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STATISTICS OF IRREDUCIBLE COMPONENTS IN LARGE TENSOR POWERS OF THE SPINOR REPRESENTATION FOR $\mathfrak{so}_{2n+1}$ AS $n \to \infty$

Abstract. We consider the Plancherel measure on irreducible components of tensor powers of the spinor representation of $\mathfrak{so}_{2n+1}$. With respect to this measure, the probability of an irreducible representation is the product of its multiplicity and dimension, divided by the total dimension of the tensor product. We study the limit shape of the highest weight as the tensor power N and the rank n of the algebra tend to infinity with N/n fixed.

Dedicated to the memory of Vladimir Dmitrievich Lyakhovsky (1942–2020)

Introduction

The limit shape problem for Young diagrams in the decomposition of a tensor power of a representation of a semisimple Lie algebra has been extensively studied for the Nth tensor power of the vector fundamental representation of $\mathfrak{sl}_{n+1}$. In this case, the multiplicities of irreducible components are the dimensions of irreducible representations of the symmetric group S_N, due to the Schur–Weyl duality. The Plancherel-type measure associated with this decomposition was first considered by Kerov [10], and its asymptotic behavior has been studied in three regimes: $N \to \infty$ with n fixed, $N \to \infty, n \to \infty$ with N/n fixed, and $N, n \to \infty$ with N/n^2 fixed. The first case was studied in [10] and later generalized to all simple Lie algebras in [17, 18, 20]. In the second case, Kerov discovered that the Vershik–Kerov–Logan–Shepp limit shape of Young diagrams with respect to the Plancherel measure on S_N as $N \to \infty$ also appears as the limit

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shape with respect to this measure. Later, P. Biane [1, 2] described the limit shapes in the third case.

In the present paper, we consider statistics of irreducible components in the Nth tensor power of the spinor representation V^{ω_n} of the algebra $\mathfrak{so}_{2n+1}$ in the limit as $N, n \to \infty$. We give an explicit formula for the limit shape in the limit as $N \to \infty$ and N/n is finite in terms of boundaries of generalized Young diagrams [9, 15] (see Fig. 3). The main theorem of the present paper can be formulated as follows.

Theorem 1. *As* $n \to \infty$, $N \to \infty$, $c = \lim_{n,N\to\infty} \frac{N+2n-1}{n} = \mathrm{const}$, *the upper boundary* f_n *of a rotated and scaled generalized Young diagram for a highest weight in the decomposition of the tensor power of the spinor representation* $(V^{\omega_n})^{\otimes N}$ *of the simple Lie algebra* $\mathfrak{so}_{2n+1}$ *into irreducible representations converges in probability in the supremum norm* $||\cdot||_\infty$ *to the limit shape given by the formula* $f(x) = 1 + \int\limits_0^x (1 - 4\rho(t))\mathrm{d}t$, *where the limit density* $\rho(x)$ *can be written explicitly as*

$$\rho(x) = \begin{cases} \frac{\theta(\sqrt{2c-4}-|x|)}{4\pi}\left[\arctan\left(\frac{-c(x-4)-8}{(c-4)\sqrt{2c-4-x^2}}\right) + \arctan\left(\frac{c(x+4)-8}{(c-4)\sqrt{2c-4-x^2}}\right)\right], & c \geqslant 4, \\ \frac{1}{2} - \frac{\theta(\sqrt{2c-4}-|x|)}{4\pi}\left[\arctan\left(\frac{-c(x-4)-8}{(4-c)\sqrt{2c-4-x^2}}\right) + \arctan\left(\frac{c(x+4)-8}{(4-c)\sqrt{2c-4-x^2}}\right)\right], & c \in [2,4], \end{cases} \tag{1}$$

where $\theta(\sqrt{2c-4} - |x|)$ *is the Heaviside step function.*

That is, for all $\varepsilon > 0$ *we have*

$$\mathbb{P}\left(\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| > \varepsilon\right) \xrightarrow[n\to\infty]{} 0.$$

The paper is organized as follows. In Sec. 1, we introduce the probability measure, the required notation and describe the generalized Young diagrams and their boundaries f_n that are used to state Theorem 1. In Sec. 2, we give a sketch of the proof. The detailed proof of the convergence and the derivation of formula (1) will be presented in the separate paper [16]. In conclusion we state open problems related to the presented results.

§1. Definitions and notation

Consider the Lie algebra $\mathfrak{g} = \mathfrak{so}_{2n+1}$. In the standard orthogonal basis, the simple roots are $\{\alpha_i = e_i - e_{i+1} | i = 1, \ldots, n-1\} \cup \{\alpha_n = e_n\}$. The root system B_n consists of the roots $\Delta = \{\pm e_i \pm e_j | \, i \neq j\} \cup \{\pm e_i\}$, the positive roots are $\Delta^+ = \{e_i + e_j | i < j\} \cup \{e_i\} \cup \{e_j - e_i | j < i\}$. The fundamental weights of B_n in the same basis are given by the following formulas: $\omega_1 = e_1$, $\omega_2 = e_1 + e_2$, $\ldots$, $\omega_{n-1} = e_1 + \cdots + e_{n-1}$, $\omega_n = \frac{1}{2}(e_1 + \cdots + e_n)$. Dominant integral weights λ are linear combinations of fundamental weights with nonnegative integer coefficients l_n, which are called Dynkin labels: $\lambda = \sum_{i=1}^{n} l_n \omega_n$. In the orthogonal coordinates, such a weight is written as $\lambda = \sum_{i=1}^{n} \left(l_i + l_{i+1} + \cdots + \frac{l_n}{2}\right) e_i$.

Dominant integral weights can be depicted by generalized Young diagrams ("diagrams"). For the algebras $\mathfrak{so}_{2n+1}$, it is convenient to use diagrams with boxes of two different widths, one being twice the other [9, 15] (see also [8]). In the present case, the analog of the Littlewood–Richardson rule for the tensor product decomposition is more subtle than that for ordinary Young diagrams (for $\mathfrak{sl}_n$), and the number of boxes in a diagram is not equal to the tensor power N. Since there are boxes of two different widths, it is important to distinguish between the number of boxes in a row and the length of the row. The length of the row λ_i is equal to the corresponding orthogonal coordinate. The number of boxes is equal to $\sum_{j=i}^{n} l_i$. In such diagrams, the first l_n boxes are of width $1/2$. See an example in Fig 1.

For convenience, we use the coordinates $\{a_i\}$ given by the formula

$$a_i = 2\sum_{j=i}^{n-1} l_j + l_n + 2(n-i) + 1.$$

These coordinates are positive integers for integral dominant weights, and $a_i > a_j$ for $i < j$. The coordinates $\{a_i\}_{i=1}^{n}$ have a natural interpretation if we scale the diagram by the factor $2\sqrt{2}$, rotate it 45° counterclockwise, and shift it in such a way that the lowest point has the coordinate $(2n)$. Then the upper boundary of the diagram is the graph of a piecewise linear function, and a_i is the x-coordinate of the middle of the ith decreasing segment, if we count these segments from the right. See Fig. 2.

We will be interested in the decomposition of tensor powers $(V^{\omega_n})^{\otimes N}$ of the last fundamental representation ω_n (also known as the spinor representation) into irreducible representations with highest weights λ. A tensor power of the representation can be decomposed as

$$(V^{\omega_n})^{\otimes N} \cong \bigoplus M_\lambda^N V^\lambda,$$

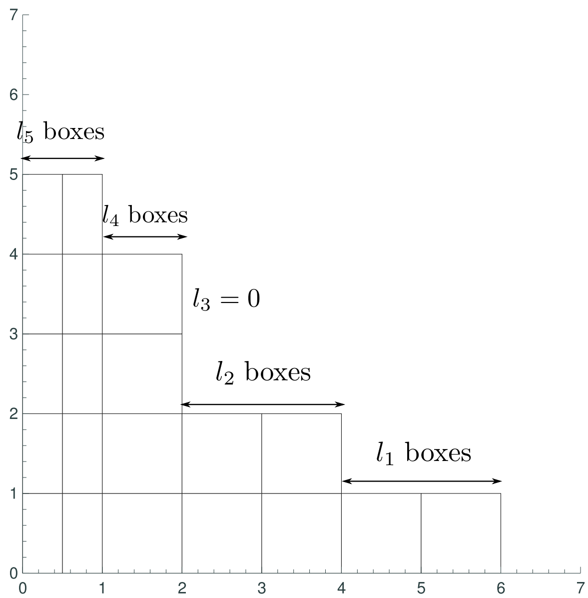

Fig. 1. A generalized Young diagram for the $\mathfrak{so}_{11}$ weight λ with orthogonal coordinates $[6, 4, 2, 2, 1]$ and Dynkin labels $(2, 2, 0, 1, 2)$.

where the sum is taken over the irreducible components of the tensor product and M_λ^N is the multiplicity of the component V^λ. In terms of dimensions, it reads $(\dim V^{\omega_n})^N = \sum M_\lambda^N \dim V^\lambda$. Since $\dim V^{\omega_n} = 2^n$, we can introduce the following probability measure on the set of dominant integral weights:

$$\mu_{n,N}(\lambda) = \frac{M_\lambda^N \dim V^\lambda}{2^{nN}}. \tag{2}$$

By analogy with the representation theory of symmetric groups, we call it the Plancherel measure.

An explicit formula for M_λ^N for the decomposition of $(V^{\omega_n})^{\otimes N}$ was derived by P. P. Kulish, V. D. Lyakhovsky, and O. V. Postnova using the

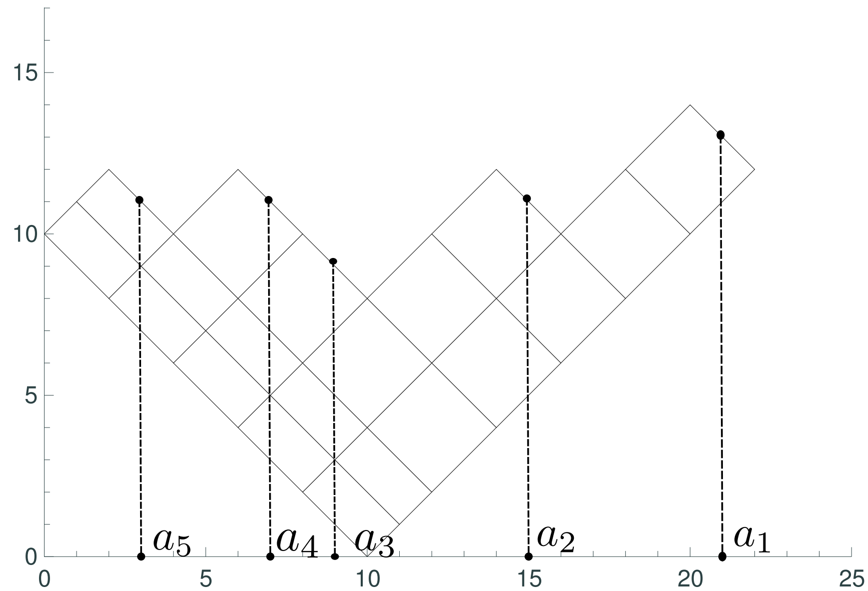

Fig. 2. A rotated and scaled generalized Young diagram and the geometric meaning of the coordinates $\{a_i\}_{i=1}^n$.

Weyl group symmetry and recurrence relations [11–14]:

$$\tilde{M}_{\lambda(a_1 \dots a_n)}^{\omega_n, N} = \prod_{k=0}^{n-1} \frac{(N+2k)!}{2^{2k} \left(\frac{N+a_{k+1}+2n-1}{2}\right)! \left(\frac{N-a_{k+1}+2n-1}{2}\right)!} \prod_{l=1}^{n} a_l \prod_{i<j} \left(a_i^2 - a_j^2\right).$$

Note that there are two congruence classes of weights, one is parameterized by even values of a_i, while the other one, by odd values. A class is determined by the parity of N. For N even we get a_i odd, and vice versa. The factors in the numerator vanish at the boundaries of Weyl chambers shifted by the Weyl vector $-\rho = -\sum_{i=1}^{n} \omega_i$, and the denominator ensures that $\tilde{M}_{\lambda}^{\omega_1, N}$ satisfies the boundary conditions and that the whole expression is anti-invariant with respect to Weyl group transformations.

An expression for $\dim V^\lambda$ can be obtained using the Weyl dimension formula:

$$\dim V^\lambda = \prod_{\alpha\in\Delta^+} \frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)} = \frac{2^{-n^2+2n} n!}{(2n)!(2n-2)!\dots 2!} \cdot \prod_{i<j} (a_i^2 - a_j^2) \prod_{l=1}^{n} a_l.$$

Finally, the explicit expression for the density function of the discrete probability measure (or the probability mass function) is

$$\begin{aligned}\mu_{n,N}(\lambda) = \mu_{n,N}(\{a_i\}) &= \frac{\tilde{M}^{\omega_n,N}_{\lambda(a_1\dots a_n)} \dim V^\lambda}{(2^n)^N}\\ &= \prod_{k=0}^{n-1} \frac{(N+2k)!}{2^{2k}\left(\frac{N+a_{k+1}+2n-1}{2}\right)!\left(\frac{N-a_{k+1}+2n-1}{2}\right)!}\\ &\qquad \times \prod_{i<j}(a_i^2-a_j^2)^2 \prod_{l=1}^{n} a_l^2 \cdot \frac{2^{-n^2+2n-nN} n!}{(2n)!(2n-2)!\dots 2!}. \end{aligned} \tag{3}$$

Now consider the limit as $N, n \to \infty$ so that the ratio of n and N tends to a finite constant:

$$c = \lim_{N,n\to\infty} \frac{N+2n-1}{n}, \quad c = \text{const}.$$

We are interested in the limiting probability distribution on irreducible components of the tensor power decomposition. Since dominant integral weights are depicted by diagrams, the measure (2) can be seen as a probability measure on diagrams. Therefore, we are interested in the limit shape of generalized Young diagrams with respect to the measure $\mu_{n,N}$.

We scale the diagram by the factor $\frac{\sqrt{2}}{n}$, rotate it 45° counterclockwise, and shift it along the x-axis in such a way that the lowest point has coordinates $(1,0)$. This corresponds to a rescaling of the coordinates $\{a_i\}$: $x_i = \frac{a_i}{2n}$. See Fig. 3 for an example of the most probable diagram for $n = 20$, $N = 200$ and the limit shape for $c = 12$. The upper boundary of the diagram is the graph of a piecewise linear function $f_n(x)$, which is almost everywhere differentiable and satisfies $f_n'(x) = \pm 1$ if $x \neq \frac{i}{2n}$. We will prove that the piecewise linear functions $f_n(x)$ converge in probability with respect to the probability measure (2) to a continuous smooth function $f(x)$ as $n \to \infty$.

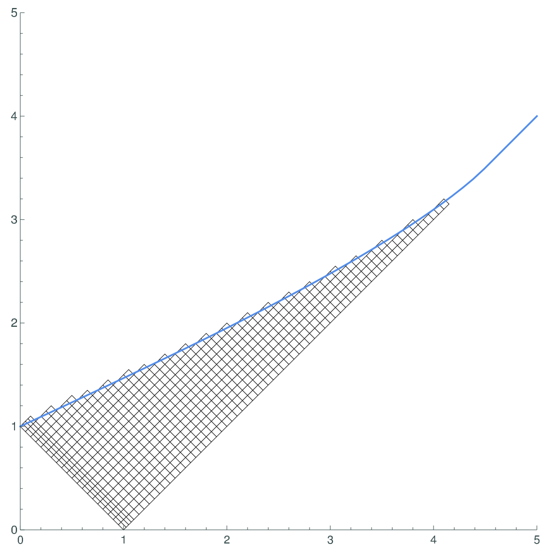

Fig. 3. The most probable diagram for the algebra $\mathfrak{so}_{41}$ and $N = 200$ and the limit shape $f(x)$ of generalized Young diagrams for $c = \frac{N}{n} + 2 = 12$.

To derive the limit shape, it is convenient to regard diagrams as particle configurations $\{x_i\}_{i=1}^n$. Consider the piecewise constant function $\rho_n(x) = \frac{1}{4}(1 - f_n'(x))$. It is equal to 0 on an interval of length $\frac{1}{n}$ if there is no particle in the middle of the interval, and is equal to $\frac{1}{2}$ if there is a particle in the middle of the interval, which means that there is a particle on one of the two intervals of length $\frac{1}{2n}$ constituting it. So, the function $\rho_n(x)$ can be called the particle density. The convergence of diagrams to a limit shape entails the convergence of the particle density functions ρ_n to a limit particle density $\rho(x)$.

Due to our choice of normalization, the limit density $\rho(x)$ is related to the derivative of the limit function $f(x)$ by the formula

$$f'(x) = 1 - 4\rho(x), \tag{4}$$

and the limit shape can be recovered from the explicit expression for $\rho(x)$ by the formula

$$f(x) = 1 + \int\limits_0^x (1 - 4\rho(t))\,\mathrm{dt}. \tag{5}$$

It is more convenient to solve the variational problem for the limit density $\rho(x)$.

§2. Sketch of the proof

To prove the convergence of generalized Young diagrams to the limit shape, we regard the upper boundaries of the rotated diagrams as functions with bounded derivative. We rewrite the probability (2) as the exponential of a quadratic functional on the boundaries of rotated diagrams and find the minimizer of this functional as a solution to the corresponding variational problem. Then we prove the convergence in the space of such functions with respect to a certain distance.

We denote by $a_{-i}, i > 0$, the "mirror image" of a_i:

$$a_{-i} \equiv -a_i,$$

and rewrite the formula for the probability measure in the form more convenient for analysis:

$$\begin{aligned} &\mu_n(\{a_i\}_{i=-n,i\neq 0}^n) \\ &= \frac{1}{Z_n} \prod_{i<j;i,j\neq 0;i,j=-n}^{n} |a_i - a_j| \cdot \prod_{l=-n,l\neq 0}^{n} \exp\left[-(2n)V_0\left(\frac{a_l}{2n}\right) - e_n\left(a_l\right)\right], \end{aligned} \tag{6}$$

where

$$V_0(u) = \frac{1}{4}\left[\left(\frac{c}{2}+u\right)\log\left(\frac{c}{2}+u\right) + \left(\frac{c}{2}-u\right)\log\left(\frac{c}{2}-u\right)\right],$$

$e_n(u) = \frac{1}{4}\log\left((cn)^2 - u^2\right) + \frac{1}{2}\log|u| + \mathcal{O}\left(\frac{1}{n}\right)$, and Z_n does not depend on a_l.

Indeed, the a_l-depending factorials in (3) can be written as exponentials, combined in one exponential with the factor $|a_l|$, and then expanded using Stirling's formula. Inside this exponential, we denote by $V_0(u)$ (one half of) the main contribution with $u = \frac{a_l}{2n}$, and by e_n (one half of) the remainder.

Now we can rewrite the probability of a highest weight λ in the limit as $N, n \to \infty$, $N \sim n$ and of the corresponding diagram as the exponential of a functional of the boundary $f_n(x)$ of the rotated diagram:

$$\mu_n(\lambda) = \mu_{n,N}\left(\{a_i\}_{i=1}^n\right) = e^{-(2n)^2 J[f_n] + \mathcal{O}(n \log n)}. \tag{7}$$

Note also that in order to obtain expression (7), we had to continue the function $\rho_n(x)$ to negative values of x so that it becomes an even function. This corresponds to the continuation of the boundary f_n such that f_n' is even and f_n is continuous at $x = 0$. The continuation is shown in Fig. 4.

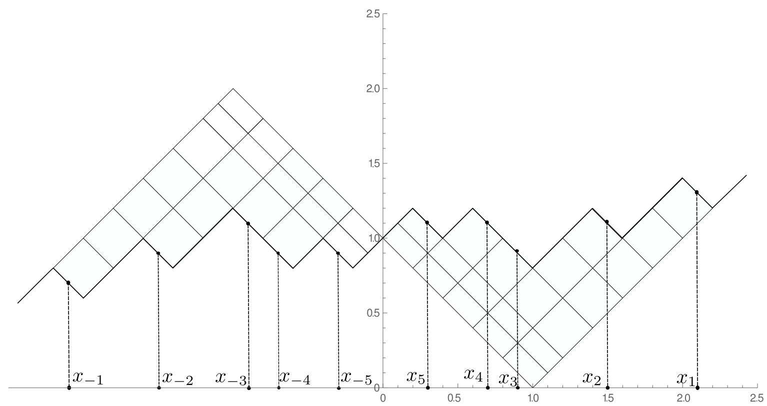

Fig. 4. A rotated and scaled diagram for $n = 5$ and its continuation to negative values of the coordinate x. The function $f_n(x)$ is shown in solid black, the points $x_i = \frac{a_i}{2n}$ are the midpoints of intervals where $f_n'(x) = -1$.

We can write the functional $J[f]$ explicitly:

$$J[f_n] = \frac{1}{2} \int_{-c/2}^{c/2} \int_{-c/2}^{c/2} \frac{1}{16} f_n'(x) f_n'(y) \log|x-y|^{-1} \mathrm{dx}\, \mathrm{dy} + C,$$

where f_n is the upper boundary of the rotated and scaled diagram for $\lambda = \lambda(a_1, \ldots, a_n)$, and the constant C is given by the formula

$$C = -\frac{1}{32} c^2 \log c + \frac{(c-2)^2}{16} \log(c-2) + \frac{c-1}{4} \log 2 - \frac{3}{64}(c-4)^2.$$

The integral in the definition of the functional J can be written as a sum of two integrals, a double integral and a single integral which contains the potential $V_0(x)$ and is taken over the interval $(-\frac{c}{2}, \frac{c}{2})$. The exponentials in (6) can be interpreted as Riemann sums for these integrals, with the double integral approximated using the first-order Taylor expansion. The integral with $V_0(x)$ corresponds to the leading a_i-dependent part of expression (6).

The functional $J[f]$ is clearly quadratic. Rewriting it in terms of densities and searching for its minimum, we arrive at the following variational problem. The particle density $\rho(x)$, which is related to the limit shape $f(x)$ by the formula $f'(x) = 1 - 4\rho(x)$, is the minimizer of the functional

$$\frac{1}{2}\int_{-\frac{c}{2}}^{\frac{c}{2}}\int_{-\frac{c}{2}}^{\frac{c}{2}} \rho(x)\rho(y)\log|x-y|^{-1}\mathrm{dx}\,\mathrm{dy} \\ + \frac{1}{4}\int_{-\frac{c}{2}}^{\frac{c}{2}} \rho(x)\left[\left(\frac{c}{2}+x\right)\log\left(\frac{c}{2}+x\right) + \left(\frac{c}{2}-x\right)\log\left(\frac{c}{2}-x\right)\right]\mathrm{dx}. \quad (8)$$

Our functional is strictly convex (see [5, Theorem 6.27]), therefore, the minimizer is unique. We construct the minimizer by an explicit integral formula, which is obtained as a solution of a Riemann–Hilbert problem, as described in the book by P. Deift [5]. Namely, we take the derivative of the corresponding Euler–Lagrange equation with respect to x and get the equilibrium condition:

$$-\int_{-a}^{a} \frac{\rho(y)\,\mathrm{dy}}{y-x} + V_0'(x) = 0.$$

The integral multiplied by $-i$ is the Hilbert transform $G(z)$ of $\rho(y)$. It is analytic on $\mathbb{C}\setminus[-a,a]$ and has limit values $G_\pm(x) = \lim_{\varepsilon\to 0}\frac{1}{i}\int\frac{\rho(y)\mathrm{dy}}{y-(x\pm i\varepsilon)}$. One can formulate a Riemann–Hilbert problem for $G(z)$, but it appears in a nonstandard form with the sum of $G_+(x)$ and $G_-(x)$ instead of their difference. By considering $\tilde{G}(z) = \frac{G(z)}{\sqrt{z^2-a^2}}$, one can formulate a standard

Riemann–Hilbert problem:

$$\tilde{G}_+(x) - \tilde{G}_-(x) = \frac{2iV_0'(x)}{\left(\sqrt{x^2 - a^2}\right)_+}, \quad z \in [-a, a],$$
$$\tilde{G}_+(z) - \tilde{G}_-(z) = 0, \quad z \notin [-a, a],\ \tilde{G}(z) \to 0, \quad z \to \infty.$$

The solution $\tilde{G}(z)$ of this problem is given by the Plemelj formula. The formula for $G(z)$ follows. To find the support of ρ, it is sufficient to consider the asymptotics of $G(z)$ as $z \to \infty$ and expand the Plemelj formula into a series. In this way one can obtain necessary conditions on the integrals and determine conditions on integration limits. Then for $c \geqslant 4$ one can obtain $a = \sqrt{2c-4}$, and use $\rho(x) = \frac{1}{\pi}\Re[G_+(x)]$ to write an explicit formula for the minimizer of the functional (8):

$$\rho(x) = \frac{1}{\pi^2}\Re\left[\sqrt{2c-4-x^2}\int\limits_{-\sqrt{2c-4}}^{\sqrt{2c-4}} \frac{\frac{1}{4}\left(\log\left(\frac{c}{2}+s\right) - \log\left(\frac{c}{2}-s\right)\right)}{\sqrt{2c-4-s^2}(s-x)}\,\mathrm{ds}\right]. \tag{9}$$

Similarly, for $2 \leqslant c \leqslant 4$ we obtain

$$\rho(x) = \frac{1}{2} - \frac{1}{\pi^2}\Re\left[\sqrt{x^2-2c+4}\int\limits_{-\sqrt{2c-4}}^{\sqrt{2c-4}} \frac{\frac{1}{4}\left(\log\left(\frac{c}{2}+s\right) - \log\left(\frac{c}{2}-s\right)\right)}{\left(\sqrt{s^2-2c+4}\right)_+(s-x)}\,\mathrm{ds}\right].$$

The factor $\tilde{g} = \frac{1}{\pi}\log\left|\frac{s-c/2}{s+c/2}\right|$ can be extracted from (9) as the Hilbert transform of the indicator function $g = \mathbf{1}_{[-\mathbf{c/2},\mathbf{c/2}]}$. Then, in order to compute the integral in (9), it suffices to use the following well-known relation (see, for example, [6]):

$$\int\limits_{-\infty}^{\infty} f(s)\tilde{g}(s)\,\mathrm{ds} = -\int\limits_{-\infty}^{\infty} \tilde{f}(s)g(s)\,\mathrm{ds}, \tag{10}$$

where $\tilde{f}$ is the Hilbert transform of f and $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$ with $\frac{1}{p}+\frac{1}{q} = 1$. Thus, the problem is reduced to computing the Hilbert transform for the remaining function $f(y)$. This integral can be computed explicitly by a change of variables.

Finally, one can write an expression for $\rho(x)$ in terms of inverse trigonometric functions:

$$\rho(x)=\begin{cases}\frac{\theta(\sqrt{2c-4}-|x|)}{4\pi}\left[\arctan\left(\frac{-c(x-4)-8}{(c-4)\sqrt{2c-4-x^2}}\right)+\arctan\left(\frac{c(x+4)-8}{(c-4)\sqrt{2c-4-x^2}}\right)\right], & c\geqslant 4,\\ \frac{1}{2}-\frac{\theta(\sqrt{2c-4}-|x|)}{4\pi}\left[\arctan\left(\frac{-c(x-4)-8}{(4-c)\sqrt{2c-4-x^2}}\right)+\arctan\left(\frac{c(x+4)-8}{(4-c)\sqrt{2c-4-x^2}}\right)\right], & c\in[2,4],\end{cases}$$

where $\theta(\sqrt{2c-4}-|x|)$ is the Heaviside step function. The graphs of the densities are presented in Fig. 5.

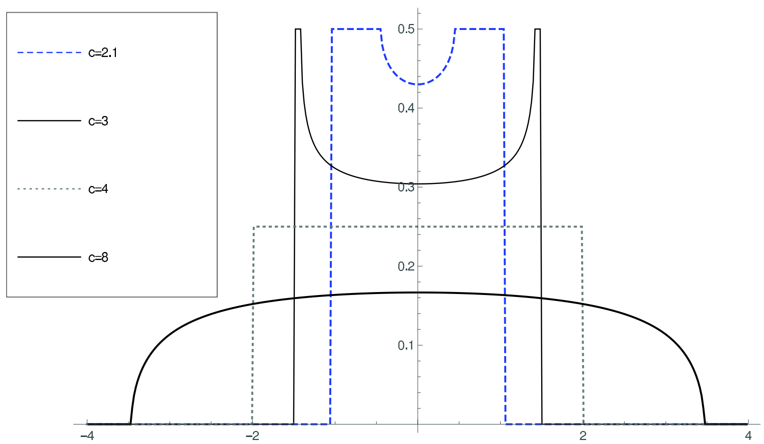

Fig. 5. Plots of the density $\rho(x)$ given by formula (1) for $c=2.1$ (dashed), $c=3$ (solid), $c=4$ (dotted), $c=8$ (bold).

We have shown that the probability of a weight is given by a quadratic functional $J[f_n]$ of a rotated diagram boundary f_n. Our approach to the proof of the convergence is similar to the proof of the Vershik–Kerov–Logan–Shepp theorem in the book by Dan Romik [19]. We write

$$J[f_n]=Q[f_n]+C,\quad Q[f_n]=\frac{1}{2}\int\limits_{-c/2}^{c/2}\int\limits_{-c/2}^{c/2}\frac{1}{16}f_n'(x)f_n'(y)\log|x-y|^{-1}\mathrm{dx}\,\mathrm{dy},$$

and use the functional to introduce the norm $||f||_Q=Q[f]^{1/2}$ for a compactly supported Lipschitz function $f:\mathbb{R}\to[0,\infty)$ (since Q is positive definite, see [19]). This allows us to introduce a pseudo-distance

$d_Q(f_1, f_2) = ||f_1 - f_2||_Q$ on the space of 1-Lipschitz functions with bounded derivative.

We can use the equivalence of different norms in the Sobolev space $H^{1/2}$ to show that the value of the functional $J[f]$ on the limit shape f is non-negative (see Proposition 1.15 in [19] for the connection of Q with this Sobolev space).

And we can easily estimate the probability of weights with a given deviation with respect to this distance: for a highest weight λ with the boundary $f_n(x)$ of the corresponding diagram such that $d(f_n, f) = \varepsilon$, the probability satisfies the inequality

$$\mu_n(\lambda) \leqslant c_1 e^{-n^2\varepsilon^2 + \mathcal{O}(n \log n)}.$$

It follows that the probability of deviation from the limit shape with respect to the d_Q-distance goes to zero as n goes to infinity. The total number $S(n, N)$ of dominant integral weights in the reducible representation $(V^{\omega_n})^{\otimes N}$ does not exceed the number of partitions inside the $n \times N$ rectangle. We consider the regime $N \approx (c-2)n$, so the total number of boxes grows as $(c-2)n^2$. Now it is straightforward to obtain an exponential upper bound $S(n, N) \leqslant c_2 e^{c_3 n}$ for some constants c_2, c_3 using the Hardy–Ramanujan formula; therefore, the total probability of deviation from the limit shape is exponentially small.

As the last ingredient of the proof, we use the fact that the quadratic part of the functional is the same as in the case of the Vershik–Kerov–Logan–Shepp limit shape and conclude that the convergence with respect to the pseudo-distance entails the convergence in the supremum norm $||f||_\infty = \sup_x |f(x)| \leqslant c_4 Q[f]^{1/4}$, where c_4 is some constant (Lemma 1.21 in [19]). Theorem 1 follows.

Conclusion

We have proven the convergence of the irreducible components of tensor powers of the spinor representation of $\mathfrak{so}_{2n+1}$ to a limit shape. One can give an alternative proof based on general results on discrete β-ensembles presented in the book by Alice Guionnet [7]. This approach (see also [3,4]) can also be used to prove the central limit theorem for the global fluctuations around the limit shape. These results, as well as the full proofs of the theorems of the present paper, will be presented in a separate publication by the authors, see the preprint [16]. Similar limit shapes can be obtained for tensor powers of certain reducible representations of $\mathfrak{gl}_n, \mathfrak{so}_{2n}, \mathfrak{sp}_{2n}$. This

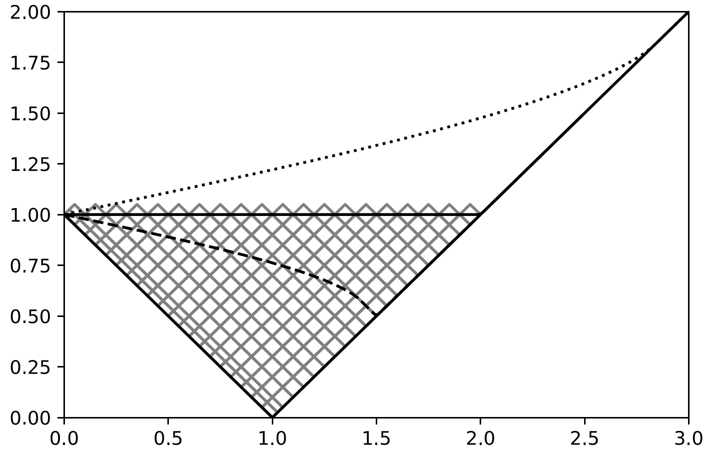

Fig. 6. The limit shapes for Young diagrams for $c = 3$ (dashed), $c = 4$ (solid), and $c = 6$ (dotted) and the most probable diagram for $n = 20, N = 40$. The limit shapes are computed via the explicit formulas (1), (5).

result will be presented in an upcoming paper by A. Nazarov, O. Postnova, and T. Scrimshaw.

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