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SEMIFINITE HARMONIC FUNCTIONS ON BRANCHING GRAPHS

Abstract. We study semifinite harmonic functions on arbitrary branching graphs. We give a detailed exposition of an algebraic method which allows one to classify semifinite indecomposable harmonic functions on some multiplicative branching graphs. It was suggested by A. Wassermann in terms of operator algebras, but we rephrase, clarify, and simplify the main arguments working only with combinatorial objects. This work was inspired by the theory of traceable factor representations of the infinite symmetric group $S(\infty)$.

1. Introduction

The classical character theory of finite and compact groups can be generalized to other classes of groups and algebras in different ways. For groups and C^*-algebras *not* of type I, the character theory deals *not* with irreducible representations, but with normal factor representations, i.e., homomorphisms to von Neumann algebras with a finite or semifinite trace. For AF-algebras, one can reformulate the character theory in a combinatorial-algebraic language, speaking about nonnegative harmonic functions on infinite graded graphs of a special type. Equivalently, one can treat these harmonic functions as central measures on the space of monotone paths in the graph. This approach was developed by A. M. Vershik and S. V. Kerov in the late 70s – early 80s. Harmonic functions that take finite values only lead to probability measures on the path space and correspond to factor representations of finite types I_n and II_1. The connection between harmonic functions and normal factor representations motivates us to study the so-called *semifinite* harmonic functions, which correspond to normal factor representations of types I_∞ and II_∞. These functions must take the value $+\infty$ and satisfy some natural condition, see Definition 3.5 below.

Key words and phrases: branching graphs, AF-algebras, semifinite traces, semifinite harmonic functions.

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A. M. Vershik and S. V. Kerov classified semifinite harmonic functions on the Young and Kingman graphs, see [7, 9]. They solved this problem using the so-called ergodic method, which involves an evaluation of a nontrivial limit. This method can be applied to any branching graph, but the main difficulty, which is not always easy to get over, is to compute that limit. There is another approach, developed by A. Wassermann. In his dissertation [18], he suggested to use a bijection between the faithfull factor representations of a primitive C^*-algebra and those of any closed two-sided ideal of this C^*-algebra, see [18, p. 143, Theorem 7]. Another ingredient of Wassermann's method requires that the K_0-group of the corresponding AF-algebra admit a compatible ring structure, see [18, p. 146, Theorem 8]. Therefore, Wassermann's method is applicable to some multiplicative graphs only, for which it may be extremely useful. A. Wassermann applied his method to determine all indecomposable semifinite harmonic functions on the Young graph and thereby proved the theorem of Vershik and Kerov without the ergodic method or any other complicated analytic computations.

This paper contains a detailed exposition of Wassermann's method in terms of algebraic combinatorics, unlike the original work [18, Chap. III, Sec. 6], where the language of operator algebras was used. The combinatorial setup allows us to clarify and simplify the main arguments of [18, Chap. III, Sec. 6]. Furthermore, we work with a generalization of branching graphs, namely, we consider branching graphs with formal nonnegative multiplicities on edges. Crucial statements of Wassermann's method can be found in [18, Chap. III, Sec. 6], [5,16], and [7,10,11]. We prove them in a completely combinatorial way. These statements, together with the original argument of A.Wassermann, constitute a powerful method for determining the indecomposable semifinite harmonic functions on those multiplicative graphs for which the limit from the Vershik–Kerov ergodic method turns out to be too complicated for an evaluation. The Macdonald graph, which corresponds to the simplest Pieri rule for the Macdonald symmetric functions, is a good example of such a graph. Using Wassermann's method, one can obtain an exhaustive list of semifinite indecomposable harmonic functions on it, see Remark 5.8.

1.1. Organization of the paper. In Sec. 2, we introduce graded graphs and discuss their ideals and coideals. In Sec. 3, we introduce semifinite harmonic functions and prove some general facts about them. Section 4 deals with semifinite harmonic functions on multiplicative branching graphs only. Section 5 contains a combinatorial analog of an observation due to R. P. Boyer. In Appendix A, we discuss finite harmonic functions on the product of branching graphs.

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2. Ideals and coideals of graded graphs

In this section, we recall the main notions related to branching graphs, ideals, and coideals.

Definition 2.1. By a *graded graph* we mean a pair $(\Gamma, \varkappa)$ where Γ is a graded set $\Gamma = \bigsqcup_{n\geqslant 0} \Gamma_n$, Γ_n are finite sets, and $\varkappa$ is a function $\Gamma \times \Gamma \to \mathbb{R}_{\geqslant 0}$ that satisfies the following constraints:

(i) if $\lambda \in \Gamma_n$ and $\mu \in \Gamma_m$, then $\varkappa(\lambda, \mu) = 0$ for $m - n \neq 1$;
(ii) for any vertex $\lambda \in \Gamma_n$ there exists $\mu \in \Gamma_{n+1}$ with $\varkappa(\lambda, \mu) \neq 0$.

Edges of a graded graph $(\Gamma, \varkappa)$ are, by definition, pairs of vertices (λ, μ) with $\varkappa(\lambda, \mu) > 0$. Thus, we may treat $\varkappa(\lambda, \mu)$ as a formal multiplicity of the edge.

If $\lambda \in \Gamma_n$, then the number n is uniquely defined. We denote it by $|\lambda|$. We write $\lambda \nearrow \mu$ if $|\mu| - |\lambda| = 1$ and $\varkappa(\lambda, \mu) \neq 0$. In this case, we say that *there is an edge from λ to μ of multiplicity $\varkappa(\lambda, \mu)$.*

Condition (i) from Definition 2.1 means that we allow edges only between adjacent levels, and condition (ii) means that each vertex must be connected by an edge with some vertex from the next level.

A *path* in a graded graph Γ is a (finite or infinite) sequence of vertices $\lambda_1, \lambda_2, \lambda_3, \ldots$ such that $\lambda_i \nearrow \lambda_{i+1}$ for every i. We write $\nu > \mu$ if $|\nu| > |\mu|$ and there is a path that connects μ and ν. We write $\nu \geqslant \mu$ if $\nu = \mu$ or $\nu > \mu$. The relation $\geqslant$ turns Γ into a poset.

Let $\mu, \nu \in \Gamma$ and $|\nu| - |\mu| = n \geqslant 1$. Then the expression

$$\dim(\mu, \nu) = \sum_{\substack{\lambda_0, \ldots, \lambda_n \in \Gamma: \\ \mu = \lambda_0 \nearrow \lambda_1 \nearrow \ldots \nearrow \lambda_{n-1} \nearrow \lambda_n = \nu}} \varkappa(\lambda_0, \lambda_1)\varkappa(\lambda_1, \lambda_2) \ldots \varkappa(\lambda_{n-1}, \lambda_n) \tag{1}$$

is the "weighted" number of paths from μ to ν. By definition we also set $\dim(\mu,\mu) = 1$ and $\dim(\mu,\nu) = 0$ if $\nu \not\geqslant \mu$. The function

$$\dim(\,\cdot\,,\,\cdot\,)\colon \Gamma \times \Gamma \to \mathbb{R}_{\geqslant 0}$$

is called the *shifted dimension*. Note that $\dim(\mu,\nu) = \varkappa(\mu,\nu)$ if $\mu \nearrow \nu$, and for $\mu \in \Gamma_m$, $\nu \in \Gamma_n$, and any k such that $m \leqslant k \leqslant n$, we have

$$\dim(\mu,\nu) = \sum_{\lambda:\lambda\in\Gamma_k} \dim(\mu,\lambda)\dim(\lambda,\nu). \tag{2}$$

Definition 2.2. A *branching graph* is defined as a graded graph $(\Gamma,\varkappa)$ that satisfies the following conditions:

- $\Gamma_0 = \{\varnothing\}$ is a singleton,
- for any $\lambda \in \Gamma_n$ with $n \geqslant 1$ there exists $\mu \in \Gamma_{n-1}$ such that $\mu \nearrow \lambda$.

For a branching graph $(\Gamma,\varkappa)$, we denote the expression $\dim(\varnothing,\lambda)$ by $\dim(\lambda)$ and call it the *dimension* of λ.

Definition 2.3. A subset I of vertices of a graded graph Γ is called an *ideal* if for any vertices $\lambda \in I$ and $\mu \in \Gamma$ such that $\mu > \lambda$, we have $\mu \in I$. A subset $J \subset \Gamma$ is called a *coideal* if for any vertices $\lambda \in J$ and $\mu \in \Gamma$ such that $\mu < \lambda$ we have $\mu \in J$.

Remark 2.4. Our terminology differs from the terminology of poset theory. Namely, our ideals and coideals are usually called *filters* and *ideals*, respectively, [15].

There is a bijective correspondence $I \leftrightarrow \Gamma\backslash I$ between ideals and coideals. Let J be a coideal and $I = \Gamma\backslash J$ be the corresponding ideal. Then the following conditions are equivalent:

(i) if $\{\mu \mid \lambda \nearrow \mu\} \subset I$, then $\lambda \in I$;
(ii) for any $\lambda \in J$ there exists a vertex $\mu \in J$ such that $\lambda \nearrow \mu$.

Definition 2.5. An ideal I and the corresponding coideal J are said to be *saturated* if they satisfy the conditions above. A saturated ideal I is said to be *primitive* if for any saturated ideals I_1, I_2 such that $I = I_1 \cap I_2$, we have $I = I_1$ or $I = I_2$. A saturated coideal J is said to be *primitive* if for any saturated coideals J_1, J_2 such that $J = J_1 \cup J_2$, we have $J = J_1$ or $J = J_2$.

The bijection $I \leftrightarrow \Gamma\backslash I$ maps primitive saturated ideals to primitive saturated coideals and vice versa. We will also use the fact that ideals and saturated coideals are graded graphs themselves.

Let Γ be a branching graph. The space of infinite paths in Γ starting at $\varnothing$ will be denoted by $\mathcal{T}(\Gamma)$. To every path $\tau = (\varnothing, \lambda_1 \nearrow \lambda_2 \nearrow \ldots) \in \mathcal{T}(\Gamma)$ we associate the saturated primitive coideal $\Gamma_\tau = \bigcup_{n\geqslant 1} \{\lambda \in \Gamma \mid \lambda \leqslant \lambda_n\}$.

In the next proposition we give a combinatorial characterization of saturated primitive coideals of an arbitrary graded graph, see [5]. Moreover, for branching graphs we describe all such coideals in terms of path coideals Γ_τ, see [16] and [18, p. 129].

Proposition 2.6. 1. *A saturated coideal J of a graded graph is primitive if and only if for any two vertices $\lambda_1, \lambda_2 \in J$ we can find a vertex $\mu \in J$ such that $\mu \geqslant \lambda_1, \lambda_2$.*

2. *Every saturated primitive coideal of a branching graph is of the form $J = \Gamma_\tau$ for some path $\tau \in \mathcal{T}(\Gamma)$.*

Proof. Let $J \subset \Gamma$ be a saturated coideal. Suppose that there exist vertices $\lambda_1, \lambda_2 \in J$ that do not possess a common majorant. Let us prove that J can be presented as a union of two distinct proper saturated coideals. We need to introduce some notation. For any $\lambda \in J$, the subset of vertices of J that lie above λ will be denoted by J^λ, i.e., $J^\lambda = \{\mu \in J \mid \mu \geqslant \lambda\}$. For any subset $A \subset J$, we define $\downarrow A$ as the subset of vertices of J that lie below some vertex of A, i.e., $\downarrow A = \{\mu \in J \mid \mu \leqslant \lambda \text{ for some } \lambda \in A\}$. Finally, for any ideal I of J, the symbol $\operatorname{sat}(I)$ stands for the minimal saturated ideal that contains I. In other words, $\operatorname{sat}(I)$ consists of all vertices of I and all vertices $\lambda \in J$ such that $\{\mu \mid \lambda \nearrow \mu\} \subset I$. With this notation in mind, we set $J_1 = \downarrow (J^{\lambda_1})$, $J_2 = J\backslash \operatorname{sat}(J^{\lambda_1})$. It is not difficult to see that J_1 and J_2 are saturated coideals and their union coincides with J. Obviously, $\lambda_1 \in J_1$ and $\lambda_1 \notin J_2$. Next, we use the fact that the vertices λ_1 and λ_2 do not possess a common majorant to show that $\lambda_2 \in J_2$ and $\lambda_2 \notin J_1$. Thus, J_1 and J_2 are proper distinct coideals of J.

Now suppose that for any vertices $\lambda_1, \lambda_2 \in J$ there exists $\mu \in J$ with $\mu \geqslant \lambda_1, \lambda_2$. We will show that $J = \Gamma_\tau$ for some path $\tau \in \mathcal{T}(\Gamma)$. Let us denote by $x_1, x_2, \ldots$ all the vertices of J enumerated in any (fixed) order. Since J is primitive, it follows that we can construct a sequence of vertices

$y_1 \leqslant y_2 \leqslant \dots$ of J with the following properties:

$$\begin{array}{llllll} & y_2 \geqslant y_1, & y_3 \geqslant y_2, & \dots & y_n \geqslant y_{n-1}, & \dots \\ y_1 = x_1, & y_2 \geqslant x_2, & y_3 \geqslant x_3, & \dots & y_n \geqslant x_n, & \dots \\ & y_2 \in J, & y_3 \in J, & \dots & y_n \in J, & \dots \end{array}$$

Let $\tau \in \mathcal{T}(\Gamma)$ be any path that passes through the vertices $y_1, y_2, \dots$. Obviously, $J = \Gamma_\tau$. □

Remark 2.7. One can formulate an obvious analog of the second part of Proposition 2.6 for arbitrary graded graphs, but this is of no particular importance to us.

Definition 2.8. A graded graph Γ is said to be *primitive* if it is primitive as a coideal, i.e., for any vertices $\lambda_1, \lambda_2 \in \Gamma$ there exists a vertex $\mu \in \Gamma$ such that $\mu \geqslant \lambda_1, \lambda_2$.

3. Semifinite harmonic functions

Definition 3.1. Let $(\Gamma, \varkappa)$ be a graded graph. A function

$$\varphi\colon \Gamma \to \mathbb{R}_{\geqslant 0} \cup \{+\infty\}$$

is said to be *harmonic* if it enjoys the following property:

$$\varphi(\lambda) = \sum_{\mu:\lambda\nearrow\mu} \varkappa(\lambda,\mu)\varphi(\mu) \text{ for every } \lambda \in \Gamma.$$

Throughout the paper, we use the following conventions:

- $x + (+\infty) = +\infty$ for any $x \in \mathbb{R}$,
- $(+\infty) + (+\infty) = +\infty$,
- $0 \cdot (+\infty) = 0$.

Definition 3.2. The set of all vertices $\lambda \in \Gamma$ with $\varphi(\lambda) < +\infty$ is called the *finiteness ideal* of φ. We denote the *zero ideal* $\{\lambda \in \Gamma \mid \varphi(\lambda) = 0\}$ of φ by $\ker\varphi$ and its *support* $\{\lambda \in \Gamma \mid \varphi(\lambda) > 0\}$ by $\operatorname{supp}\varphi$.

Note that the zero set $\ker(\varphi)$ is a saturated ideal and $\operatorname{supp}(\varphi)$ is a saturated coideal of Γ, while $\ker(\varphi) \cup \operatorname{supp}(\varphi) = \Gamma$. Furthermore, we can restrict φ to any ideal or saturated coideal that contains $\operatorname{supp}(\varphi)$. The restriction is a harmonic function on that ideal or coideal, respectively.

The symbol $\mathrm{K}_0(\Gamma)$ stands for the $\mathbb{R}$-vector space spanned by the vertices of Γ subject to the following relations:

$$\lambda = \sum_{\mu:\lambda\nearrow\mu} \varkappa(\lambda,\mu) \cdot \mu \quad \text{for every} \quad \lambda \in \Gamma.$$

The symbol $\mathrm{K}_0^+(\Gamma)$ denotes the positive cone in $\mathrm{K}_0(\Gamma)$ generated by the vertices of Γ, i.e., $\mathrm{K}_0^+(\Gamma) = \mathrm{span}_{\mathbb{R}_{\geqslant 0}}(\lambda \mid \lambda \in \Gamma)$. The partial order defined by the cone $\mathrm{K}_0^+(\Gamma)$ is denoted by $\geqslant_K$. Thus, $a \geqslant_K b \iff a - b \in \mathrm{K}_0^+(\Gamma)$. For instance, if $\lambda \leqslant \mu$, then $\lambda \geqslant_K \dim(\lambda, \mu) \cdot \mu$.

Remark 3.3. The notation $\mathrm{K}_0(\Gamma)$ is motivated by the following fact. If all formal multiplicities of edges are integers, then the vector space $\mathrm{K}_0(\Gamma)$ can be identified with the Grothendieck K_0-group of the corresponding AF-algebra. Under this bijection, the cone $\mathrm{K}_0^+(\Gamma)$ gets identified with the cone of true modules [10, Theorem 13, p. 32].

Observation 3.4. If $b \in \mathrm{K}_0^+(\Gamma)$ and $b \leqslant_K \lambda$, then b has the form

$$b = \sum_{\mu\colon |\mu| = N} b_\mu \mu$$

for some N and some real numbers b_μ subject to the following constraints: $0 \leqslant b_\mu \leqslant \dim(\lambda, \mu)$. In particular, $b_\mu = 0$ if $\mu \not\geqslant \lambda$.

The $\mathbb{R}_{\geqslant 0}$-linear map $\mathrm{K}_0^+(\Gamma) \to \mathbb{R}_{\geqslant 0} \cup \{+\infty\}$ defined by a harmonic function φ will be denoted by the same symbol φ. Note that this map is monotone in the sense of the partial order. Namely, if $a \geqslant_K b$, then $\varphi(a) \geqslant \varphi(b)$.

Definition 3.5. A harmonic function φ is said to be *semifinite* if it is not finite and the map $\varphi\colon \mathrm{K}_0^+(\Gamma) \to \mathbb{R}_{\geqslant 0} \cup \{+\infty\}$ enjoys the following property:

$$\varphi(a) = \sup_{\substack{b \in \mathrm{K}_0^+(\Gamma)\colon b \leqslant_K a, \\ \varphi(b) < +\infty}} \varphi(b) \quad \text{for every} \quad a \in \mathrm{K}_0^+(\Gamma). \tag{3}$$

If $\varphi(a) < +\infty$, then condition (3) turns into the trivial identity $\varphi(a) = \varphi(a)$.

Condition (3) arises in a natural way in the theory of operator algebras [4, Definition 1.8].

Remark 3.6. A harmonic function φ is semifinite if and only if there exists an element $a \in \mathrm{K}_0^+(\Gamma)$ with $\varphi(a) = +\infty$ and for any such a we can find a sequence $\{a_n\}_{n \geqslant 1} \subset \mathrm{K}_0^+(\Gamma)$ such that

- $a_n \leqslant_K a$,
- $\varphi(a_n) < +\infty$,
- $\lim\limits_{n \to +\infty} \varphi(a_n) = +\infty$.

We will call this sequence $\{a_n\}_{n\geqslant 1}$ an *approximating sequence.*

Proposition 3.7. *A harmonic function φ is semifinite if and only if it is not finite and for any vertex $\lambda \in \Gamma$ the following equality holds:*

$$\varphi(\lambda) = \lim_{N\to\infty} \sum_{\substack{\mu:\ \mu\geqslant\lambda,\ |\mu|=N \\ 0<\varphi(\mu)<+\infty}} \dim(\lambda,\mu)\varphi(\mu). \tag{4}$$

Proof. If (4) is fulfilled, then φ is semifinite, since the prelimit sums give us an approximating sequence. If φ is semifinite and $\varphi(\lambda) < +\infty$, then (4) is a trivial consequence of Definition 3.1. If $\varphi(\lambda) = +\infty$, then we can find an approximating sequence and Observation 3.4 implies that the prelimit expression is unbounded in N. We are left to prove that the limit exists. In fact, we will show that the prelimit sequence is nondecreasing in N. Let us denote the prelimit expression by ψ_N.

Next, the function

$$\phi(\lambda) = \begin{cases} \varphi(\lambda) \text{ if } 0 < \varphi(\lambda) < +\infty, \\ 0 \text{ otherwise} \end{cases}$$

is *subharmonic*:

$$\phi(\lambda) \leqslant \sum_{\mu:\ \lambda\nearrow\mu} \varkappa(\lambda,\mu)\,\phi(\mu).$$

Then from the equality

$$\psi_N = \sum_{\mu:\ |\mu|=N} \dim(\lambda,\mu)\phi(\mu)$$

and (2) it follows that $\psi_1 \leqslant \psi_2 \leqslant \psi_3 \leqslant \dots$. □

Corollary 3.8. *If φ is a semifinite harmonic function on a graded graph Γ, then for any vertex $\lambda \in \Gamma$ with $\varphi(\lambda) = +\infty$ there exists a vertex $\mu \geqslant \lambda$ such that $0 < \varphi(\mu) < +\infty$.*

Remark 3.9. Let $\{c_\mu\}_{\mu\in\Gamma}$ be a tuple of nonnegative real "numbers" $c_\mu \in \mathbb{R}_{\geqslant 0} \cup \{+\infty\}$ such that for every vertex $\lambda \in \Gamma$ there exists a limit

$$\lim_{N\to\infty} \sum_{\mu\in\Gamma_N} \dim(\lambda,\mu)c_\mu,$$

which may be infinite. For instance, we may take $c_\mu = \psi(\mu)$ where ψ is a *subharmonic* function: $\psi(\lambda) \leqslant \sum\limits_{\mu:\lambda\nearrow\mu} \varkappa(\lambda,\mu)\psi(\mu)$. Then the function

$$\overline{c}(\lambda) = \lim_{N\to\infty} \sum_{\mu\in\Gamma_N} \dim(\lambda,\mu) c_\mu$$

is harmonic, cf. [12, p. 4], see also [7, formula (47)].

Definition 3.10. A semifinite harmonic function φ is said to be *indecomposable* if for any finite or semifinite harmonic function φ' that does not vanish identically on the finiteness ideal of φ and satisfies the inequality $\varphi' \leqslant \varphi$, we have $\varphi' = \mathrm{const}\cdot\varphi$ on the finiteness ideal of φ.

At first glance, it seems that the finiteness ideal of φ' might be bigger than that of φ, but the next remark shows that this is not the case.

Remark 3.11. If φ and φ' from Definition 3.10 are proportional on the finiteness ideal of φ, then they are proportional on the whole graph Γ. Indeed, by virtue of Proposition 3.7, we may write

$$\begin{aligned}\varphi(\lambda) &= \mathrm{const}^{-1}\cdot \lim_{N\to\infty} \sum_{\substack{\mu:\ \mu\geqslant\lambda,\ |\mu|=N\\ 0<\varphi(\mu)<+\infty}} \dim(\lambda,\mu)\varphi'(\mu)\\ &\leqslant \mathrm{const}^{-1}\cdot \lim_{N\to\infty} \sum_{\substack{\mu:\ \mu\geqslant\lambda,\ |\mu|=N\\ 0<\varphi'(\mu)<+\infty}} \dim(\lambda,\mu)\varphi'(\mu) = \mathrm{const}^{-1}\cdot\varphi'(\lambda).\end{aligned}$$

Thus, $\varphi' \leqslant \varphi \leqslant \mathrm{const}^{-1}\cdot\varphi'$, and the finitiness ideals of φ and φ' coincide.

Notation. The set of all indecomposable finite (not identically zero) and semifinite harmonic functions on a graded graph Γ is denoted by $\mathcal{H}_{\mathrm{ex}}(\Gamma)$. The subset of $\mathcal{H}_{\mathrm{ex}}(\Gamma)$ consisting of strictly positive functions is denoted by $\mathcal{H}^\circ_{\mathrm{ex}}(\Gamma)$.

Lemma 3.12. *Let I be an ideal of a graded graph Γ. Assume that a function $\varphi \in \mathcal{H}_{\mathrm{ex}}(\Gamma)$ does not vanish on I identically. Then the following equality holds:*

$$\varphi(\lambda) = \lim_{N\to\infty} \sum_{\substack{\mu:\mu\in I\\ |\mu|=N}} \dim(\lambda,\mu)\varphi(\mu), \quad \lambda\in\Gamma. \tag{5}$$

Moreover, for any element $a \in \mathrm{K}_0^+(\Gamma)$ we have $\varphi(a) = \sup\limits_{\substack{b\in\mathrm{K}_0^+(I):\ b\leqslant_K a,\\ \varphi(b)<+\infty}} \varphi(b)$.

Remark 3.13. If we omit the assumption that φ is indecomposable, then the equality above should be replaced by the inequality

$$\varphi(\lambda) \geqslant \lim_{N\to\infty} \sum_{\substack{\mu:\mu\in I\\ |\mu|=N}} \dim(\lambda,\mu)\varphi(\mu).$$

Proof of Lemma 3.12. First of all, we note that there exists a vertex $\nu \in I$ such that $0 < \varphi(\nu) < +\infty$. Indeed, φ does not vanish identically on I, hence we can find a vertex $\nu' \in I$ such that $\varphi(\nu') > 0$. If $\varphi(\nu') = +\infty$, then, by Corollary 3.8, we can find another vertex $\nu > \nu'$ with $0 < \varphi(\nu) < +\infty$, which necessarily lies in I.

Note that the function

$$\phi(\lambda) = \begin{cases} \varphi(\lambda) \text{ if } \lambda \in I,\\ 0 \text{ otherwise}\end{cases}$$

is subharmonic on Γ. Then, by Remark 3.9, the right-hand side of (5) defines a harmonic function on Γ. From Observation 3.4 and Remark 3.6 it follows that the restriction of φ to the ideal I is a finite or semifinite harmonic function on I. Then the harmonic function on Γ defined by the right-hand side of (5) is finite or semifinite as well. Next, by the very definition of harmonic functions, the prelimit expression is majorized by φ for any N. Then the harmonic function that is defined as the $N \to +\infty$ limit is also majorized by φ. Finally, the indecomposibility of φ implies that φ and the right-hand side of (5) are proportional, but φ and the right-hand side of (5) coincide on the ideal I. Thus, they coincide on the whole graph Γ, since there exists $\nu \in I$ with $0 < \varphi(\nu) < +\infty$. $\square$

Now we are ready to prove the most crucial result of Wassermann's method. The following theorem is a combinatorial analog of a result which is well known in the context of C^*-algebras, see [18, Theorem 7 on p. 143, Corollary on p. 144] and [1, II.6.1.6, p. 102].

Theorem 3.14. *Let I be an ideal of a graded graph Γ.*

1. *There is a bijective correspondence between $\{\varphi \in \mathcal{H}_{\mathrm{ex}}(\Gamma) \colon \varphi|_I \neq 0\}$ and $\mathcal{H}_{\mathrm{ex}}(I)$, defined by the following mutually inverse maps:*

$$\mathrm{Res}_I^\Gamma : \left\{\varphi \in \mathcal{H}_{\mathrm{ex}}(\Gamma) \colon \varphi|_I \neq 0\right\} \to \mathcal{H}_{\mathrm{ex}}(I), \quad \varphi \mapsto \varphi\big|_I,$$

$$\mathrm{Ext}_I^\Gamma : \mathcal{H}_{\mathrm{ex}}(I) \to \left\{\varphi \in \mathcal{H}_{\mathrm{ex}}(\Gamma) \colon \varphi|_I \neq 0\right\}, \quad \varphi(\cdot) \mapsto \lim_{N\to\infty} \sum_{\substack{\mu:\mu\in I\\ |\mu|=N}} \dim(\cdot,\mu)\varphi(\mu).$$

Furthermore, for any element $a \in \mathrm{K}_0^+(\Gamma)$ *we have*

$$\operatorname{Ext}_I^\Gamma(\varphi)(a) = \sup_{\substack{b \in \mathrm{K}_0^+(I)\colon b \leqslant_K a, \\ \varphi(b) < +\infty}} \varphi(b).$$

2. *If* Γ *is a primitive graded graph, then the bijection above preserves strictly positive harmonic functions:* $\mathcal{H}_{\mathrm{ex}}^\circ(I) \longleftrightarrow \mathcal{H}_{\mathrm{ex}}^\circ(\Gamma)$.

Proof. Suppose that $\varphi \in \mathcal{H}_{\mathrm{ex}}(\Gamma)$ and $\varphi|_I \neq 0$. Then from Observation 3.4 and Remark 3.6 it follows that $\operatorname{Res}_I^\Gamma(\varphi) = \varphi|_I$ is a finite or semifinite harmonic function on I. Lemma 3.12 implies that $\operatorname{Res}_I^\Gamma(\varphi)$ is indecomposable.

Now let $\varphi \in \mathcal{H}_{\mathrm{ex}}(I)$. From the proof of Proposition 3.7 it follows that the limit from the definition of $\operatorname{Ext}_I^\Gamma$ exists and $\operatorname{Ext}_I^\Gamma(\varphi)$ is a finite or semifinite harmonic function on Γ. Note that $\operatorname{Ext}_I^\Gamma(\varphi)$ is strictly positive for $\varphi \in \mathcal{H}_{\mathrm{ex}}^\circ(I)$ because of the following simple fact, which holds for any primitive graded graph. For any vertex $\lambda \in \Gamma$ there exists a vertex $\mu \in I$ such that $\mu \geqslant \lambda$.

Let us show that the harmonic function $\operatorname{Ext}_I^\Gamma(\varphi)$ is indecomposable for any $\varphi \in \mathcal{H}_{\mathrm{ex}}(I)$. Suppose that $\operatorname{Ext}_I^\Gamma(\varphi) \geqslant \psi$ for some ψ that does not vanish on the finiteness ideal of $\operatorname{Ext}_I^\Gamma(\varphi)$ identically. We denote the ideal by $\widetilde{I}$. The finiteness ideal of φ is denoted by I^φ. Let us introduce more notation: $\psi_1 = \psi|_{\widetilde{I}}$ and $\psi_2 = \operatorname{Ext}_I^{\widetilde{I}}(\varphi) - \psi_1$. Then ψ_1 and ψ_2 are finite harmonic functions on $\widetilde{I}$. Note that $\operatorname{Ext}_I^{\widetilde{I}}(\varphi) = \operatorname{Ext}_{I \cap I^\varphi}^{\widetilde{I}}(\varphi)$. On the one hand, we have $\operatorname{Ext}_{I \cap I^\varphi}^{\widetilde{I}}(\varphi) = \psi_1 + \psi_2$. On the other hand, $\varphi = \psi_1 + \psi_2$ on $I \cap I^\varphi$, hence

$$\operatorname{Ext}_{I \cap I^\varphi}^{\widetilde{I}}(\varphi) = \operatorname{Ext}_{I \cap I^\varphi}^{\widetilde{I}}(\psi_1) + \operatorname{Ext}_{I \cap I^\varphi}^{\widetilde{I}}(\psi_2) \leqslant \psi_1 + \psi_2,$$

where the last inequality follows from Remark 3.13. Therefore, $\psi_1 = \operatorname{Ext}_{I \cap I^\varphi}^{\widetilde{I}}(\psi_1)$ and $\psi_2 = \operatorname{Ext}_{I \cap I^\varphi}^{\widetilde{I}}(\psi_2)$. Let us rewrite the first equality in the form $\psi|_{\widetilde{I}} = \operatorname{Ext}_{I \cap I^\varphi}^{\widetilde{I}}(\psi)$. Then we see that the function $\psi|_{I^\varphi}$ does not vanish identically. Now, the indecomposability of φ implies that φ and ψ are proportional on I^φ. Thus, from $\psi|_{\widetilde{I}} = \operatorname{Ext}_{I \cap I^\varphi}^{\widetilde{I}}(\psi)$ and $\operatorname{Ext}_I^{\widetilde{I}}(\varphi) = \operatorname{Ext}_{I \cap I^\varphi}^{\widetilde{I}}(\varphi)$ it follows that $\operatorname{Ext}_I^\Gamma(\varphi)$ and ψ are proportional on $\widetilde{I}$.

Therefore, the maps $\operatorname{Res}_I^\Gamma$ and $\operatorname{Ext}_I^\Gamma$ are well defined, and the identity $\operatorname{Res}_I^\Gamma \circ \operatorname{Ext}_I^\Gamma = \mathrm{id}$ holds. The remaining identity $\operatorname{Ext}_I^\Gamma \circ \operatorname{Res}_I^\Gamma = \mathrm{id}$ immediately follows from Lemma 3.12. $\square$

Remark 3.15. Let $I_1 \subset I_2$ be ideals of Γ. Then $\mathrm{Ext}^{\Gamma}_{I_2} \circ \mathrm{Ext}^{I_2}_{I_1} = \mathrm{Ext}^{\Gamma}_{I_1}$.

Proposition 3.16 ([10, p. 35, Lemma 12])**.** *Let* Γ *be a graded graph. If* $\varphi \in \mathcal{H}_{\mathrm{ex}}(\Gamma)$*, then* $\mathrm{supp}(\varphi)$ *is a primitive coideal.*

Proof. Let $\lambda_1, \lambda_2 \in \mathrm{supp}(\varphi)$. Then Lemma 3.12 implies that

$$\varphi(\lambda_2) = \lim_{N\to\infty} \sum_{\substack{\mu:\mu\in\Gamma^{\lambda_1}\\ |\mu|=N}} \dim(\lambda_2,\mu)\varphi(\mu),$$

where $\Gamma^{\lambda_1} = \{\nu \in \Gamma \mid \nu \geqslant \lambda_1\}$. Then the inequality $\varphi(\lambda_2) > 0$ implies that there exists a vertex μ such that $\mu \geqslant \lambda_1, \lambda_2$ and $\varphi(\mu) \neq 0$. Thus, by virtue of Proposition 2.6, the coideal $\mathrm{supp}(\varphi)$ is primitive. □

4. Multiplicative branching graphs

In this section, we recall some basic notions related to multiplicative branching graphs [8,10]. For such graphs, we prove a theorem which states that some multiplicative branching graphs admit no strictly positive semifinite indecomposable harmonic functions [18, Theorem 8, p. 146]. We call this theorem Wassermann's forbidding theorem. We also prove a semifinite analog of the Vershik–Kerov ring theorem [11, p. 144].

Definition 4.1 ([10, p. 40])**.** A branching graph Γ is said to be *multiplicative* if there exists an associative $\mathbb{Z}_{\geqslant 0}$-graded $\mathbb{R}$-algebra $A = \bigoplus_{n\geqslant 0} A_n$, $A_0 = \mathbb{R}$, with a distinguished basis of homogeneous elements $\{a_\lambda\}_{\lambda\in\Gamma}$ that satisfy the following conditions:

1) $\deg a_\lambda = |\lambda|$,
2) $a_\varnothing$ is the identity in A,
3) for $\widehat{a} = \sum_{\nu\in\Gamma_1} \varkappa(\varnothing,\nu)a_\nu$ and any vertex $\lambda \in \Gamma$, we have

$$\widehat{a}\cdot a_\lambda = \sum_{\mu:\lambda\nearrow\mu} \varkappa(\lambda,\mu)a_\mu.$$

Moreover, we assume that the structure constants of A with respect to the basis $\{a_\lambda\}_{\lambda\in\Gamma}$ are nonnegative.

Let $(\Gamma, \varkappa)$ be the multiplicative graph that is related to an algebra A and a basis $\{a_\lambda\}_{\lambda\in\Gamma}$. We denote the quotient algebra $A/(\widehat{a}-1)$ by R, the canonical homomorphism $A \twoheadrightarrow R$ by $[\,\cdot\,]$, and the positive cone in R consisting of all elements that can be written in the form $\sum_{\lambda\in\Gamma_n} c_\lambda[a_\lambda]$

for a sufficiently large n and some $c_\lambda \geqslant 0$, by R^+. The correspondence $[\lambda] \mapsto [a_\lambda]$ defines an isomorphism of $\mathbb{R}$-vector spaces $\mathrm{K}_0(\Gamma) \xrightarrow{\sim} R$. The image of the cone $\mathrm{K}_0^+(\Gamma) \subset \mathrm{K}_0(\Gamma)$ under this map coincides with R^+.

Consider the positive cone $A^+ \subset A$ consisting of all elements of A that can be written as a linear combination of basis elements a_λ with non-negative coefficients. For any semifinite harmonic function $\varphi \in \mathcal{H}(\Gamma)$, we may speak about the $\mathbb{R}_{\geqslant 0}$-linear map $\varphi\colon A^+ \to \mathbb{R}_{\geqslant 0} \cup \{+\infty\}$.

Let us now formulate the Vershik–Kerov ring theorem [11, p. 134], see also [6, Proposition 8.4].

Definition 4.2. A harmonic function φ on a branching graph Γ is said to be *normalized* if $\varphi(\varnothing) = 1$.

Theorem 4.3 (Vershik–Kerov ring theorem [11, p. 134])**.** *A finite normalized harmonic function φ on a multiplicative branching graph Γ is indecomposable if and only if the corresponding functional on A is multiplicative: $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$ for any $a, b \in A$.*

The following semifinite analog of the ring theorem holds.

Theorem 4.4 ([11, p. 144])**.** *For any semifinite indecomposable harmonic function φ on a multiplicative branching graph Γ there exists a finite normalized indecomposable harmonic function ψ such that $\varphi(a\cdot b) = \psi(a)\cdot\varphi(b)$ for any $a, b \in A^+$ with $\varphi(b) < +\infty$.*

Proof. Note that

$$(\widehat{a})^n = \sum_{\nu\colon \nu\in\Gamma_n} \dim(\nu) \cdot a_\nu.$$

Then $\varphi\Big((\widehat{a})^n\, a_\mu\Big) = \varphi(a_\mu) \geqslant \dim(\lambda)\varphi(a_\lambda \cdot a_\mu)$ and $\varphi^\lambda(\mu) = \varphi(a_\lambda a_\mu)$ is a finite harmonic function on the finiteness ideal of φ. Since the restriction of φ to its finiteness ideal is an indecomposable harmonic function (see Lemma 3.12), it follows that there exists $c_\lambda \in \mathbb{R}_{\geqslant 0}$ such that $\varphi(a_\mu \cdot a_\lambda) = c_\lambda \varphi(a_\mu)$. We set $\psi(\lambda) = c_\lambda$ by definition. One can check that ψ is a harmonic function and that the functional on A defined by ψ is multiplicative. Then the Vershik–Kerov ring theorem implies that ψ is indecomposable. $\square$

From Theorem 4.4 it follows that the subspace

$$I = \mathrm{span}_{\mathbb{R}}\,(a_\lambda \mid \lambda\colon \varphi(\lambda) < +\infty) \subset A$$

is an ideal for any semifinite indecomposable harmonic function φ. However, the proof shows that this is true for an arbitrary harmonic function φ without any additional assumptions.

The following theorem imposes some restrictions on multiplicative graphs that possess strictly positive indecomposable semifinite harmonic functions, [18, Theorem 8, p. 146].

Theorem 4.5 (Wassermann's forbidding theorem)**.** *If $a_\lambda a_\mu \neq 0$ for any $\lambda, \mu \in \Gamma$, then the graph Γ admits no strictly positive semifinite indecomposable harmonic functions.*

Proof. Let φ be a strictly positive indecomposable semifinite harmonic function. The argument at the beginning of the proof of Theorem 4.4 shows that the function φ^μ defined by $\varphi^\mu(\lambda) = \varphi\,(a_\lambda a_\mu)$ is a finite harmonic function on Γ, while $\varphi(\mu) < +\infty$. Furthermore, the following inequality holds: $\varphi \geqslant \text{const} \cdot \varphi^\mu$. Next, observe that φ^μ is strictly positive, since $a_\lambda a_\mu \neq 0$ and the structure constants of A are nonnegative with respect to the basis $\{a_\lambda\}_{\lambda \in \Gamma}$. Therefore, φ and φ^μ are proportional. Thus, φ is finite. $\square$

Corollary 4.6 ([3, p. 371, the paragraph just before Theorem 3.5])**.** *If Γ admits a strictly positive indecomposable finite harmonic function, then it possesses no strictly positive semifinite indecomposable harmonic functions.*

Proof. Suppose that φ is a strictly positive indecomposable finite harmonic function and $a_\lambda a_\mu = 0$ for some $\lambda, \mu \in \Gamma$. Then $\varphi(a_\lambda a_\mu) = \varphi(0) = 0$, and Theorem 4.3 implies that $\varphi(\lambda)\varphi(\mu) = 0$, which contradicts the strict positivity of φ. $\square$

5. Boyer's lemma

In this section, we discuss a very useful result related to arbitrary harmonic functions on a graded graph. It allows one to determine the finiteness ideal of an indecomposable semifinite harmonic function in several concrete situations. This principle, which was first observed by R. P. Boyer and published only in 1983, see [4, Theorem 1.10, Example on p. 212], was also stated by Wassermann [18, Boyer's lemma, p. 149] two years before the paper [4]. We formulate and prove a slightly involved generalization of Wassermann's concise argument. It turns out to be a combinatorial analog of [4, Theorem 1.10]. After that, we consider a couple of examples, which immediately follow from the general claim. Boyer's lemma from [18] becomes a part of the first example, see Remark 5.6.

5.1. General statement. Recall that the set of vertices at the nth level of a graded graph Γ is denoted by Γ_n. Below we work with arbitrary harmonic functions and do *not* assume that they are finite or semifinite.

Definition 5.1. A harmonic function φ is said to be *semifinite at a vertex* λ if $\varphi(\lambda) = +\infty$ and there exists a sequence $\{a_n\}_{n\geqslant 1} \subset \mathrm{K}_0^+(\Gamma)$ such that

- $a_n \leqslant_K \lambda$,
- $\varphi(a_n) < +\infty$,
- $\lim\limits_{n\to+\infty} \varphi(a_n) = +\infty$.

The sequence $\{a_n\}_{n\geqslant 1}$ will be called an *approximating sequence for the vertex* λ.

Observation 5.2. If φ is semifinite at a vertex λ, then for any vertex $\mu \leqslant \lambda$ the function φ is semifinite at the vertex μ too.

Proposition 5.3 (generalized Boyer's lemma)**.** *Let $(\Gamma, \varkappa)$ be a graded graph and φ be a harmonic function on it. Assume that $I \subset \Gamma$ is an ideal, $J = \Gamma\backslash I$ is the corresponding coideal, and we are given a fixed vertex $\lambda \in J_n$. Suppose that there exists a positive integer $m = m(\lambda)$ and a tuple of nonnegative real numbers $\{\beta_\nu\}_{\nu\in I_m}$, which may depend on λ, such that the following conditions are satisfied:*

- *there exists a vertex $\nu \in I_m$ with $\beta_\nu \neq 0$ and $\varphi(\nu) > 0$,*
- *for any sufficiently large l and any vertex $\eta \in I_{n+l+1}$, the following inequality holds:*

$$\sum_{\mu\in J_{n+l}} \dim(\lambda,\mu)\varkappa(\mu,\eta) \geqslant \sum_{\nu\in I_m} \beta_\nu \dim(\nu,\eta). \tag{6}$$

Then $\varphi(\lambda) = +\infty$. If, additionally, $\varphi(\nu) < +\infty$ for any $\nu \in I_m$ such that $\beta_\nu \neq 0$, then φ is semifinite at the vertex λ.

Remark 5.4. Condition (6) is a refinement of some condition on the "number" of paths in the graph Γ, which admits a graphical interpretation, see condition (14) from Corollary 5.5 and Fig. 2.

Proof of Proposition 5.3. Let us multiply (6) by $\eta \in K_0(\Gamma)$ and sum over all $\eta \in I_{n+l+1}$. Then we get

$$\sum_{\substack{\eta\in I_{n+l+1}\\ \mu\in J_{n+l}}} \dim(\lambda,\mu)\varkappa(\mu,\eta)\cdot\eta \geqslant_K \sum_{\substack{\eta\in I_{n+l+1}\\ \nu\in I_m}} \beta_\nu \dim(\nu,\eta)\cdot\eta, \tag{7}$$

where both sides of the inequality are regarded as elements of $\mathrm{K}_0(\Gamma)$ and the partial order on $\mathrm{K}_0(\Gamma)$ defined by the cone $\mathrm{K}_0^+(\Gamma)$ is denoted by $\geqslant_K$. Furthermore, the right-hand side of inequality (7) equals $\sum\limits_{\nu\in I_m} \beta_\nu \nu$. Let us denote it by b_λ. Then $\varphi(b_\lambda) > 0$ and

$$\sum_{\substack{\eta\in I_{n+l+1}\\ \mu\in J_{n+l}}} \dim(\lambda,\mu)\varkappa(\mu,\eta)\eta \geqslant_K b_\lambda.$$

The only thing we still have to do is to reproduce the original argument of A. Wassermann [18, p. 149, the proof of Boyer's lemma] in our context:

$$\lambda = \sum_{\overline{\eta}\in\Gamma_{n+N+1}} \dim(\lambda,\overline{\eta})\overline{\eta} \geqslant_K \sum_{\overline{\eta}\in I_{n+N+1}} \dim(\lambda,\overline{\eta})\overline{\eta}. \tag{8}$$

Note that if $\lambda\in J_n$ and $\overline{\eta}\in I_{n+N+1}$, then

$$\dim(\lambda,\overline{\eta}) = \sum_{l=0}^{N} \sum_{\substack{\eta\in I_{n+l+1}\\ \mu\in J_{n+l}}} \dim(\lambda,\mu)\varkappa(\mu,\eta)\dim(\eta,\overline{\eta}). \tag{9}$$

Substitute (9) into (8):

$$\lambda \geqslant_K \sum_{l=0}^{N} \sum_{\substack{\eta\in I_{n+l+1}\\ \mu\in J_{n+l}}} \sum_{\overline{\eta}\in I_{n+N+1}} \dim(\lambda,\mu)\varkappa(\mu,\eta)\dim(\eta,\overline{\eta})\overline{\eta}.$$

Now sum over $\overline{\eta}$:

$$\lambda \geqslant_K \sum_{l=0}^{N} \sum_{\substack{\eta\in I_{n+l+1}\\ \mu\in J_{n+l}}} \dim(\lambda,\mu)\varkappa(\mu,\eta)\eta \geqslant_K b_\lambda\cdot N. \tag{10}$$

Compare (10) with (1.10.1) and (1.10.2) from [4, Theorem 1.10].

Thus, (10) yields $\varphi(\lambda) \geqslant \varphi(b_\lambda)\cdot N$ for any N, hence $\varphi(\lambda) = +\infty$. Moreover, the sequence $a_N = b_\lambda\cdot N$ is an approximating sequence for the vertex λ if $\varphi(b_\lambda) < +\infty$. □

5.2. Example 1. Consider graded graphs $(\Gamma_1,\varkappa_1)$ and $(\Gamma_2,\varkappa_2)$ and suppose that we are given a graded map $\Gamma_1\to\Gamma_2$, $\lambda\mapsto\lambda'$. Let $(\Gamma,\varkappa)$ be still another graded graph that satisfies the following requirements:

$$(\Gamma)_n = (\Gamma_1)_n \sqcup (\Gamma_2)_{n-1} \text{ for } n\geqslant 1,\ (\Gamma)_0 = (\Gamma_1)_0\,; \tag{11}$$

$$\begin{aligned} \varkappa(\lambda, \mu) &= \varkappa_1(\lambda, \mu) \ \text{ if } \lambda, \mu \in \Gamma_1, \\ \varkappa(\lambda, \mu) &= \varkappa_2(\lambda, \mu) \ \text{ if } \lambda, \mu \in \Gamma_2; \end{aligned} \tag{12}$$

$$\varkappa(\lambda, \mu) = 0 \text{ if } \lambda \in \Gamma_2, \mu \in \Gamma_1. \tag{13}$$

Condition (13) means that Γ_2 is an ideal of Γ. For simplicity, one may assume that edges from Γ_1 to Γ_2 can go from λ to λ' only, see Fig. 1. But we will not use this in what follows.

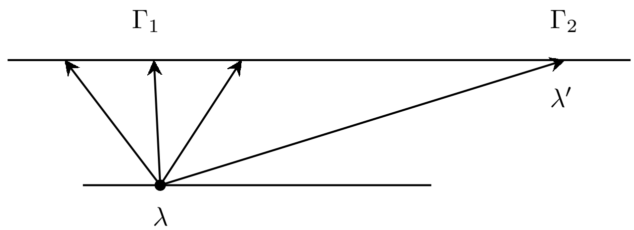

Fig. 1. An example of a branching rule for Γ.

Corollary 5.5. *Assume that the map $\nu \mapsto \nu'$ is surjective, and let $\lambda \in (\Gamma_1)_n$ be a fixed vertex. Suppose that for any sufficiently large l and any vertex $\mu \in (\Gamma_1)_{n+l}$, the following inequality holds:*

$$\dim_1(\lambda, \mu)\varkappa(\mu, \mu') \geqslant \dim_2(\lambda', \mu'), \tag{14}$$

where $\dim_1(\,\cdot\,,\,\cdot\,)$ and $\dim_2(\,\cdot\,,\,\cdot\,)$ are the shifted dimensions for $(\Gamma_1, \varkappa_1)$ and $(\Gamma_2, \varkappa_2)$. Now let φ be a harmonic function on Γ with $\varphi(\lambda') > 0$. Then $\varphi(\lambda) = +\infty$, and φ is semifinite at the vertex λ if $\varphi(\lambda') < +\infty$.

Proof. Recall that Γ_2 is an ideal of Γ. Therefore, we may apply Proposition 5.3 for $I = \Gamma_2$, $J = \Gamma_1$, $m = |\lambda| + 1$, and $\beta_\nu = \delta_{\nu,\lambda'}$. Then we bound the sum in the left-hand side of (6) from below in terms of one of its summands and use (14). □

Remark 5.6. If the map $\lambda \mapsto \lambda'$ is a branching graph morphism, i.e., $\varkappa(\lambda, \mu) = \varkappa(\lambda', \mu')$, then condition (14) means that $\varkappa(\mu, \mu') \geqslant 1$. If the equality holds identically, then we obtain the original formulation of Boyer's lemma [18, p. 149, Boyer's lemma].

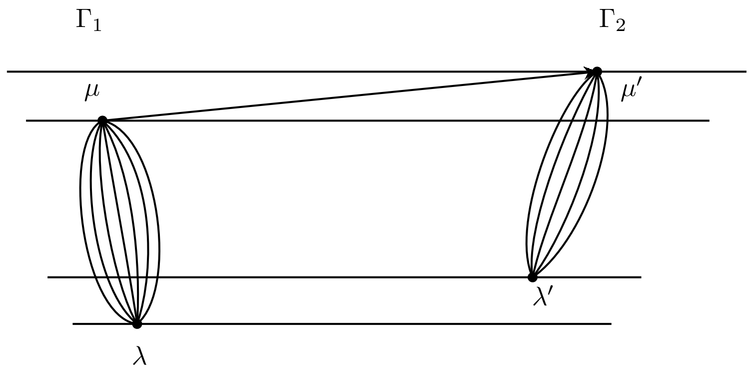

Fig. 2. Condition (14) means that the "number" of paths from λ to μ' that pass through μ is not less than the "number" of arbitrary paths from λ' to μ'.

5.3. Example 2. Consider graded graphs $(\Gamma_1, \varkappa_1)$ and $(\Gamma_2, \varkappa_2)$ and suppose that we are given a graded map $\Gamma_1 \to \Gamma_2$, $\lambda \mapsto \lambda'$. Let $(\Gamma, \varkappa)$ be another graded graph that satisfies the condition $(\Gamma)_n = (\Gamma_1)_n \sqcup (\Gamma_2)_n$ for $n \geqslant 0$ and conditions (12), (13). Recall that the last condition means that Γ_2 is an ideal of Γ. For simplicity, one may assume that vertices $\lambda \in \Gamma_1$ and $\mu \in \Gamma_2$ are joined by an edge if and only if $\lambda' \nearrow \mu$, as shown in Fig. 3.

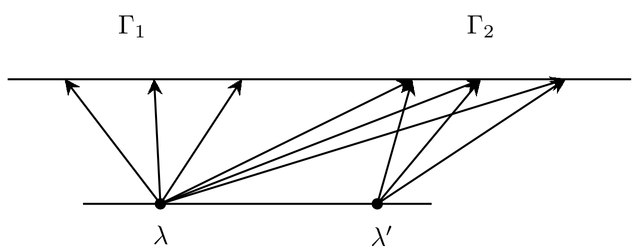

Fig. 3. An example of a branching rule for Γ.

Corollary 5.7. *Suppose that the map $\lambda \mapsto \lambda'$ is surjective. Let $\lambda \in \Gamma_1$ be a fixed vertex, and assume that the following inequalities hold for any $\mu \in \Gamma_1$:*

$$\varkappa(\lambda, \mu) \geqslant \varkappa(\lambda', \mu'),$$

$$\varkappa(\lambda,\mu') \geqslant \varkappa(\lambda',\mu').$$

Then $\varphi(\lambda) = +\infty$ *for any harmonic function* φ *on* Γ *such that* $\varphi(\lambda') > 0$. *Moreover,* φ *is semifinite at the vertex* λ *if* $0 < \varphi(\lambda') < +\infty$.

Proof. Let us take $I = \Gamma_2$, $J = \Gamma_1$, $m = |\lambda|$, and $\beta_\nu = \delta_{\nu,\lambda'}$ in Proposition 5.3 and prove that $\sum_{\mu\in\Gamma_1} \dim(\lambda,\mu)\varkappa(\mu,\eta) \geqslant \dim(\lambda',\eta)$ for any $\eta \in \Gamma_2$. In order to do so, we check that $\dim(\lambda,\mu) \geqslant \dim(\lambda',\mu')$ and write

$$\frac{\sum_{\mu\in\Gamma_1} \dim(\lambda,\mu)\varkappa(\mu,\eta)}{\dim(\lambda',\eta)} \geqslant \frac{\sum_{\mu\in\Gamma_1} \dim(\lambda',\mu')\varkappa(\mu',\eta)}{\dim(\lambda',\eta)}$$
$$\geqslant \frac{\sum_{\overline{\mu}\in\Gamma_2} \dim(\lambda',\overline{\mu})\varkappa(\overline{\mu},\eta)}{\dim(\lambda',\eta)} = 1.$$

For each of these inequalities, we have used the fact that the map $\lambda \mapsto \lambda'$ is surjective. □

Remark 5.8. As it was pointed out in the introduction, one can obtain an exhaustive list of indecomposable semifinite harmonic functions on the Macdonald graph, which corresponds to the simplest Pieri rule for the Macdonald symmetric functions, by applying Wassermann's method. This list turns out to be very similar to that for the Young graph, see [18, Theorem 9, p. 150]. For instance, the space of classification parameters is an obvious (q,t)-deformation of the parameter space for the Young graph. Namely, we should deform only the continuous part of the data in the same way as it deforms in the case of finite harmonic functions, replacing the ordinary Thoma simplex with the (q,t)-deformed Thoma simplex, see Theorem 1.4 and Proposition 1.6 from [13], while the discrete part remains the same. This result easily follows from the original argument of A. Wassermann, Theorem 1.4 and Proposition 1.6 from [13], Proposition 2.6, Theorem 3.14, Proposition 3.16, and Corollary 5.5. Instead of using Theorem 4.5, we must apply a similar argument obtained with the help of a trick due to K. Matveev [13, §6, proof of Proposition 1.6].

Appendix A. Direct product of branching graphs

In this appendix, we describe indecomposable finite harmonic functions on the product of branching graphs in terms of harmonic functions on the factors. This result is not related to semifinite harmonic functions in a straightforward way, but it turns out to be very useful for describing semifinite harmonic functions on some branching graphs, such as the Gnedin–Kingman graph [14] and the zigzag graph. The latter was studied in the paper [6]. One can treat the main result of this appendix, Proposition A.4, as a generalization of the well-known de Finetti theorem [2, Theorems 5.1 and 5.2]. The difference between Proposition A.4 (the case $n = 2$) and the de Finetti theorem is that we replace two sides of the Pascal triangle, which correspond to two embeddings $\mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \mathbb{Z}$ along the first and the second components, with arbitrary branching graphs. Note that the case where one of these graphs is a line consisting of one vertex at each level was already known, see [17, Theorem 2.8]. Observe that in this theorem one should consider only strictly positive harmonic functions (or, equivalently, central measures) instead of arbitrary ones.

Let us provide some motivation for the main definition of the present section. If A and B are unital $\mathbb{Z}_{\geqslant 0}$-graded $\mathbb{R}$-algebras, then their tensor product (over $\mathbb{R}$) is a unital graded algebra too. Namely, if $A = \bigoplus_{n\geqslant 0} A_n$, $A_0 = \mathbb{R}$ and $B = \bigoplus_{n\geqslant 0} B_n$, $B_0 = \mathbb{R}$, then $A \otimes_{\mathbb{R}} B = \bigoplus_{k\geqslant 0} (A \otimes_{\mathbb{R}} B)_k$, where

$$(A \otimes_{\mathbb{R}} B)_k = \bigoplus_{\substack{n,m\geqslant 0:\\ n+m=k}} A_n \otimes_{\mathbb{R}} B_m.$$

Furthermore, $\mathbf{1}_{A\otimes B} = \mathbf{1}_A \otimes \mathbf{1}_B$ and $(A \otimes_{\mathbb{R}} B)_0 = \mathbb{R}\cdot\mathbf{1}_{A\otimes B}$. This simple fact, together with Definition 4.1, motivates us to consider the *direct product* of two graded graphs.

Definition A.1. By the *direct product* of graded graphs $(\Gamma_1, \varkappa_1)$ and $(\Gamma_2, \varkappa_2)$ we mean the graded graph $(\Gamma_1 \times \Gamma_2, \varkappa_1 \times \varkappa_2)$ where

$$(\Gamma_1 \times \Gamma_1)_k = \bigsqcup_{\substack{n,m\geqslant 0:\\ n+m=k}} (\Gamma_1)_n \times (\Gamma_2)_m$$

and

$$(\varkappa_1 \times \varkappa_2)\Big((\lambda_1, \mu_1); (\lambda_2, \mu_2)\Big) = \begin{cases} \varkappa_1(\lambda_1, \lambda_2) & \text{if } \mu_1 = \mu_2, \\ \varkappa_2(\mu_1, \mu_2) & \text{if } \lambda_1 = \lambda_2, \\ 0 & \text{otherwise.} \end{cases}$$

The next lemma ties together some properties of the direct product of graded graphs.

The subset $\Gamma_\lambda = \{\mu \in \Gamma \mid \mu \leqslant \lambda\}$ of a graded graph Γ is called the *principal coideal* associated to $\lambda \in \Gamma$.

Lemma A.2. *Let Γ_1 and Γ_2 be graded graphs.*

1. *The graph $\Gamma_1 \times \Gamma_2$ is primitive if and only if Γ_1 and Γ_2 are primitive.*

2. *If Γ_1 and Γ_2 are branching graphs and $J \subset \Gamma_1 \times \Gamma_2$ is a saturated primitive coideal, then there exist coideals $J_1 \subset \Gamma_1$ and $J_2 \subset \Gamma_2$ such that $J = J_1 \times J_2$ and*
   - *J_1, J_2 are saturated and primitive, or*
   - *J_1 is principal and J_2 is saturated and primitive, or*
   - *J_1 is saturated and primitive and J_2 is principal.*

   Moreover, coideals J_1 and J_2 are uniquely defined.

3. *Let $\lambda, \lambda' \in \Gamma_1$ and $\mu, \mu' \in \Gamma_2$. Then*
$$\dim\Big((\lambda,\mu),(\lambda',\mu')\Big) = \binom{|\lambda'|-|\lambda|+|\mu'|-|\mu|}{|\lambda'|-|\lambda|} \dim_1\big(\lambda,\lambda'\big)\dim_2\big(\mu,\mu'\big),$$
   where $\binom{n}{k}$ denotes the binomial coefficient and $\dim_1(\,\cdot\,,\,\cdot\,)$, $\dim_2(\cdot,\cdot)$ are the shifted dimensions for Γ_1 and Γ_2, see formula (1) on p. 116.

Proof. The first and the second assertions follow from Proposition 2.6 immediately, and the third one is obvious. $\square$

Note that we can easily generalize the statement of Lemma A.2 to the case of $n > 2$ graded graphs. Furthermore, the direct product of multiplicative graphs is multiplicative too. For the direct product of two multiplicative graphs, the corresponding algebra is the tensor product of the original algebras, the distinguished basis is the tensor product of the bases, and the element that was denoted by $\widehat{a}$ in Definition 4.1 is $\widehat{a} \otimes_{\mathbb{R}} \mathbf{1}_B + \mathbf{1}_A \otimes_{\mathbb{R}} \widehat{b}$, where $\widehat{a}$ and $\widehat{b}$ are the same elements for the original algebras. Thus, we can define the direct product of finitely many graded graphs, and the product of multiplicative graphs is multiplicative as well.

Recall that a harmonic function φ on a branching graph Γ is said to be normalized if $\varphi(\varnothing) = 1$.

Remark A.3. Let $\Gamma_1, \ldots, \Gamma_n$ be branching graphs; let $\varphi_1, \ldots, \varphi_n$ be finite normalized harmonic functions on these graphs and $w_1, \ldots, w_n$ be real positive numbers such that $w_1 + \ldots + w_n = 1$. Then the function

$$\varphi\colon \Gamma_1 \times \ldots \times \Gamma_n \to \mathbb{R}_{\geqslant 0}$$

defined by

$$\varphi(\lambda_1, \ldots, \lambda_n) = w_1^{|\lambda_1|} \ldots w_n^{|\lambda_n|} \varphi_1(\lambda_1) \ldots \varphi_n(\lambda_n) \tag{A.1}$$

is harmonic and normalized.

Note that we can recover these $\varphi_1, \ldots, \varphi_n$ and $w_1, \ldots, w_n$ from φ as follows. Let us set

$$\mathrm{MC}(a_1, \ldots, a_n) = \binom{a_1 + \ldots + a_n}{a_1, \ldots, a_n} = \frac{(a_1 + \ldots + a_n)!}{a_1! \ldots a_n!}.$$

Then

$$\begin{aligned} \varphi_i(\mu) = \sum_{\substack{\lambda_j \in \Gamma_j, j \neq i \\ j=1,\ldots,n}} & \mathrm{MC}\Big(|\lambda_1|, \ldots, |\lambda_{i-1}|, |\mu| - 1, |\lambda_{i+1}|, \ldots, |\lambda_n|\Big) \\ & \times \prod_{\substack{j=1 \\ j \neq i}}^{n} \dim(\lambda_j) \cdot \varphi(\lambda_1, \ldots, \lambda_{i-1}, \mu, \lambda_{i+1}, \ldots, \lambda_n) \end{aligned} \tag{A.2}$$

for $|\mu| \geqslant 1$ and

$$w_1^{k_1} \ldots w_n^{k_n} = \sum_{\substack{\lambda_i \in \Gamma_i, |\lambda_i| = k_i \\ i=1,\ldots,n}} \dim(\lambda_1) \ldots \dim(\lambda_n) \cdot \varphi(\lambda_1, \ldots, \lambda_n) \tag{A.3}$$

for any positive integers $k_1, \ldots, k_n$.

Compare (A.2) and (A.3) with the first two formulas from the proof of Theorem 2.8 in [17].

Notation. Let $(\Gamma, \varkappa)$ be a branching graph. We denote by $\mathcal{FH}_{\mathrm{ex}}(\Gamma)$ the set of all finite normalized harmonic functions on Γ, and by $\mathcal{FH}^{\circ}_{\mathrm{ex}}(\Gamma)$ the subset of all strictly positive functions.

Proposition A.4. *Let $\Gamma_1, \ldots, \Gamma_n$ be branching graphs and Δ_n^0 be the interior of the $(n-1)$-dimensional simplex, i.e.,*

$$\Delta_n^0 = \{(w_1, \ldots, w_n) \mid w_1 + \ldots + w_n = 1,\ w_i > 0\}.$$

1) *There is a bijection between the sets $\mathcal{FH}^{\circ}_{\mathrm{ex}}(\Gamma_1 \times \ldots \times \Gamma_n)$ and*

$$\mathcal{FH}^{\circ}_{\mathrm{ex}}(\Gamma_1) \times \ldots \times \mathcal{FH}^{\circ}_{\mathrm{ex}}(\Gamma_n) \times \Delta_n^0$$

defined by (A.1).

2) *There is a bijection between the sets* $\mathcal{FH}_{\mathrm{ex}}(\Gamma_1 \times \ldots \times \Gamma_n)$ *and*

$$\bigsqcup_{\substack{I:\ I\subset\{1,2,\ldots,n\}\\ I\neq\varnothing}} \Delta^0_{|I|} \times \bigtimes_{i\in I} \mathcal{FH}_{\mathrm{ex}}(\Gamma_i).$$

More precisely, for any harmonic function $\varphi \in \mathcal{FH}_{\mathrm{ex}}(\Gamma_1 \times \ldots \times \Gamma_n)$ *there exist a nonempty set* $I \subset \{1, 2, \ldots, n\}$*, harmonic functions* $\varphi_i \in \mathcal{FH}_{\mathrm{ex}}(\Gamma_i)$*, which are indexed by* $i \in I$*, and* $w \in \Delta^0_{|I|}$ *such that for any* n*-tuple of vertices* $\lambda_1 \in \Gamma_1, \ldots, \lambda_n \in \Gamma_n$ *the following identity holds:*

$$\varphi\left(\lambda_1, \ldots, \lambda_n\right) = \begin{cases} \prod\limits_{i\in I} w_i^{|\lambda_i|}\varphi_i(\lambda_i) \ \textit{if } \lambda_j = \varnothing \textit{ for every } j \in \{1, 2, \ldots, n\}\backslash I, \\ 0 \ \textit{otherwise}. \end{cases}$$

Moreover, these I*,* φ_i*, and* w *are uniquely defined.*

Remark A.5. For multiplicative graphs, Proposition A.4 is a straightforward consequence of the Vershik–Kerov ring theorem (Theorem 4.3). Namely, we should apply this theorem to the following elementary fact:

$$\operatorname{Hom}\Big(A_1 \otimes_{\mathbb{R}} \ldots \otimes_{\mathbb{R}} A_n, \mathbb{R}\Big) = \bigtimes_{i=1}^{n} \operatorname{Hom}\left(A_i, \mathbb{R}\right),$$

where Hom stands for the set of algebra homomorphisms. Indeed, to prove the first part of the proposition, we note that there are two mutually inverse maps

$$\begin{gathered}\Phi_{\rightarrow}\colon \mathcal{FH}^{\circ}_{\mathrm{ex}}(\Gamma_1) \times \ldots \times \mathcal{FH}^{\circ}_{\mathrm{ex}}(\Gamma_n) \times \Delta^0_n \longrightarrow \mathcal{FH}^{\circ}_{\mathrm{ex}}(\Gamma_1 \times \ldots \times \Gamma_n),\\ (\varphi_1, \ldots, \varphi_n, w) \mapsto (\varphi_1 \circ r_{w_1}) \otimes \ldots \otimes (\varphi_n \circ r_{w_n}),\end{gathered}$$

and

$$\begin{gathered}\Phi_{\leftarrow}\colon \mathcal{FH}^{\circ}_{\mathrm{ex}}(\Gamma_1 \times \ldots \times \Gamma_n) \longrightarrow \mathcal{FH}^{\circ}_{\mathrm{ex}}(\Gamma_1) \times \ldots \times \mathcal{FH}^{\circ}_{\mathrm{ex}}(\Gamma_n) \times \Delta^0_n,\\ \varphi \mapsto \left(\varphi\,|_{A_1} \circ r^{-1}_{w_1}, \ldots, \varphi\,|_{A_n} \circ r^{-1}_{w_n}, w\right).\end{gathered}$$

Here r_u denotes the automorphism of a graded algebra defined on homogeneous elements as $a \mapsto u^{\deg a} a$, and $\varphi\,|_{A_i}$ is the restriction of the map $\varphi\colon A_1 \otimes \ldots \otimes A_n \to \mathbb{R}$ to the subalgebra $1^{\otimes i-1} \otimes A_i \otimes 1^{\otimes n-i} \simeq A_i$. Furthermore, the n-tuple $w = (w_1, \ldots, w_n)$ that appears in the definition of the map $\Phi_{\leftarrow}$ has the following form: $w_i = \varphi\left(1^{\otimes i-1} \otimes \widehat{a}^{(i)} \otimes 1^{\otimes n-i}\right)$. Recall that the element $\widehat{a}^{(i)} \in A_i$ defines the branching rule for Γ_i, see Definition 4.1.

Proof of Proposition A.4. We prove the first part of the proposition for $n = 2$ only. The case $n > 2$ can be dealt with in the same manner. One can prove the second part of the proposition applying essentially the same argument, Proposition 3.16, and the second part of Lemma A.2.

One can check that for any harmonic function φ on $\Gamma_1 \times \Gamma_2$, the right-hand side of (A.2) defines a harmonic function on Γ_i. Thus, the function φ defined by (A.1) is indecomposable if φ_1 and φ_2 are indecomposable. Then (A.1) defines an injective map $\mathcal{FH}^\circ_{\mathrm{ex}}(\Gamma_1) \times \mathcal{FH}^\circ_{\mathrm{ex}}(\Gamma_2) \times \Delta_2^0 \longrightarrow \mathcal{FH}^\circ_{\mathrm{ex}}(\Gamma_1 \times \Gamma_2)$. Using the Vershik–Kerov ergodic method, see [9, p. 20, Theorem 2], [8, p. 60], we will show that this map is also surjective. Let φ be a finite strictly positive normalized indecomposable harmonic function on $\Gamma_1 \times \Gamma_2$. Then, by [8, p. 60], there exists a path

$$\tau = \Big((\varnothing, \varnothing), (\lambda_1, \mu_1), \dots\Big) \in \mathcal{T}(\Gamma_1 \times \Gamma_2)$$

such that

$$\varphi(\lambda, \mu) = \lim_{N \to +\infty} \frac{\dim\Big((\lambda, \mu), (\lambda'_N, \mu'_N)\Big)}{\dim\Big((\lambda'_N, \mu'_N)\Big)}.$$

From the last part of Lemma A.2 it follows that

$$\frac{\dim\Big((\lambda, \mu), (\lambda'_N, \mu'_N)\Big)}{\dim\Big((\lambda'_N, \mu'_N)\Big)} = \frac{\Big(|\lambda'_N|\Big)^{\downarrow|\lambda|} \cdot \Big(|\mu'_N|\Big)^{\downarrow|\mu|}}{\Big(|\lambda'_N| + |\mu'_N|\Big)^{\downarrow(|\lambda|+|\mu|)}} \cdot \frac{\dim_1(\lambda, \lambda'_N)}{\dim_1(\lambda'_N)} \cdot \frac{\dim_2(\mu, \mu'_N)}{\dim_2(\mu'_N)},$$

where $x^{\downarrow k} = x(x-1)\dots(x-k+1)$. Then the strict positivity of φ implies that $|\lambda'_N| \to +\infty$ and $|\mu'_N| \to +\infty$ as $N \to +\infty$. Therefore, passing to appropriate subsequences, we may assume that the following limits exist:

$$\lim_{N \to +\infty} \frac{\dim_1(\lambda, \lambda'_N)}{\dim_1(\lambda'_N)}, \quad \lim_{N \to +\infty} \frac{\dim_2(\mu, \mu'_N)}{\dim_2(\mu'_N)},$$

$$\lim_{N \to +\infty} \frac{|\lambda'_N|}{|\lambda'_N| + |\mu'_N|}, \quad \lim_{N \to +\infty} \frac{|\mu'_N|}{|\lambda'_N| + |\mu'_N|}.$$

Denoting them by $\varphi_1(\lambda)$, $\varphi_2(\mu)$, w_1, and w_2, we obtain a desired element of $\mathcal{FH}^\circ_{\mathrm{ex}}(\Gamma_1) \times \mathcal{FH}^\circ_{\mathrm{ex}}(\Gamma_2) \times \Delta_2^0$. Note that these φ_1 and φ_2 are indecomposable, since φ is indecomposable. $\square$

Example A.6. Let us take $\Gamma_1 = \ldots = \Gamma_n = \mathbb{Z}_{\geqslant 0}$ and assume that all edges are simple and go from k to $k+1$ for $k \geqslant 0$. Then $\mathcal{FH}^\circ_{\mathrm{ex}}(\Gamma_i) = \mathcal{FH}_{\mathrm{ex}}(\Gamma_i)$ is a singleton and $\Gamma_1 \times \ldots \times \Gamma_n$ is the Pascal pyramid $\mathbb{P}_n$. Then from Proposition A.4 it follows that $\mathcal{FH}^{\circ}_{\mathrm{ex}}(\mathbb{P}_n) = \Delta^0_n$ and $\mathcal{FH}_{\mathrm{ex}}(\mathbb{P}_n) = \bigsqcup_{\substack{I\colon I\subset\{1,2,\dots,n\}\\ I\neq\varnothing}} \Delta^0_{|I|} = \Delta_n$, which is the $(n-1)$-dimensional simplex.

Remark A.7. Proposition A.4 gives us the following view on Kerov's construction [6, §4]. The comultiplication provides us with a linear map $\mathrm{K}_0(\Gamma) \to \mathrm{K}_0(\underbrace{\Gamma\times\ldots\times\Gamma}_{n})$, and we take the composition of this map with an indecomposable harmonic function on $\underbrace{\Gamma\times\ldots\times\Gamma}_{n}$ to obtain an indecomposable harmonic function on Γ.

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