C}. \tag{3.2}$$

[2]The choice of this sequence, instead of, for example, monic polynomials, is justified by the combinatorial significance of this particular sequence. Also, the generating function of this sequence becomes very elegant.

Remark 2. Even though the inner product is meaningful only for $N \geqslant 0$, it is evident that the recurrence and the generating function can be treated as well for $N < 0$. It may, however, be true that there are no important combinatorial interpretations of Krawtchouk polynomials with negative N, but in this study, we show that their zeros share the same real part.

By applying (2.7) to the recurrence (3.1), we obtain the following representation:

$$K_k^{(N)}(z) = \begin{vmatrix} -2z+N & 1 & 0 & 0 & \cdots & 0 \\ \frac{N}{2} & \frac{-2z+N}{2} & 1 & 0 & \cdots & 0 \\ 0 & \frac{N-1}{3} & \frac{-2z+N}{3} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{-2z+N}{k-1} & 1 \\ 0 & 0 & 0 & \cdots & \frac{N-k}{k} & \frac{-2z+N}{k} \end{vmatrix}$$

$$= \frac{1}{k!} \begin{vmatrix} -2z+N & 1 & 0 & \cdots & 0 \\ N & -2z+N & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -2z+N & k-1 \\ 0 & 0 & \cdots & N-k & -2z+N \end{vmatrix}, \qquad (3.3)$$

i.e, the diagonals of this tridiagonal matrix are, ignoring the factor $\frac{1}{k!}$, as follows: $\mathbf{u} = (j)$, $\mathbf{m} = (-2z+N)$, $\mathbf{l} = (N-j+1)$, where j runs from 1 to $k-1$.

3.1. An analog of the Riemann hypothesis for the Krawtchouk polynomials.

Theorem 3. *For $N = -1$, the zeros of the Krawtchouk polynomials lie on the line* $\mathrm{Re}(z) = -\frac{1}{2}$.

Proof. For $N = -1$, formula (3.3) becomes

$$K_k^{(-1)}(z) = \frac{1}{k!} \begin{vmatrix} -2z-1 & 1 & 0 & 0 & \ldots & 0 \\ -1 & -2z-1 & 2 & 0 & \ldots & 0 \\ 0 & -2 & -2z-1 & 3 & \ldots & 0 \\ 0 & 0 & -3 & -2z-1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \end{vmatrix}, \qquad (3.4)$$

and upon substituting $w = 2z+1$ the equality $K_k^{(-1)}(z) = 0$ becomes equivalent to

$$\begin{vmatrix} -w & 1 & 0 & 0 & 0 & \dots & 0 \\ -1 & -w & 2 & 0 & 0 & \dots & 0 \\ 0 & -2 & -w & 3 & 0 & \dots & 0 \\ 0 & 0 & -3 & -w & 4 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \end{vmatrix} = 0. \tag{3.5}$$

But (3.5) is, in fact, the eigenvalue equation of a tridiagonal matrix $[\boldsymbol{u}, \mathbf{0}, \boldsymbol{l}]$. By the Jacobi theorem, the above matrix is similar to some skew-symmetric matrix, and hence all its eigenvalues are purely imaginary. It follows that each w which satisfies the above equation is of the form $i\lambda$ where $\lambda \in \mathbb{R}$. Hence, $z = \frac{1}{2}(w-1) = -\frac{1}{2} + i\frac{\lambda}{2}$. $\square$

§4. On the discrete Chebyshev polynomials

For the definition and properties of the discrete Chebyshev polynomials $D_r^{(N)}$, we refer to [3]. Here again we point out three facts important for our considerations:

(1) The inner product giving rise to the discrete Chebyshev polynomials is

$$(p,q) = \sum_{i=0}^{N} p(i)q(i).$$

(2) The recurrence relation for the discrete Chebyshev polynomials is

$$r^2 D_r^{(N)}(z) = (2r-1)(N-2z)D_{r-1}^{(N)}(z) - (N+r)(N-r+2)D_{r-2}^{(N)}(z), \tag{4.1}$$

with initial terms $D_0^{(N)}(z) = 1$, $D_1^{(N)}(z) = N - 2z$.

(3) The generating function is known and can be represented in two equivalent forms [4]:

$$T_{N,z}(t) = (1+t)^{N-2z} \sum_k \binom{z}{k}\binom{z-N-1}{k} t^{2k} \tag{4.2}$$

and

$$T_{N,z}(t) = (1-t)^{2z-N} \sum_k \binom{N-z}{k}\binom{-z-1}{k} t^{2k}. \tag{4.3}$$

4.1. An analog of the Riemann hypothesis for the discrete Chebyshev polynomials. By applying (2.7) to the recurrence (4.1), we obtain the following representation:

$$D_k^{(N)}(z) = \begin{vmatrix} -2z+N & 1 & 0 & \cdots & 0 \\ \frac{N+2}{4}N & \frac{3}{4}(-2z+N) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{2k-3}{(k-1)^2}(-2z+N) & 1 \\ 0 & 0 & \cdots & \frac{N+k}{k^2}(N-k-2) & \frac{2k-1}{k^2}(-2z+N) \end{vmatrix}$$
$$= \frac{1}{(k!^2)} \times \begin{vmatrix} -2z+N & 1 & 0 & \cdots & 0 \\ (N+2)N & 3(-2z+N) & 4 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & (2k-3)(-2z+N) & (k-1)^2 \\ 0 & 0 & \cdots & (N+k)(N-k-2) & (2k-1)(-2z+N) \end{vmatrix}. \tag{4.4}$$

Theorem 4. *For $N = -1$, the zeros of the discrete Chebyschev polynomials lie on the line* $\mathrm{Re}(z) = -\frac{1}{2}$.

Proof. For $N = -1$, formula (4.4) becomes

$$(k!)^2 D_k^{(-1)}(z) = \begin{vmatrix} -(2z+1) & 1 & 0 & \cdots & 0 \\ -1\cdot 3 & -3(2z+1) & 4 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -(2k-3)(2z+1) & (k-1)^2 \\ 0 & 0 & \cdots & (1-k)(k+3) & -(2k-1)(2z+1) \end{vmatrix}$$
$$= 3\cdot\ldots\cdot(2k-3)\cdot(2k-1)\times \begin{vmatrix} -(2z+1) & 1 & 0 & \cdots & 0 \\ -1 & -(2z+1) & \frac{4}{3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -(2z+1) & \frac{(k-1)^2}{2k-3} \\ 0 & 0 & \cdots & \frac{(1-k)(k+3)}{2k-1} & -(2z+1) \end{vmatrix}. \tag{4.5}$$

By substituting $w = 2z + 1$, we see, as in the case of the Krawtchouk polynomials, that the equality $D_k^{(-1)}(z) = 0$ is equivalent to the eigenvalue equation of a tridiagonal matrix satisfying the conditions of the Jacobi theorem. Hence, the eigenvalues are purely imaginary, and the conclusion follows exactly as in the case of the Krawtchouk polynomials. □

§5. Functional equations

Letting $M(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2})\Gamma(s)$, we can rewrite the functional equation (1.2) again in the form

$$\zeta(s) = M(s)\zeta(1-s).$$

Here we demonstrate that the Krawtchouk polynomials and discrete Chebyshev polynomials also satisfy a functional equation.

Theorem 5. *The Krawtchouck polynomials $K_n^{(-1)}(s)$ satisfy the functional equation*

$$K_n^{(-1)}(-\frac{1}{2}+\sigma) = (-1)^n K_n^{(-1)}(-\frac{1}{2}-\sigma).$$

Proof. For $N = -1$, the generating function of the Krawtchouk polynomials becomes

$$T_{-1,z}(t) = (1+t)^{-1-z}(1-t)^z = \sum_{r=0}^{\infty} K_r^{(-1)}(z)t^r,$$

and, clearly,

$$T_{-1,1-z}(t) = (1+t)^{-1-(1-z)}(1-t)^{1-z} = (1+t)^{-2+z}(1-t)^{1-z}.$$

Also,

$$T_{-1,z-2}(-t) = (1-t)^{-1-(z-2)}(1+t)^{z-2} = (1+t)^{-2+z}(1-t)^{-z+1},$$

hence $T_{-1,1-z}(t) = T_{-1,z-2}(-t)$, meaning that

$$\sum_{r=0}^{\infty} K_r^{(-1)}(1-z)t^r = \sum_{r=0}^{\infty} K_r^{(-1)}(z-2)(-t)^r.$$

Therefore, we have $K_r^{(-1)}(1-z) = (-1)^r K_r^{(-1)}(z-2)$. Letting $z = \frac{3}{2} - \sigma$ yields the claim. □

Theorem 6. *The discrete Chebyshev polynomials $D_n^{(-1)}(s)$ satisfy the functional equation*

$$D_n^{(-1)}(-\frac{1}{2}+\sigma) = (-1)^n D_n^{(-1)}(-\frac{1}{2}-\sigma).$$

Proof. Similar to the proof for the Krawtchouk polynomials. □

Conclusion

We have shown that a family of mathematically interesting polynomials satisfies the Riemann hypothesis property. Analogous results have been presented earlier (see, e.g., [2]), but our methods are far more straightforward and hence of larger interest.

The extension of the method is, evidently, applicable to other values of $N < 0$ and also to other orthogonal polynomials for which β_n/α_n in the recurrence (2.2) is constant. Studying the literature, we learn that this condition is true for quite interesting families of orthogonal polynomials: in fact, the Hahn, Meixner, and Charlier polynomials are included.

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