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A. H. Morales, I. Pak, G. Panova

HOOK FORMULAS FOR SKEW SHAPES IV. INCREASING TABLEAUX AND FACTORIAL GROTHENDIECK POLYNOMIALS

Abstract. We present a new family of hook-length formulas for the number of *standard increasing tableaux* which arise in the study of factorial Grothendieck polynomials. In the case of straight shapes, our formulas generalize the classical *hook-length formula* and the *Littlewood formula*. For skew shapes, our formulas generalize the *Naruse hook-length formula* and its q-analogs, which were studied in previous papers of the series.

§1. Introduction

1.1. Foreword. There is more than one way to explain a miracle. First, one can show how it is made, a step-by-step guide to perform it. This is the most common yet the least satisfactory approach as it takes away the joy and gives you nothing in return. Second, one can investigate away every consequence and implication, showing that what appears to be miraculous is actually both reasonable and expected. This takes nothing away from the miracle except for its shining power, and puts it in the natural order of things. Finally, there is a way to place the apparent miracle as a part of the general scheme. Even, or especially, if this scheme is technical and unglamorous, the underlying pattern emerges with the utmost clarity.

The *hook-length formula* (HLF) is long thought to be a minor miracle, a product formula for the number of certain planar combinatorial arrangements, which emerges where one would expect only a determinant formula. Despite its numerous proofs and generalizations, including some by the authors (see §7.1), it continues to mystify and enthrall. The goal of this paper is to give new curious generalizations of the HLF by using *Grothendieck polynomials*. The resulting formulas are convoluted enough to be unguessable yet retain the hook product structure to be instantly recognizable.

Key words and phrases: hook-length formula, factorial symmetric functions, Grothendieck polynomials, standard Young tableaux, increasing tableaux.

1.2. Straight shapes. Recall some classical results in the area. Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$ be an *integer partition* of n with $\ell = \ell(\lambda)$ parts, and let $f^\lambda := |\operatorname{SYT}(\lambda)|$ be the number of *standard Young tableaux* of shape λ. The *hook-length formula* by Frame–Robinson–Thrall [16] states that

$$f^\lambda \, = \, n! \prod_{u \in \lambda} \frac{1}{h(u)}\,, \tag{HLF}$$

where $h(u) = \lambda_i - i + \lambda'_j - j + 1$ is the *hook-length* of the square $u = (i,j) \in \lambda$.

Similarly, let $\operatorname{SSYT}(\lambda)$ denote the set of *semi-standard Young tableaux* of shape λ. For a tableau $T \in \operatorname{SSYT}(\lambda)$, let $|T|$ denote the sum of its entries. The *Littlewood formula*, a special case of the *Stanley hook-content formula*, states that

$$\sum_{T \in \operatorname{SSYT}(\lambda)} q^{|T|} \, = \, q^{b(\lambda)} \prod_{u \in \lambda} \frac{1}{1 - q^{h(u)}}\,, \tag{q-HLF}$$

where

$$b(\lambda) \, := \sum_{(i,j) \in \lambda} (i-1) \, = \, \sum_{i=1}^{\ell(\lambda)} (i-1)\lambda_i\,,$$

see, e.g., [59, §7.21]. Note that (q-HLF) implies (HLF) by taking the limit $q \to 1$ and using a geometric argument, see [50, §2], or the *P-partition theory*, see [59, §3.15]. We are now ready to state the first two results of the paper, which generalize (HLF) and (q-HLF), respectively.

For a tableau $T \in \operatorname{SSYT}(\lambda)$, let $T_k = \{u \in \lambda \,:\, T(u) = k\}$ be the set of tableau entries equal to k. Define $T_{\leqslant k} = \{u \in \lambda \,:\, T(u) \leqslant k\}$, $T_{\geqslant k} = \{u \in \lambda \,:\, T(u) \geqslant k\}$, and $T_{<k} = T_{\leqslant k+1}$ similarly. Finally, let $\nu(T_k)$, $\nu(T_{<k})$, and $\nu(T_{\geqslant k})$ be the shapes of these tableaux.

We say that T is a *standard increasing tableau* if it is strictly increasing in rows and columns, and T_k is nonempty for all $1 \leqslant k \leqslant m$, where $m = m(T)$ is the maximal entry in T. Note that the (usual) standard Young tableaux are exactly the standard increasing tableaux T with $m(T) = n$. Denote by $\operatorname{SIT}(\lambda)$ the set of standard increasing tableaux of shape λ. By definition, for $T \in \operatorname{SIT}(\lambda)$, we have $0 \leqslant \nu_i(T_{\leqslant k}) \leqslant \lambda_i$, where $\nu_i(T_{\leqslant k})$ is the number of elements in $T_{\leqslant k}$ in the ith row of λ.

Theorem 1.1. *Fix $d \geqslant 1$. In the notation above, for every $\lambda \vdash n$ with $\ell(\lambda) \leqslant d$, we have:*

$$\sum_{T \in \mathrm{SIT}(\lambda)} \prod_{k=1}^{m(T)} \left(\left[\prod_{i=1}^{d} \frac{1+\beta\big(\nu_i(T_{<k})+d-i+1\big)}{1+\beta\big(\lambda_i+d-i+1\big)} \right] - 1 \right)^{-1}$$
$$= \frac{1}{(-\beta)^n} \prod_{i=1}^{\ell(\lambda)} \big(1+\beta(\lambda_i+d-i+1)\big)^{\lambda_i} \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)}\,. \qquad \text{(K-HLF)}$$

Here "K" in (K-HLF) stands for *K-theory*, see below. Note that (K-HLF) implies (HLF) by taking the limit $\beta \to 0$, see Proposition 4.8.

To state the K-theory analog of (q-HLF), we need a few more notation. For a strictly increasing tableau $T \in \mathrm{SIT}(\lambda)$, denote by $T_{\geqslant k}$ the skew subtableau of integers $\geqslant k$, and let $a(T_{\geqslant k}) := |\nu(T_{\geqslant k})|$ denote the number of such integers. This should not be confused with $|T_{\geqslant k}|$, which is the sum of such integers. Finally, denote

$$s(\lambda) \,:=\, \sum_{(i,j)\in\lambda} (i+j-1) \,=\, b(\lambda)+b(\lambda')+|\lambda|\,.$$

Corollary 1.2. *In the notation above, for every $\lambda \vdash n$, we have:*

$$\sum_{T\in\mathrm{SIT}(\lambda)} q^{|T|} \prod_{k=1}^{m(T)} \frac{1}{1-q^{a(T_{\geqslant k})}} \;=\; q^{s(\lambda)} \prod_{(i,j)\in\lambda} \frac{1}{1-q^{h(i,j)}}\,. \tag{1.1}$$

The relationship between (K-HLF) and (1.1) is somewhat indirect, and both follow from a more general equation (4.5) by taking limits.

Remark 1.3. Denote by $\mathrm{RPP}(\lambda)$ the set of *reverse plane partitions*, which are Young tableaux with entries $\geqslant 0$ weakly increasing in rows and columns. Similarly, denote by $\mathrm{IT}(\lambda)$ the set of *increasing tableaux*, which are Young tableaux with entries $\geqslant 1$ strictly increasing in rows and columns. Thus:

$$\mathrm{SYT}(\lambda) \subset \mathrm{SIT}(\lambda) \subset \mathrm{IT}(\lambda) \subset \mathrm{SSYT}(\lambda) \subset \mathrm{RPP}(\lambda). \tag{1.2}$$

It is well known, and easily follows from (q-HLF), that

$$\sum_{T\in\mathrm{IT}(\lambda)} q^{|T|} \;=\; q^{s(\lambda)} \sum_{T\in\mathrm{RPP}(\lambda)} q^{|T|} \;=\; q^{s(\lambda)} \prod_{(i,j)\in\lambda} \frac{1}{1-q^{h(i,j)}}\,. \tag{1.3}$$

Note that both (1.1) and (1.3) have identical RHS, but the LHS of (1.1) has an extra product term. In fact, there is a similar direct way to derive (1.1) from (q-HLF) by subtracting a constant from the entries in each anti-diagonal of the tableau. However, this approach does not extend to skew shapes, see Theorem 1.5 below and §7.9.

1.3. Skew shapes. We start with the *Naruse hook-length formula* (NHLF), the subject of the previous papers in this series [41, 42, 43]. Here we omit some definitions; precise statements are given in §5.

Let λ/μ be a *skew Young diagram* (skew shape), and let

$$f^{\lambda/\mu} = |\mathrm{SYT}(\lambda/\mu)$$

be the number of standard Young tableaux of shape λ/μ. Then

$$f^{\lambda/\mu} \; = \; |\lambda/\mu|! \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{u\in\lambda\setminus D} \frac{1}{h(u)} \;, \tag{NHLF}$$

where $h(u)$ is the (usual) hook-length of the square $u \in \lambda$ and $\mathcal{E}(\lambda/\mu)$ denotes the set of *excited diagrams* of shape λ/μ. Note that when $\mu = \varnothing$, there is a unique generalized excited diagram $D = \varnothing$, and (NHLF) reduces to (HLF).

The q-analog of (NHLF) generalizing Littlewood's formula (q-HLF) to skew shapes was given by the authors in [41]:

$$\sum_{T\in\mathrm{SSYT}(\lambda/\mu)} q^{|T|} \; = \; \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in\lambda\setminus D} \frac{q^{\lambda'_j-i}}{1-q^{h(i,j)}} \;. \tag{q-NHLF}$$

In Remark 1.6, we discuss another notable q-analog as a summation over $\mathrm{RPP}(\lambda/\mu)$. The following results respectively generalize Theorem 1.1 and Corollary 1.2 to skew shapes, thus giving advanced generalizations of (HLF).

Let $\mu \subset \lambda$ be two integer partitions. Define the set $\mathrm{SIT}(\lambda/\mu)$ of *standard increasing tableaux of skew shape* λ/μ again as the set of Young tableaux T which strictly increase in rows and columns and have nonempty T_k for all $1 \leqslant k \leqslant m(T)$. In this case, the *generalized excited diagrams* were introduced by Graham–Kreiman [18] and Ikeda–Naruse [25]. We denote the set of such diagrams by $\mathcal{D}(\lambda/\mu)$, and postpone their definition until the next section.

Theorem 1.4. *Fix $d \geqslant 1$. In the notation above, for every $\mu \subset \lambda$ with $\ell(\lambda) \leqslant d$, we have:*

$$\sum_{T \in \mathrm{SIT}(\lambda/\mu)} \prod_{k=1}^{m(T)} \left(\left[\prod_{i=1}^{d} \frac{1+\beta\big(\nu_i(T_{<k}) + d - i + 1\big)}{1+\beta(\lambda_i + d - i + 1)} \right] - 1 \right)^{-1}$$

$$= \sum_{D \in \mathcal{D}(\lambda/\mu)} (-\beta)^{|D| - |\lambda|} \prod_{(i,j) \in \lambda \setminus D} \frac{\beta(\lambda_i + d - i + 1) + 1}{h(i,j)} . \tag{K-NHLF}$$

See §6.4 for a completely different generalization of (HLF) to skew shapes, which also has a q-analog and K-theory analog (Theorem 6.8). Finally, Corollary 1.2 extends to skew shapes as follows.

Theorem 1.5. *In the notation above, for every $\mu \subset \lambda$, we have:*

$$\sum_{T \in \mathrm{SIT}(\lambda/\mu)} q^{|T|} \prod_{k=1}^{m(T)} \frac{1}{1 - q^{a(T_{\geqslant k})}} = \sum_{D \in \mathcal{D}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{q^{h(i,j)}}{1 - q^{h(i,j)}} . \tag{1.4}$$

Again, equation (1.4) reduces to (1.1) by taking $\mu = \varnothing$ and noting that

$$\sum_{(i,j) \in \lambda} h(i,j) = \sum_{(i,j) \in \lambda} (\lambda'_j - i + 1) + \sum_{(i,j) \in \lambda} (\lambda_i - j) = s(\lambda).$$

Remark 1.6. While the inclusions in (1.2) continue to hold for skew shapes, the natural analog of (1.3) is no longer straightforward. In fact, for

$$I_{\lambda/\mu}(q) := \sum_{T \in \mathrm{IT}(\lambda/\mu)} q^{|T|} \quad \text{and} \quad R_{\lambda/\mu}(q) := \sum_{T \in \mathrm{RPP}(\lambda/\mu)} q^{|T|} ,$$

the theory of P-partition gives:

$$I_{\lambda/\mu}(-q) = q^N R_{\lambda/\mu}(1/q) \quad \text{for some } N \geqslant 0, \quad \text{see [59, §3.15]}. \tag{1.5}$$

On the other hand, the summation formula for $R_{\lambda/\mu}(q)$ given in [41, Theorem 1.5] gives yet another generalization of (NHLF), but is summing over a different, albeit related, set of *pleasant diagrams* (see §5.2):

$$\sum_{T \in \mathrm{RPP}(\lambda/\mu)} q^{|T|} = \sum_{S \in \mathcal{P}(\lambda/\mu)} \prod_{(i,j) \in S} \frac{q^{h(i,j)}}{1 - q^{h(i,j)}} . \tag{1.6}$$

As we explain in §6, equation (K-NHLF) is really a generalization of (1.6) rather than (q-NHLF). A connection can also be seen through yet another summation formula for $R_{\lambda/\mu}(q)$ given in [41, Corollary 6.17] in terms of (ordinary) excited diagrams and subsets $\pi(\lambda/\mu)$ of *excited peaks* (see the definition in §5.2):

$$\sum_{T \in \mathrm{RPP}(\lambda/\mu)} q^{|T|} \;=\; \sum_{D \in \mathcal{E}(\lambda/\mu)} q^{c(D)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{1 - q^{h(i,j)}}, \tag{1.7}$$

where $c(D) := \sum_{(i,j) \in \pi(\lambda/\mu)} h(i,j)$. Finally, let us mention that the corresponding summation formula for $I_{\lambda/\mu}(q)$, implied by (1.5) and (1.7), is obtained in (6.8) more directly.

1.4. Methodology. While all results in this paper can be understood as enumeration of certain Young tableaux, both the motivation and the proofs are algebraic. This is routine in algebraic combinatorics, of course, and goes back to the most basic and classical results in the area.

For example, for the LHS of (HLF), we have $f^\lambda = \dim \mathbb{S}^\lambda$, the dimension of the corresponding irreducible S_n-module, with standard Young tableaux giving a natural basis. On the other hand, the LHS in (q-HLF) is equal to the evaluation of the Schur function $s_\lambda(1, q, q^2, \ldots)$, and counts multiplicities of $\mathbb{S}^\lambda$ in the natural action on the symmetric algebra $\mathbb{C}[x_1, \ldots, x_n]$ graded by the degree. The connection between the two is then provided by the combination of the Burnside and Chevalley theorems.

One can similarly define the standard Young tableaux of skew shapes, excited diagrams, etc., even if the explanations become more technical and involved with each generalization. A tremendous amount of work by many authors went into developments of this theory, making a proper overview for a paper of this scope impossible. Instead, we skip to the end of the story and briefly describe the motivation behind our new enumerative results.

Before we proceed to the recent work, it is worth pausing and pondering on how the results in the area come about. First, there are algebraic areas (representation theory, enumerative algebraic geometry, etc.) which provide the source of key algebraic objects (characters, Schubert cells, characteristic classes, etc.). Second, in order to build the theory of these objects and be able to compute them, combinatorial objects are extracted which are able to characterize the algebraic objects (Schur functions, Schubert polynomials, etc.).

Third, algebraic combinatorialists join the party and introduce the theory of these combinatorial objects without regard to their algebraic origin. Along the way they introduce a plethora of new combinatorial tools (Young tableaux, reduced decompositions, RSK, etc.) which substantially enhance and clarify the resulting combinatorial structures. This is still the same theory, of course, but the self-contained presentation and rich yet to be understood combinatorics allows an easy access to people not algebraically inclined.

All this leads to the fourth wave, by enumerative combinatorialists who are able to use tools and ideas from algebraic combinatorics to study purely combinatorial problems. This is where we find ourselves in this paper, staring with an amazement at new enumerative results we obtain following this course that we would not be able to dream up otherwise, yet grasping for understanding of what these results really mean in the grand scheme of things.

1.5. Motivation and background. The main result of this paper is an unusual *β-deformation* of many known hook formulas. Notably, our β-deformation (K-HLF) of (HLF), see Theorem 1.1, remains concise and multiplicative even if it is quite cumbersome at first glance. By comparison, it is unlikely that $g^\lambda := |\,\mathrm{SIT}(\lambda)|$ has a closed formula (cf. §7.10), so a product formula for the *weighted* enumeration of SITs is both a minor miracle and testament to the intricate nature of such tableaux.

The same pattern extends to other, more general, hook formulas, suggesting that (K-HLF) is not an accident, that the β-deformation is a far-reaching generalization, on par with the *"q-analog," "shifted analog,"* etc. We expect further results in this direction in the future.

In the combinatorial context, *standard increasing tableaux* (without the restriction on the values of the entries) appear as byproducts of the classical *Edelman–Greene insertion* [11, 22] aimed at understanding Stanley's theorem on reduced factorizations of *Grassmannian permutations* (permutations with at most one descent, see, e.g., [36]). They also appear in a more general setting of the *Hecke insertion* [6].

More recently, standard increasing tableaux have appeared in the context of the K-theoretic version of the *jeu de taquin* of Thomas and Yong [62, 64] and *K-promotion* in *K-theoretic Schubert calculus* [53]. Closely related *semistandard set-valued tableaux* were defined by Buch [4], and have also been studied in a number of papers.

In the algebraic context, the K-theoretic Schubert calculus of the Grassmannian was introduced by Lascoux and Schützenberger [32]. There, they defined the *Grothendieck polynomials* as representatives for K-theory classes determined by structure sheaves of Schubert varieties. The theory has been rapidly developed in the past two decades. We refer to [3, 5] for early surveys of the subject, as reviewing the extensive recent literature is beyond the scope of this paper.

In this paper, the key role is played by the *factorial Grothendieck polynomials* [37, 28], which generalize both the well-studied *Grothendieck polynomials* and *factorial symmetric functions*. The latter were also first introduced by Lascoux and Schützenberger [31] in the guise of double Schubert polynomials for Grassmannian permutations, and have been systematically studied by Macdonald [35], see also [7] for further background.

Finally, let us mention the *excited diagrams*, *pleasant diagrams*, and the *generalized excited diagrams*, which all arise in the context of hook formulas of skew shapes, introduced by Ikeda–Naruse [24], by us [41], and by Naruse–Okada [47], respectively. These diagrams provide a combinatorial language needed to state our results.

1.6. Proof ideas. For us, the story starts with our proof in [41] of equations (NHLF) and (q-NHLF) using evaluations of factorial Schur functions and the *Chevalley type formulas*, see [39]. Naruse's (unpublished) approach was likely similar, cf. [46]. After our paper, Naruse–Okada [47] rederived and further generalized to d-complete posets our $\mathrm{RPP}(\lambda/\mu)$ generalization (1.7) of (NHLF) using the Billey-type and the Chevalley-type formulas from the *equivariant K-theory*. Note that our own proof of the $\mathrm{RPP}(\lambda/\mu)$ summation (1.6) given in [41] is completely combinatorial, and based on a generalization of the *Hillman–Grassl bijection*.

Our proofs in this paper combine our earlier proof technique in [42] with that of Naruse–Okada. Namely, we study evaluations of factorial Grothendieck polynomials in two different ways. First, we use the Pieri rule for the factorial Grothendieck polynomials to obtain the LHS of the equations in terms of increasing tableaux. In the skew case, we combine these with the Chevalley type formulas. We also use the Naruse–Okada characterization of *generalized excited diagrams* in terms of usual excited diagrams (see Proposition 5.1), to obtain equation (6.7) and its generalizations. We also prove that these diagrams have a lattice path interpretation, which we exploit in §5.3 to obtain an upper bound on their number.

Second, for the RHS of our hook formulas, we use the vanishing property of the evaluation for the case of straight shapes. Finally, we use formulas in terms of excited diagrams of Graham–Kreiman [18] for the case of skew shapes.

1.7. Paper structure. We begin with preliminary Sections 2 and 3, where we review basic definitions and properties of permutation classes, Young tableaux, increasing tableaux, and factorial Grothendieck polynomials. We then proceed to present proofs of all our hook formulas via more general multivariate formulas.

Namely, in §4, we prove Theorem 4.2, the main result of the straight shape case, which implies Theorem 1.1 and Corollary 1.2. In §5 we review the technology of excited diagrams, which was unnecessary for the straight shape. We also relate our notation and results to further clarify the combinatorics of the double Grothendieck polynomials of vexillary permutations for devotees of the subject. Then, in §6, we prove Theorem 6.5, the main and most general result of this paper, which similarly implies both Theorems 1.4 and 1.5.

Let us emphasize that this paper is not self-contained by any measure, as we are freely using results from the area and from our previous papers in this series. We tried, however, to include all necessary definitions and results, so the paper can be read by itself. This governed the style of the paper: we covered the straight shape case first, as it requires less of a background and can be understood by a wider audience. This also helped to set up the more general skew shape case which followed. We conclude with final remarks and open problems in §7.

§2. Permutations, Dyck paths, and Young tableaux

2.1. Basic notation. Let $\mathbb{N} = \{0, 1, \ldots\}$ and $[n] = \{1, \ldots, n\}$.

2.2. Permutations. We write permutations of $[n]$ as $w = w_1 w_2 \ldots w_n \in S_n$, where w_i is the image of i. The *Rothe diagram* of a permutation w is the subset of $[n] \times [n]$ given by $\mathrm{R}(w) := \big\{(i, w_j) \mid i < j, w_i > w_j\big\}$. The *essential set* of a permutation w is the subset of $\mathrm{R}(w)$ given by $\mathrm{Ess}(w) := \big\{(i,j) \in R(w) \mid (i+1,j), (i,j+1), (i+1,j+1) \notin \mathrm{R}(w)\big\}$, see, e.g., [36, §2.1–2].

A permutation $w \in S_n$ is called *Grassmannian* if it has a unique descent, say at position k. Such a Grassmannian permutation corresponds to a partition $\mu = \mu(w)$ with $\ell(\mu) \leqslant k$, and $\mu_1 \leqslant n - k$. Grassmannian permutations w can also be characterized as having $\mathrm{Ess}(w)$ contained in one row, the last row of $\mathrm{R}(w)$, and $\mu(w)$ can be read from the number of boxes of $\mathrm{R}(w)$ in each row bottom to top.

A permutation $w \in S_n$ is called *vexillary* if it is 2143-avoiding. Vexillary permutations can also be characterized as permutations w where $\mathrm{R}(w)$ is, up to permuting rows and columns, the Young diagram of a partition $\mu = \mu(w)$. Given a vexillary permutation w, let $\lambda = \lambda(w)$ be the smallest partition containing the diagram $\mathrm{R}(w)$. We call this partition the *supershape* of w and note that $\mu(w) \subseteq \lambda(w)$. The Young diagram of $\lambda(w)$ can also be obtained by taking the union over $i \times j$ rectangles with NW and SE corners $(1,1)$ and (i,j) for each (i,j) in $\mathrm{Ess}(w)$. Note also that Grassmannian permutations are examples of vexillary permutations.

2.3. Lattice paths. A *lattice path* contained in a Young diagram λ is a path with steps $(1,0)$ and $(0,1)$ along the square grid centered at the centers of the cells of λ.

A *Dyck path* γ of length $2n$ is a lattice path from $(0,0)$ to $(2n,0)$ with steps $(1,0)$ and $(1,-1)$ that stays on or above the x-axis. The set of Dyck paths of length $2n$ is denoted by $\mathsf{Dyck}(n)$. For a Dyck path γ, a *peak* is a point (c,d) such that $(c-1,d-1)$ and $(c+1,d-1)$ are in γ. A peak (c,d) is called a *high peak* if $d > 1$. The set of high peaks of a Dyck path γ is denoted by $\mathcal{HP}(\gamma)$, and its size by $\mathrm{hp}(\gamma)$. Note that a Dyck path, upon rotation and rescaling, is also a lattice path in the Young diagram of $\delta_n = (n+1, n, \ldots, 1)$.

For general lattice paths γ above a certain base path γ', we can also define the set of *high peaks* relative to γ' as the set of points (c,d) such that $(c,d-1),(c+1,d) \in \gamma$ and $(c,d) \notin \gamma'$. We will also denote this set by $\mathcal{HP}(\gamma)$.

2.4. Plane partitions and Young tableaux. We use the standard English notation for drawing integer partitions, Young diagrams, and Young tableaux, see, e.g., [59, §7].

To simplify the notation, we use the same letter to denote both an *integer partition* and the corresponding *Young diagram* $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, where $\ell = \ell(\lambda)$ is the *number of parts* of λ. The *skew shape* (skew Young diagram) λ/μ is given by a pair of Young diagrams such that $\mu \subset \lambda$. Denote by $|\lambda/\mu|$ the *size* of the skew shape.

A *reverse plane partition* of skew shape λ/μ is an array $A = (a_{ij})$ of nonnegative integers of shape λ/μ that is weakly increasing in rows and columns. A *semistandard Young tableau* (SSYT) of shape λ/μ is a reverse plane partition of shape λ/μ that is strictly increasing in columns and has entries $\geqslant 1$. We denote these sets of tableaux by $\mathrm{RPP}(\lambda/\mu)$ and $\mathrm{SSYT}(\lambda/\mu)$, respectively.

A *standard Young tableau* of shape λ/μ is a reverse plane partition T of shape λ/μ which contains entries $1, \ldots, |\lambda/\mu|$ exactly once. We denote this set by $\mathrm{SYT}(\lambda/\mu)$, and let $f^{\lambda/\mu} := |\,\mathrm{SYT}(\lambda/\mu)|$ be the *number of standard Young tableaux* of shape λ/μ.

In less standard notation, for a tableau $T \in \mathrm{RPP}(\lambda)$, we define tableaux T_k, $T_{\leqslant k}$, and $T_{\geqslant k}$ as in the introduction. The (skew) shape of a tableau Q is denoted by $\nu(Q)$. We are using $a(Q) := |\nu(Q)|$ to denote the size (the number of entries) in Q. As in the introduction, we write $|T|$ to denote the sum of entries in the tableau T.

2.5. Increasing and set-valued Young tableaux. An *increasing tableau* of shape λ/μ is a row strict semistandard Young tableau of shape λ/μ. A *standard increasing tableau*[1] is an increasing tableau of shape λ/μ whose entries are exactly $[m]$, for some $m \leqslant |\lambda/\mu|$. As in the introduction, we denote by $m(T) := m$ the maximal entry in T.

Denote by $\mathrm{IT}(\lambda/\mu)$ the set of increasing tableaux, and by $\mathrm{SIT}(\lambda/\mu)$ the set of standard increasing tableaux of shape λ/μ. Let $g^{\lambda/\mu} := |\,\mathrm{SIT}(\lambda/\mu)|$ be the *number of standard increasing tableaux* of shape λ/μ.

A tableau $T \in \mathrm{SIT}(\lambda/\mu)$ is called a *barely standard Young tableau* of shape λ/μ if $m(T) = |\lambda/\mu| - 1$. In other words, these are the standard increasing tableaux with exactly one entry appearing twice (cf. §7.4). We denote the set of these tableaux by $\mathrm{BSYT}(\lambda/\mu)$. We also denote by $\mathrm{BSYT}_k(\lambda/\mu)$ the tableaux in $\mathrm{BSYT}(\lambda/\mu)$ with entry k appearing twice.

Finally, for a positive integer n, a *semistandard set-valued tableau* of shape λ/μ is an assignment of nonempty subsets of $[n]$ to the cells of λ/μ such that for the set $T(u)$ in a cell $u \in \lambda$, we have:

- $\max T(u) \leqslant \min T(u')$, where u' is the cell to the right of u in the same row, and
- $\max T(u) < \min T(u')$, where u' is the cell below u in the same column.

[1] In the literature, these tableaux are sometimes called (just) *increasing tableaux* or *packed increasing tableaux* [54].

We use $\mathrm{ne}(T)$ to denote the *number of entries* of T, and $\mathrm{SSVT}_n(\lambda/\mu)$ to denote the set of such tableaux. When we draw set-valued tableaux, we place all integers in $T(u)$ inside the corresponding square u.

2.6. Examples. To illustrate the definitions, in the figure below we have $\lambda = 442$, $\mu = 21$, $A \in \mathrm{RPP}(\lambda/\mu)$, $B \in \mathrm{SSYT}(\lambda/\mu)$, $C \in \mathrm{SYT}(\lambda/\mu)$, $D \in \mathrm{SSVT}_5(\lambda/\mu)$, $E \in \mathrm{IT}(\lambda/\mu)$, $F \in \mathrm{SIT}(\lambda/\mu)$ with $m(F) = 5$, and $G \in \mathrm{BSYT}_3(\lambda/\mu)$. Note that $\mathrm{ne}(D) = 9$.

$$A = \begin{array}{|c|c|c|c|}\hline \cellcolor{gray} & \cellcolor{gray} & 0 & 1 \\ \hline \cellcolor{gray} & 0 & 0 & 1 \\ \hline 1 & 2 \\ \cline{1-2}\end{array}, \quad B = \begin{array}{|c|c|c|c|}\hline \cellcolor{gray} & \cellcolor{gray} & 1 & 1 \\ \hline \cellcolor{gray} & 1 & 2 & 3 \\ \hline 3 & 3 \\ \cline{1-2}\end{array}, \quad C = \begin{array}{|c|c|c|c|}\hline \cellcolor{gray} & \cellcolor{gray} & 2 & 5 \\ \hline \cellcolor{gray} & 1 & 4 & 6 \\ \hline 3 & 7 \\ \cline{1-2}\end{array}, \quad D = \begin{array}{|c|c|c|c|}\hline \cellcolor{gray} & \cellcolor{gray} & 1 & 1,4 \\ \hline \cellcolor{gray} & 1 & 3 & 5 \\ \hline 1,2 & 2 \\ \cline{1-2}\end{array}$$

$$E = \begin{array}{|c|c|c|c|}\hline \cellcolor{gray} & \cellcolor{gray} & 1 & 3 \\ \hline \cellcolor{gray} & 1 & 2 & 4 \\ \hline 2 & 7 \\ \cline{1-2}\end{array}, \quad F = \begin{array}{|c|c|c|c|}\hline \cellcolor{gray} & \cellcolor{gray} & 1 & 3 \\ \hline \cellcolor{gray} & 1 & 2 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}, \quad G = \begin{array}{|c|c|c|c|}\hline \cellcolor{gray} & \cellcolor{gray} & 2 & 3 \\ \hline \cellcolor{gray} & 1 & 4 & 5 \\ \hline 3 & 6 \\ \cline{1-2}\end{array}$$

In this case, we have $|F| = 18$, $\nu(F_{\leqslant 0}) = \mu = 21$, $\nu(F_{\leqslant 1}) = 32$, $\nu(F_{\leqslant 2}) = 331$, $\nu(F_{\leqslant 3}) = 431$, $\nu(F_{\leqslant 4}) = 441$, and $\nu(F_{\leqslant 5}) = \lambda = 442$. Similarly, $\nu(F_{\geqslant 2}) = 442/32$ and $a(F_{\geqslant 2}) = 5$.

Finally, in the notation of the introduction, $b(\lambda) = |N_\lambda|$ and $s(\lambda) = |M_\lambda|$ are the sum of the entries of the minimal reverse plane partition $N_\lambda \in \mathrm{RPP}(\lambda)$ and the minimal strictly increasing tableau $M_\lambda \in \mathrm{SIT}(\lambda)$, with entries $N_\lambda(i,j) = (i-1)$ and $M_\lambda(i,j) = (i+j-1)$, respectively. See an example in the figure below:

$$N_{442} = \begin{array}{|c|c|c|c|}\hline 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2}\end{array} \quad \text{and} \quad M_{442} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 4 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$$

In this case, $b(\lambda) = |N_{442}| = 8$ and $s(\lambda) = |M_{442}| = 31$.

2.7. Special cases. To further clarify the definitions, let us give a quick calculation of the number of increasing tableaux for the *two row shape* (n,n) and the *hook shape* $(p, 1^q)$.

Let s_n denote the nth *little Schröder number* [58, A001003] that counts the lattice paths $(0,0) \to (n,n)$ with steps $(1,0)$, $(0,1)$, and $(1,1)$ that never go below the main diagonal $x = y$ and have no $(1,1)$ steps on the diagonal.

Proposition 2.1 ([53]). *We have* $g^{(n,n)} = s_n$.

Proof. We interpret the SITs as lattice paths on the square grid. In the case $\lambda = (n,n)$, let $T \in \mathrm{SIT}(\lambda)$ correspond to the lattice path $\gamma : (0,0) \to (n,n)$ given by a sequence of steps for $i = 1, \ldots, 2n$:
$(1,0)$ if the entry i appears only in the first row of T,
$(0,1)$ if the entry i appears only in the second row of T, and
$(1,1)$ if the entry i appears in both rows.
The increasing columns condition forces the paths γ not to cross below the diagonal, with all $(1,1)$ steps strictly above the diagonal, as desired. □

Similarly, let $\mathrm{D}(m,n)$ denote the *Delannoy number* [58, A008288] that counts the lattice paths $(0,0) \to (m,n)$ with steps $(0,1)$, $(1,0)$, and $(1,1)$. We call these *Delannoy steps*.

Proposition 2.2 (cf. [56]). *For the hook shape* $\lambda = (p, 1^q)$, *we have* $g^{(p,1^q)} = \mathrm{D}(p-1, q)$.

The proof follows verbatim the argument above, but the lattice paths with Delannoy steps no longer have a diagonal constraint. We omit the details.

§3. Factorial Grothendieck polynomials

Recall the following operators first introduced in [13, 15]:

$$x \oplus y \ := \ x + y + \beta x y, \qquad x \ominus y \ := \ \frac{(x-y)}{(1+\beta y)}\,, \qquad \ominus x \ := \ 0 \ominus x,$$
$$\text{and} \qquad [x \,|\, \mathbf{y}]^k \ := \ (x \oplus y_1)(x \oplus y_2) \,\cdots\, (x \oplus y_k),$$

where $\mathbf{y} = (y_1, y_2, \ldots)$.

Definition/Theorem 3.1 (McNamara [37]). The factorial Grothendieck polynomials *are defined by either of the following formulas:*

$$G_\mu(x_1, \ldots, x_d \,|\, \mathbf{y}) := \sum_{T \in \mathrm{SSVT}_d(\mu)} \beta^{\mathrm{ne}(T) - |\mu|} \prod_{u \in \mu,\ r \in T(u)} \big(x_r \oplus y_{r + c(u)}\big)$$
$$= \det\Big([x_i \,|\, \mathbf{y}]^{\mu_j + d - j} (1 + \beta x_i)^{j-1}\Big)_{i,j=1}^{d} \prod_{1 \leqslant i < j \leqslant d} \frac{1}{(x_i - x_j)}\,.$$

The factorial Grothendieck polynomials are equal to the *double Grothendieck polynomials* parameterized by a Grassmannian permutation associated to the partition μ, see [38]. These in turn were defined earlier in [28], in the greater generality of all vexillary permutations, see equation (3.2) below. We postpone their definition until §5.6 (see also §7.3).

Remark 3.2. As mentioned in [37, Remark 3.2], in the literature Grothendieck polynomials sometimes appear only in the case $\beta = -1$. However, one can obtain the β case from the case $\beta = -1$ by replacing x_i with $-x_i/\beta$ and y_i with y_i/β,

$$G_\mu\big(\mathbf{x}\,|\,\mathbf{y}\big)\,\Big|_{\beta=-1} \;=\; (-\beta)^{|\mu|}\cdot G_\mu\big(-\mathbf{x}/\beta \mid \mathbf{y}/\beta\big). \tag{3.1}$$

It is easy to see that $G_\varnothing(\mathbf{x}|\mathbf{y}) = 1$. We need the following technical result.

Proposition 3.3 ([37, 38])**.** *The factorial Grothendieck polynomials $G_\varnothing(\mathbf{x}|\mathbf{y})$ satisfy:*

(i) $G_\mu(x_1,\ldots,x_d\,|\,\mathbf{y})$ *is symmetric in* $x_1,x_2,\ldots,x_d$.

(ii) *Doing the substitution $y_i \leftarrow (-y_i)$, and setting $\beta = 0$, we obtain the factorial Schur function:*

$$G_\mu\big(x_1,\ldots,x_d\,|\,-\mathbf{y}\big)\,\Big|_{\beta=0} \;=\; s_\mu(x_1,\ldots,x_d\,|\,\mathbf{y}).$$

(iii) *Setting $y_i = 0$, we obtain the ordinary Grothendieck polynomials:*

$$G_\mu(x_1,\ldots,x_d \mid \mathbf{y})\,\Big|_{y_i=0} \;=\; G_\mu(x_1,\ldots,x_d).$$

(iv) *They are equal to double Grothendieck polynomials of Grassmannian permutations:*

$$\mathfrak{G}_{w(\mu)}(\mathbf{x},\mathbf{y}) \;=\; G_\mu(x_1,\ldots,x_d \mid \mathbf{y}), \tag{3.2}$$

for $d \geqslant \ell(\mu)$, where $w(\mu)$ is the Grassmannian permutation *with descent at position d associated to μ.*

Proposition 3.4 (*vanishing property of Grothendieck polynomials* [37, Theorem 4.4])**.** *When evaluated at $\mathbf{y}_\lambda := (\ominus\, y_{\lambda_1+d}, \ominus\, y_{\lambda_2+d-1},\ldots,\ominus\, y_{\lambda_d+1})$ with $\ell(\lambda) \leqslant d$,*

$$G_\mu(\mathbf{y}_\lambda\,|\,\mathbf{y}) \;=\; \begin{cases} 0 & \text{if } \mu \not\subseteq \lambda,\\ \prod_{(i,j)\in\lambda}(y_{d+j-\lambda'_j} \ominus y_{\lambda_i+d-i+1}) & \text{if } \mu = \lambda. \end{cases} \tag{3.3}$$

To simplify the notation, we write G_1 for $G_{(1)}$. We use the notation $\nu \mapsto \mu$ when the skew shape ν/μ is nonempty and its boxes are in different rows and columns. Note that $\nu \neq \mu$ in this case. In this notation, every standard increasing tableau $T \in \mathrm{SIT}(\lambda/\mu)$ is viewed as a chain

$$\lambda = \nu\big(T_{\leqslant k}\big) \mapsto \nu\big(T_{\leqslant k-1}\big) \mapsto \ldots \mapsto \nu\big(T_{\leqslant 1}\big) \mapsto \nu\big(T_{\leqslant 0}\big) = \mu. \tag{3.4}$$

Lemma 3.5 (*Pieri rule for Grothendieck polynomials* [37, Proposition 4.8])**.**

$$G_\mu(\mathbf{x}\,|\,\mathbf{y})\big(1+\beta G_1(\mathbf{x}\,|\,\mathbf{y})\big) \ = \ \big(1+\beta G_1(\mathbf{y}_\mu\,|\,\mathbf{y})\big) \sum_{\nu \mapsto \mu} \beta^{|\nu/\mu|} G_\nu(\mathbf{x}\,|\,\mathbf{y})\,. \tag{3.5}$$

We can rewrite this Pieri rule as follows.

Proposition 3.6. *We have:*

$$G_\mu(\mathbf{x}\,|\,\mathbf{y}) \left(\frac{G_1(\mathbf{x}\,|\,\mathbf{y}) - G_1(\mathbf{y}_\mu\,|\,\mathbf{y})}{1 + \beta G_1(\mathbf{y}_\mu\,|\,\mathbf{y})} \right) \ = \ \sum_{\nu \mapsto \mu} \beta^{|\nu/\mu|-1} G_\nu(\mathbf{x}\,|\,\mathbf{y})\,. \tag{3.6}$$

Proof. We expand both sides of (3.5) and cancel the term $G_\mu(\mathbf{x}|\mathbf{y})$ giving

$$\begin{aligned} G_\mu(\mathbf{x}\,|\,\mathbf{y}) \cdot \beta G_1(\mathbf{x}\,|\,\mathbf{y}) \ = \ & \beta G_1(\mathbf{y}_\mu\,|\,\mathbf{y}) \cdot G_\mu(\mathbf{x}\,|\,\mathbf{y}) \\ & + \big(1 + \beta G_1(\mathbf{y}_\mu\,|\,\mathbf{y})\big) \sum_{\nu \mapsto \mu} \beta^{|\nu/\mu|} G_\nu(\mathbf{x}\,|\,\mathbf{y})\,. \end{aligned}$$

Now collect the terms with $G_\mu(\mathbf{x}\,|\,\mathbf{y})$ on the LHS. Dividing by

$$1 + \beta G_1(\mathbf{y}_\mu\,|\,\mathbf{y}) \neq 0$$

and β gives the desired expression. □

Remark 3.7. When we set $\beta = 0$ in the Pieri rule above, it immediately reduces to the Pieri rule for factorial Schur functions (see, e.g., [39, §3]).

Note that

$$1 + \beta G_1(\mathbf{x}\,|\,\mathbf{y}) \ = \ \prod_{j=1}^{d} \big(1 + \beta(x_j \oplus y_j)\big) \ = \ \prod_{i=1}^{d} (1 + \beta x_i) \prod_{i=1}^{d} (1 + \beta y_i)\,.$$

Evaluating both sides at $\mathbf{x} = \mathbf{y}_\lambda$, we get

$$1 + \beta G_1(\mathbf{y}_\lambda\,|\,\mathbf{y}) \ = \ \prod_{i=1}^{d} \frac{1 \,+\, \beta y_i}{1 \,+\, \beta y_{\lambda_i + d - i + 1}}\,. \tag{3.7}$$

§4. Hook formula for straight shapes

The goal of this section is to prove the multivariate Theorem 4.2 and derive its specializations Theorem 1.1 and Corollary 1.2.

4.1. Multivariate formulas. First we evaluate $\mathbf{x} = \mathbf{y}_\lambda$ in (3.6) and simplify to obtain the following expression.

Proposition 4.1. *We have:*

$$G_\mu(\mathbf{y}_\lambda \,|\, \mathbf{y}) \big(\mathrm{wt}(\lambda/\mu) - 1\big) \;=\; \sum_{\nu \mapsto \mu} \beta^{|\nu/\mu|}\, G_\nu(\mathbf{y}_\lambda \,|\, \mathbf{y})\,, \tag{4.1}$$

where

$$\mathrm{wt}(\lambda/\mu) \;:=\; \prod_{i=1}^{d} \frac{1+\beta y_{\mu_i+d-i+1}}{1+\beta y_{\lambda_i+d-i+1}}\,.$$

Proof. We evaluate (3.6) at $\mathbf{x} = \mathbf{y}_\lambda$ and multiply by β. Note that

$$\frac{\beta G_1(\mathbf{y}_\lambda \,|\, \mathbf{y}) \,-\, \beta G_1(\mathbf{y}_\mu \,|\, \mathbf{y})}{1 \,+\, \beta G_1(\mathbf{y}_\mu \,|\, \mathbf{y})} \;=\; \frac{1+\beta G_1(\mathbf{y}_\lambda \,|\, \mathbf{y})}{1 \,+\, \beta G_1(\mathbf{y}_\mu \,|\, \mathbf{y})} \,-\, 1.$$

By (3.7), this equals $\mathrm{wt}(\lambda/\mu) - 1$, as desired. □

Theorem 4.2 (multivariate K-HLF)**.** *Fix $d \geqslant 1$. For every $\lambda \vdash n$ with $\ell(\lambda) \leqslant d$, we have:*

$$\begin{aligned}
&\sum_{T \in \mathrm{SIT}(\lambda)} \prod_{k=1}^{m(T)} \left(\left[\prod_{i=1}^{d} \frac{1 \,+\, \beta y_{\nu_i(T_{<k})+d-i+1}}{1 \,+\, \beta y_{\lambda_i+d-i+1}} \right] - 1 \right)^{-1} \\
&\quad = \; \frac{1}{\beta^n} \prod_{i=1}^{d} \big(1 \,+\, \beta y_{\lambda_i+d-i+1}\big)^{\lambda_i} \prod_{(i,j)\in\lambda} \frac{1}{y_{d+j-\lambda'_j} \,-\, y_{\lambda_i+d-i+1}}\,.
\end{aligned} \tag{4.2}$$

Proof. We apply Proposition 4.1 repeatedly, by taking $\mu \leftarrow \nu(T_{\leqslant k-1})$ and $\nu \leftarrow \nu(T_{\leqslant k})$, and noting that $\nu \mapsto \mu$ by equation (3.4). Since this is a straight shape, we are starting with the empty partition $\varnothing = \nu(T_{\leqslant 0})$, until we eventually reach $\nu(T_{\leqslant k}) = \lambda$. Here we use that the vanishing property Proposition 3.4 ensures that all shapes are contained in λ. We obtain:

$$\sum_{T \in \mathrm{SIT}(\lambda)} \prod_{k=0}^{m(T)-1} \frac{\beta^{a(T_{\leqslant k+1}) \,-\, a(T_{\leqslant k})}}{\mathrm{wt}(\lambda/\nu^{(k)}) - 1} \;=\; \frac{G_\lambda(\mathbf{y}_\lambda \,|\, \mathbf{y})}{G_\varnothing(\mathbf{y}_\lambda \,|\, \mathbf{y})}\,.$$

Since $G_\varnothing = 1$ and

$$G_\lambda(\mathbf{y}_\lambda \,|\, \mathbf{y}) \;=\; \prod_{(i,j)\in\lambda} \frac{y_{d+j-\lambda'_j} - y_{\lambda_i+d-i+1}}{1+\beta y_{\lambda_i+d-i+1}}$$

by Proposition 3.4, the desired statement follows. $\square$

Proposition 4.3. *Fix $d \geqslant 1$. For every $\lambda \vdash n$ with $\ell(\lambda) \leqslant d$, we have:*

$$(-1)^n \, G_\lambda(\mathbf{y}_\lambda \,|\, \mathbf{y})\big|_{y_i=i} \;=\; \prod_{i=1}^{d} \frac{1}{(1+\beta(\lambda_i+d-i+1))^{\lambda_i}} \prod_{(i,j)\in\lambda} h(i,j)\,.$$

Proof. This follows directly from Proposition 3.4, since for $y_i = i$, $i \geqslant i$, we have:

$$\big(y_{d+j-\lambda'_j} \ominus y_{\lambda_i+d-i+1}\big) \;=\; \frac{j-\lambda'_j-\lambda_i+i-1}{1+\beta(\lambda_i+d-i+1)}$$

and $h(i,j) = \lambda'_j - i + \lambda_i - j + 1$. $\square$

Proof of Theorem 1.1. This follows from Theorem 4.2 by substituting $y_i \leftarrow i$, for all $i \geqslant 1$. Indeed, notice that

$$y_{d+j-\lambda'_j} \;-\; y_{\lambda_i+d-i+1} \;=\; -(\lambda_i - j + \lambda'_j - i + 1) \;=\; -h(i,j),$$

which implies the result. $\square$

Example 4.4. For $\lambda = (2,2) \vdash 4$, the hook lengths are $3,2,2,1$ as in the tableau H below. We have:

$$G_{22}(\mathbf{y}_{22} \,|\, \mathbf{y})\,\big|_{y_1=y_2=1} \;=\; \frac{3\cdot 2\cdot 2\cdot 1}{(1+3\beta)^2(1+4\beta)^4}\,.$$

There are three standard increasing tableaux: $\mathrm{SIT}(\lambda) = \{A, B, C\}$, as shown below:

$$H \;=\; \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 & 1 \\ \hline \end{array} \qquad A \;=\; \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}\,, \qquad B \;=\; \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}\,, \qquad C \;=\; \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array}\,.$$

The terms on the RHS of (4.2) are

$$u(A) = u(B) = \frac{(1+3\beta)^3(1+4\beta)^2}{6\beta^4(4+10\beta)}\,, \quad u(C) = -\frac{(1+3\beta)^2(1+4\beta)^2}{3\beta^3(4+10\beta)}\,,$$

and indeed we have

$$\beta^4\big(u(A)+u(B)+u(C)\big) \;=\; \frac{(1+3\beta)^2(1+4\beta)^4}{12}\,.$$

4.2. An infinite version. Next we give an equivalent expression for Theorem 1.1 in terms of increasing tableaux instead of standard increasing tableaux.

Theorem 4.5 (infinite multivariate K-HLF)**.** *Fix $d \geqslant 1$. For every $\lambda \vdash n$ with $\ell(\lambda) \leqslant d$, we have:*

$$\begin{aligned}\sum_{T\in \mathrm{IT}(\lambda)} \prod_{k=1}^{m} \prod_{i=1}^{d} \frac{1+\beta y_{\lambda_i+d-i+1}}{1+\beta y_{\nu_i(T_{<k})+d-i+1}} \\ = \frac{1}{(\beta)^n} \prod_{i=1}^{d} \big(1+\beta(\lambda_i+d-i+1)\big)^{\lambda_i} \prod_{(i,j)\in\lambda} \frac{1}{y_{d+j-\lambda'_j}-y_{\lambda_i+d-i+1}}\,.\end{aligned} \tag{4.3}$$

In contrast with (4.2), the sum on the LHS of (4.3) is infinite. This is somewhat further away from the original (HLF), but closer in spirit to (q-HLF).

Proof. Rewrite Proposition 4.1 as

$$G_\mu(\mathbf{y}_\lambda\,|\,\mathbf{y}) \;=\; \sum_{\nu\mapsto\mu \text{ or } \nu=\mu} \beta^{|\nu/\mu|}\,\frac{G_\nu(\mathbf{y}_\lambda \mid \mathbf{y})}{\mathrm{wt}(\lambda/\mu)}\,.$$

Now, as in the proof of Theorem 4.2, iterate this relation until $\nu(T_{\leqslant m})=\lambda$, where $m=m(T)$. This implies the result. $\square$

By analogy with the previous argument for SITs, we obtain the following infinite version of (K-HLF).

Corollary 4.6 (infinite K-HLF)**.** *Fix $d \geqslant 1$. For every $\lambda \vdash n$ with $\ell(\lambda) \leqslant d$, we have:*

$$\begin{aligned}\sum_{T\in \mathrm{IT}(\lambda)} \prod_{k=1}^{m(T)} \prod_{i=1}^{d} \frac{1+\beta(\lambda_i+d-i+1)}{1+\beta(\nu_i(T_{<k})+d-i+1)} \\ = \frac{1}{(-\beta)^n} \prod_{i=1}^{d} \big(1+\beta(\lambda_i+d-i+1)\big)^{\lambda_i} \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)}\,.\end{aligned} \tag{4.4}$$

The proof follows verbatim the proof above and will be omitted.

4.3. The q-analog. Let us now obtain the q-analog of (K-HLF).

Theorem 4.7 (q-K-HLF)**.** *Fix $d \geqslant 1$. For every $\lambda \vdash n$ with $\ell(\lambda) \leqslant d$, we have:*

$$\sum_{T \in \mathrm{SIT}(\lambda)} \prod_{k=1}^{m(T)} \left(\left[\prod_{i=1}^{d} \frac{1 + \beta q^{\nu_i(T_{<k})+d-i+1}}{1 + \beta q^{\lambda_i + d - i + 1}} \right] - 1 \right)^{-1} \\ = \frac{q^{m(\lambda)}}{\beta^n} \prod_{i=1}^{d} \left(1 + \beta q^{\lambda_i + d - i + 1}\right)^{\lambda_i} \prod_{(i,j) \in \lambda} \frac{1}{1 - q^{h(i,j)}} . \tag{4.5}$$

Proof. Substitute $y_i \leftarrow q^i$ for all $i \geqslant 1$ in Theorems 1.1 and 4.5. Observe that

$$y_{d+j-\lambda'_j} - y_{\lambda_i + d - i + 1} = q^{d+j-\lambda'_j}\left(1 - q^{h(i,j)}\right),$$

since $h(i,j) = (\lambda'_j - j) + (\lambda_i - i) + 1$. Following verbatim the argument above, this implies the result. $\square$

Proof of Corollary 1.2. Letting $\beta \to \infty$ in (4.5), for each term on the LHS we have:

$$\frac{1 + \beta q^{\nu_i(T_{<k})+d-i+1}}{1 + \beta q^{\lambda_i + d - i + 1}} \to q^{\nu_i(T_{<k}) - \lambda_i} = q^{-\nu_i(T_{\geqslant k})} .$$

The product of the inverses of such terms over all $1 \leqslant i \leqslant d$ gives $q^{a(T_{\geqslant k})}$. Factoring out the leading β^n terms on both sides and simplifying the formula, we obtain (1.1). $\square$

4.4. Evaluations of coefficients. We can expand the LHS in (1.1) as a power series in β and compare the coefficients on both sides. First, as mentioned in the introduction, we recover the original hook-length formula (HLF) by evaluating the constant terms.

Proposition 4.8 ($\beta = 0$ in K-HLF)**.** *The term at β^{-n} in equation* (K-HLF) *gives* (HLF)*.*

Proof. Let $\lambda \vdash n$. Extract the constant term in (K-HLF), after multiplying both sides by β^n. In the RHS, we obtain the product of hooks $\prod_{u \in \lambda} 1/h(u)$. In the LHS, since

$$\frac{1 + \beta p}{1 + \beta t} = 1 + \sum_{i=1}^{\infty} (p - t)\,(-t)^{i-1} \beta^i,$$

the constant term contains only the summands with $m(T) = n$, each with weight $1/n!$. By definition, these summands correspond to $T \in \mathrm{SYT}(\lambda)$. Thus (K-HLF) at $\beta = 0$ gives the HLF in the form

$$\sum_{T \in \mathrm{SYT}(\lambda)} \frac{1}{n!} \;=\; \prod_{u \in \lambda} \frac{1}{h(u)}\,,$$

as desired. □

We conclude with a curious corollary relating standard Young tableaux and barely standard Young tableaux (see §2.5). Here we are using

$$p_2(x_1, \ldots, x_d) \;=\; x_1^2 + \ldots + x_d^2,$$

a symmetric power sum. Other notation are the *staircase shape* $\delta_d = (d-1, \ldots, 1, 0)$ and the *harmonic number* $h_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$.

Corollary 4.9 (coefficient of β^{1-n} in K-HLF)**.** *Fix $d \geqslant 1$. For every $\lambda \vdash n$ with $\ell(\lambda) \leqslant d$, we have:*

$$\begin{aligned}\sum_{\nu \subsetneq \lambda} f^{\nu} f^{\lambda/\nu} \frac{p_2(\nu + \delta_d)}{n - |\nu|} \;-\; \sum_{k=1}^{n} (n+k-2)\, \bigl|\mathrm{BSYT}_k(\lambda)\bigr| \\ =\; f^{\lambda} \left((h_n - 1)\, p_2(\lambda + \delta_d) \;+\; \frac{n(n - d(d+1))}{2} \right).\end{aligned} \tag{4.6}$$

The proof is a lengthy but straightforward calculation of evaluating the coefficient of β^1 on both sides of (K-HLF) normalized by β^n, and will be omitted. See §7.4 for the background on BSYTs.

§5. Generalized excited diagrams

5.1. Definitions. Given a set $S \subset \lambda$, we say that a cell $(i, j) \in S$ is *active* if $(i+1, j)$, $(i, j+1)$, and $(i+1, j+1)$ are in $\lambda \setminus S$. For an active cell $u = (i, j) \in S$, define $a_u(S)$ to be the set obtained by replacing (i, j) by $(i+1, j+1)$ in S. Similarly, define $b_u(S)$ to be the set obtained by adding $(i+1, j+1)$ to S. We call $a_u(S)$ a *type I excited move* and $b_u(S)$ a *type II excited move.*

Let $\mathcal{E}(\lambda/\mu)$ be the set of diagrams obtained from μ after a sequence of type I excited moves on active cells. These are called *excited diagrams.* These diagrams are used in both the Naruse hook-length formula (NHLF) and its q-analog (q-NHLF).

Let $\mathcal{D}(\lambda/\mu)$ be the set of diagrams obtained from μ after a sequence of both types of excited moves on active cells. These are called *generalized* *excited diagrams*. For example, the skew shape $\lambda/\mu = 43/2$ has five generalized excited diagrams, three of which are the ordinary excited diagrams. These are illustrated in Fig. 3 below.

5.2. Properties. To an excited diagram $D \in \mathcal{E}(\lambda/\mu)$ we associate a subset $\pi(D) \subseteq \lambda \setminus D$ called the *excited peaks*, constructed inductively, see [41, §6.3]. For $\mu \in \mathcal{E}(\lambda/\mu)$, let $\pi(\mu) = \varnothing$. Let $D \in \mathcal{E}(\lambda/\mu)$ be an excited diagram with active cell $u = (i, j)$, and let $D' = a_u(D)$ be the result of the type I excited move $D \to D'$. Then the excited peaks of D' are defined as

$$\pi(D') := \pi(D) - (i, j+1) - (i+1, j) + (i, j),$$

see Fig. 1. It is easy to see that the set $\pi(D)$ of excited peaks is well defined and independent on the order of the moves. Naruse–Okada gave in [47, Proposition 3.7] an explicit nonrecursive description of $\pi(D)$, as well as the following characterization of generalized excited diagrams in terms of excited diagrams and excited peaks.

Proposition 5.1 ([47, Proposition 3.13]). *We have:*

$$\mathcal{D}(\lambda/\mu) = \bigcup_{D \in \mathcal{E}(\lambda/\mu)} \big\{ D \cup S \ : \ S \subseteq \pi(D) \big\},$$

so, in particular,

$$\big|\mathcal{D}(\lambda/\mu)\big| = \sum_{D \in \mathcal{E}(\lambda/\mu)} 2^{|\pi(D)|}. \tag{5.1}$$

Remark 5.2. There is a certain duality between the set $\mathcal{D}(\lambda/\mu)$ of generalized excited diagrams and the set $\mathcal{P}(\lambda/\mu)$ of *pleasant diagrams* defined in [41] to give an $\mathrm{RPP}(\lambda/\mu)$ version of (q-NHLF). In particular, the following result is a direct analog of Proposition 5.1.

Proposition 5.3 ([41, §6.2]). *We have:*

$$\mathcal{P}(\lambda/\mu) = \bigcup_{D \in \mathcal{E}(\lambda/\mu)} \big\{ \pi(D) \cup S \ : \ S \subseteq \lambda \setminus D \big\},$$

so, in particular,

$$\big|\mathcal{P}(\lambda/\mu)\big| = \sum_{D \in \mathcal{E}(\lambda/\mu)} 2^{|\lambda/\mu| - |\pi(D)|}. \tag{5.2}$$

Example 5.4. We have $|\mathcal{E}(332/21)| = 5$, see Fig. 1, giving $|\mathcal{D}(332/21)| = 11$ by (5.1). Similarly, equation (5.2) gives $|\mathcal{P}(332/21)| = 88$ pleasant diagrams in this case.

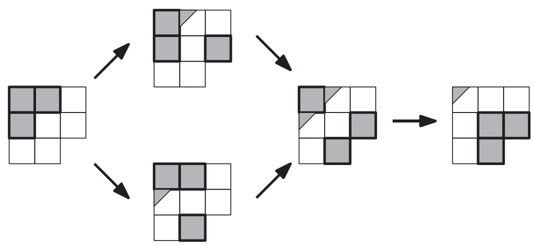

Fig. 1. Excited diagrams of shape $\lambda/\mu = 332/21$, excited moves of type I, and the corresponding excited peaks denoted by shaded triangles.

5.3. Lattice paths interpretation. Following the approach in [26, 42], these generalized excited diagrams are in bijection with certain collections of lattice paths by the following construction.

Let us cut the skew diagram λ/μ into *border strips* greedily starting from μ. Consider these strips between the diagonal starting at $(0, \ell(\mu))$ and the diagonal starting at $(\mu_1, 0)$. Within this region, let these border strips start at squares with midpoints A_i and end at squares with midpoints B_i, see Fig. 2 (left).

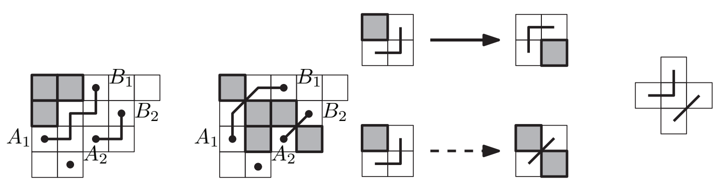

Fig. 2. Paths corresponding to two generalized excited diagrams, the flips of the paths in the type I and II excited moves, and the forbidden path configuration.

Let $\eta(A, B)$ be the number of paths $A \to B$ inside λ with endpoints at the centers of the squares of the Young diagram and Delannoy steps. We call these *Delannoy paths*. The following result interprets the generalized excited diagrams $\mathcal{D}(\lambda/\mu)$ as collections of nonintersecting Delannoy paths inside λ/μ.

Proposition 5.5. *The set $\mathcal{D}(\lambda/\mu)$ is in bijection with Delannoy path collections $\gamma_i : A_i \to B_i$ such that no two such lattice paths γ_i and γ_j intersect or have a configuration as in Fig. 2 (right). In particular, we have:*

$$\left|\mathcal{D}(\lambda/\mu)\right| \leqslant \det\left[\eta(A_i, B_j)\right]_{i,j} .$$

Proof. For the first part, take Delannoy paths in the complement as shown in Fig. 3. Observe that for the initial configuration $\mu \in \mathcal{D}$, the lowest such lattice paths traverse μ inside λ/μ. A type I excited move transforms a path by flipping a corner from $(1,0),(0,1)$ steps to $(0,1),(1,0)$ steps. A type II excited move transforms a path by changing a $(1,0),(0,1)$ corner to a $(1,1)$ step, while the SE and NW cells of that step are empty. Further, a type I excited move applied to a cell u with a diagonal step at its SE corner results in flipping this diagonal to steps $(0,1),(1,0)$ and transferring the diagonal step to the nearest SE path. A type II excited move at a cell u with a diagonal step already present results in modifying the nearest SE path as above. See Fig. 2 (middle).

The final configuration can be drawn by a greedy traverse of the non-excited cells starting from A_1 to B_1, see Fig. 3. Thus the paths pass exactly through the cells outside S, the corresponding moves are reversible on paths as long as there is no intersection and no forbidden configuration. For the second part, note that *all* nonintersecting Delannoy paths are enumerated by the determinant using the Lindström–Gessel–Viennot (LGV) lemma (see, e.g., [17, §5.4]), giving the desired determinant inequality. ☐

Example 5.6. For the skew shape $\lambda/\mu = 5442/21$ as in Fig. 2, we have:

$$23 = \left|\mathcal{D}(5442/21)\right| \leqslant \det \begin{bmatrix} 13 & 7 \\ 1 & 3 \end{bmatrix} = 32.$$

5.4. Labeled lattice paths. Kreiman [26] (see also [42, Proposition 3.6]) showed that the excited diagrams are in bijection with the complements of collections of nonintersecting lattice paths consisting of $(0,1)$ and $(1,0)$ steps, contained in λ, and with starting and ending points A_i, B_i as above. Note that in [26, 42], the starting and ending points were different, but

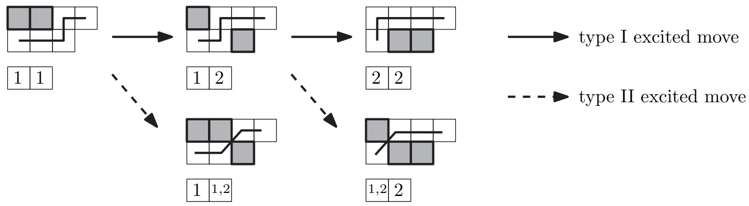

Fig. 3. The generalized excited diagrams of shape $\lambda/\mu = 43/2$, their peaks, and the corresponding flagged set tableaux (see §7.5). The complements of diagrams in $\mathcal{D}(\lambda/\mu)$ can be viewed as Delannoy paths inside λ.

the geometry actually forces the corner portions of the paths to be always fixed and hence the starting and ending points can vary.

Following the definition in §2.3, consider the *high peaks* of a collection of nonintersecting lattice paths. Here the high peaks of a path are defined as the peaks which moved from the corresponding base path cut out from the skew diagram λ/μ. As an example, in Fig. 1, there is one lattice path which corresponds to the white cells, and the inner corners which are high peaks are labeled.

Remark 5.7. Note that the high peaks are a subset of the cells on which a type I excited move was applied at some point and correspond exactly to the *excited peaks*.

Denote by $\Pi(\lambda/\mu)$ the set of such collections of paths where each high peak has been labeled 0 or 1. Similarly, denote by $\Delta(\lambda/\mu)$ the set of collections of Delannoy paths in the complement of generalized excited diagrams in $\mathcal{D}(\lambda/\mu)$.

We can now explain Proposition 5.1 via lattice paths by the following bijection $\phi : \Pi(\lambda/\mu) \to \Delta(\lambda/\mu)$ between labeled lattice and Delannoy paths. Formally, for a collection $\Upsilon \in \Pi(\lambda/\mu)$, replace each high peak labeled 1 with a $(1, 1)$ step; all other peaks and paths stay the same.

Proposition 5.8. *For a skew shape λ/μ, the map $\phi : \Pi(\lambda/\mu) \to \Delta(\lambda/\mu)$ defined above is a bijection.*

Proof. It is easy to see that for every $\Upsilon \in \Pi(\lambda/\mu)$, the paths in $\phi(\Upsilon)$ are exactly the Delannoy paths for $\Delta(\lambda/\mu)$. For the inverse map ϕ^{-1}, replace every $(1,1)$ step with $(0,1),(1,0)$ steps which would necessarily form a high peak and label it 1. This implies the result. $\square$

5.5. The thick zigzag shape. Consider the *thick zigzag shape* δ_{n+2k}/δ_n. Recall that

$$\big|\mathcal{E}(\delta_{n+2k}/\delta_n)\big| = \det\big[C_{n+i+j-2}\big]_{i,j=1}^{k}$$

and

$$\big|\mathcal{P}(\delta_{n+2}/\delta_n)\big| = 2^{\binom{k}{2}} \det\big[\widehat{s}_{n+i+j-2}\big]_{i,j=1}^{k},$$

where $\widehat{s}_n = 2^{n+2}s_n$. The first equality is proved in [42, Corollary 8.1], while the second one was originally conjectured in [41, Conjecture 9.3] and proved in [23, Theorem 1.1]. We give a similar determinant formula for the number of generalized excited diagrams of the thick zigzag shape.

Theorem 5.9. *We have:* $\big|\mathcal{D}(\delta_{n+2}/\delta_n)\big| = s_n$ *and*

$$\big|\mathcal{D}(\delta_{n+4}/\delta_n)\big| = \frac{1}{2}\big(s_n s_{n+2} - s_n^2\big).$$

More generally, we have:

$$\big|\mathcal{D}(\delta_{n+2k}/\delta_n)\big| \,=\, 2^{-\binom{k}{2}} \det\big[s_{n-2+i+j}\big]_{i,j=1}^{k} \quad \textit{for all } k \geqslant 1. \tag{5.3}$$

Proof. From [42, §3.3, §8.1], the complements of excited diagrams $D \in \mathcal{E}(\delta_{n+2k}/\delta_n)$ correspond to k-tuples $\Upsilon := (\gamma_1,\ldots,\gamma_k)$ of nonintersecting Dyck paths $\gamma_i \in \mathsf{Dyck}(n+2i-2)$, for all $1 \leqslant i \leqslant k$, whose set we denote by $\mathsf{NDyck}(n,k)$. Define $\mathcal{HP}(\Upsilon) := \bigcup_{i=1}^k \mathcal{HP}(\gamma_i)$ and $\operatorname{hp}(\Upsilon) := |\mathcal{HP}(\Upsilon)|$.

By Proposition 5.8, the diagrams $D \in \mathcal{D}(\delta_{n+2k}/\delta_n)$ correspond to the tuples (Υ, S) where $\Upsilon \in \mathsf{NDyck}(n,k)$ and $S \subseteq \mathcal{HP}(\Upsilon)$ are the high peaks labeled with 1. We conclude:

$$\big|\mathcal{D}(\delta_{n+2k}/\delta_n)\big| \;=\; \sum_{\Upsilon \in \mathsf{NDyck}(n,k)} 2^{\operatorname{hp}(\Upsilon)}. \tag{5.4}$$

Let

$$L_n(x) \,:=\, \sum_{\gamma \in \mathsf{Dyck}(n)} x^{\operatorname{hp}(\gamma)} \quad \text{and} \quad L_{n,k}(x) \,:=\, \sum_{\Upsilon \in \mathsf{NDyck}(n,k)} x^{\operatorname{hp}(\Upsilon)}.$$

Note that $s_n = L_n(2)$, see, e.g., [61]. By (5.4), we have

$$L_{n,k}(2) = \big|\mathcal{D}(\delta_{n+2k}/\delta_n)\big|.$$

Finally, by [23, Theorem 5.9], the sum $L_{n,k}(x)$ satisfies the following determinant formula:

$$x^{\binom{k}{2}} \cdot L_{n,k}(x) = \det\bigl[L_{n+i+j-2}(x)\bigr]_{i,j=1}^{k}. \tag{5.5}$$

Setting $x = 2$, we obtain the result. □

5.6. Double Grothendieck polynomials. Excited diagrams can be used to give a combinatorial model of these polynomials in the special case we need. For a definition and combinatorial models of *double Grothendieck polynomials* for all permutations, see [13, 14, 27].

In [28], Knutson–Miller–Yong gave the following formula for the Grothendieck polynomials of vexillary permutations, originally stated in terms of flagged set tableaux and restated here in terms of generalized excited diagrams. See also §7.7 for a discussion of another proof of this result.

Theorem 5.10 ([28, Theorem 5.8])**.** *Let w be a vexillary permutation of shape μ and supershape λ. Then the double Grothendieck polynomial parameterized by w can be computed as follows:*

$$\mathfrak{G}_w(\mathbf{x},\mathbf{y}) = \sum_{D\in\mathcal{D}(\lambda/\mu)} \beta^{|D|-|\mu|} \prod_{(i,j)\in D} (x_i \oplus y_j). \tag{5.6}$$

Corollary 5.11. *Let w be a vexillary permutation of shape μ and supershape λ. Then we have:*

$$\mathfrak{G}_w(\mathbf{x},\mathbf{y}) \;=\; \sum_{D\in\mathcal{E}(\lambda/\mu)} \beta^{|D|-|\mu|} \prod_{(i,j)\in\pi(D)} \bigl(1+\beta(x_i\oplus y_j)\bigr) \prod_{(i,j)\in D} \bigl(x_i \oplus y_j\bigr).$$

Proof. This follows immediately from Theorem 5.10 and Proposition 5.1. □

Example 5.12. For $w = 1432 \in S_4$, we have $\mu = 21$, $\lambda = 332$, and $|\mathcal{D}(332/21)| = 11$, see Example 5.4 and [13, Example 1]. By Corollary 5.11, for $y_i = 0$ we have:

$$\begin{aligned}\mathfrak{G}_{1432}(\mathbf{x},\mathbf{0}) &= x_1^2x_2 + x_2^2x_1\,(1+\beta x_1)\\ &\quad + x_1^2x_3\,(1+\beta x_2) + x_1x_2x_3\,(1+\beta x_1)\,(1+\beta x_2) + x_2^2x_3\,(1+\beta x_1)\\ &= x_1^2x_2 + x_2^2x_1 + x_1^2x_3 + x_1x_2x_3 + x_2^2x_3 + \beta x_1^2x_2^2 + 2\,\beta x_1^2x_2x_3\\ &\qquad + 2\beta x_2^2x_1x_3 + \beta^2x_1^2x_2^2x_3\,.\end{aligned}$$

5.7. The principal specialization. Let $\Gamma_w(\beta) := \mathfrak{G}_w(\mathbf{1},\mathbf{0})$ be the *principal specialization of the Grothendieck polynomial*. Substituting $x_i \leftarrow 1$ and $y_i \leftarrow 0$ in Corollary 5.11, we immediately obtain the following.

Corollary 5.13. *Let w be a vexillary permutation of shape μ and supershape λ. Then:*

$$\Gamma_w(\beta) \;=\; \sum_{D\in\mathcal{D}(\lambda/\mu)} \beta^{|D|-|\mu|} \;=\; \sum_{D\in\mathcal{E}(\lambda/\mu)} \beta^{|D|-|\mu|}\,(1+\beta)^{|\pi(D)|}\,. \tag{5.7}$$

Using the lattice paths interpretation from §5.3, let $\eta_\beta(A,B)$ be the weighted sum of Delannoy paths $A\to B$ with β keeping track of the number of $(1,1)$ steps. We have the following inequality for the principal specialization of the Grothendieck polynomials considered above.

Corollary 5.14. *Let w be a vexillary permutation of shape μ and supershape λ, and let $\Gamma_w(\beta)$ be the principal specialization of the Grothendieck polynomial. Then:*

$$\Gamma_w(\beta) \;\leqslant\; \det\big[\eta_\beta(A_i,B_j)\big]_{i,j}\,,$$

where $\leqslant$ means coefficient-wise inequality for polynomials in β.

Proof. The result follows immediately from Corollary 5.13, the proof of Proposition 5.5, and the proof of the LGV lemma which preserves the total number of $(1,1)$ steps under the involution. □

Finally, we give a determinant formula for the principal specialization $\Gamma_{w(n,k)}(1)$, where

$$w(n,k) := (1,2,\ldots,k,n+k,n+k-1,\ldots,k+1).$$

See [15] and [43, Corollary 5.8] for the analogous results on evaluations of Schubert polynomials of $w(n,k)$.

Corollary 5.15. *For all $n,k\geqslant 1$, in the notation above we have:*

$$\Gamma_{w(n,k)}(1) \;=\; 2^{-\binom{k}{2}}\,\det\big[s_{n-2+i+j}\big]_{i,j=1}^{k} \quad \textit{for all } k\geqslant 1.$$

Proof. The permutation $w(n,k)$ is *dominant* (132-avoiding), and hence vexillary. Denote by λ/δ_n the skew shape associated to $w(n,k)$, see [43, Fig. 6(a)]. Then Corollary 5.13 at $\beta=1$ gives:

$$\Gamma_{w(n,k)}(1) \;=\; \big|\mathcal{D}(\lambda/\delta_n)\big|.$$

From the definition of generalized excited diagrams, or from their correspondence with flagged set-valued tableaux (see §7.5), it is easy to see that $\big|\mathcal{D}(\lambda/\delta_n)\big| = \big|\mathcal{D}(\delta_{n+2k}/\delta_n)\big|$. The result then follows by Theorem 5.9. $\square$

§6. Hook formula for skew shapes

6.1. The setup. Recall the vanishing property (Proposition 3.4) of the factorial Grothendieck polynomials:

$$G_\mu(\mathbf{y}_\lambda \,|\, \mathbf{y}) \;=\; \begin{cases} 0 & \text{if } \mu \not\subseteq \lambda\,, \\ \prod_{(i,j)\in\lambda} (y_{d+j-\lambda'_j} \ominus y_{\lambda_i+d-i+1}) & \text{if } \mu = \lambda\,. \end{cases}$$

Following the approach of Ikeda–Naruse [24] and Kreiman [26] for the factorial Schur functions $s_\mu(\mathbf{y}_\lambda \,|\, \mathbf{y})$, we present a combinatorial model for the Andersen–Jentzen–Soergel [1] and Billey [2] expressions for evaluations of the factorial Grothendieck polynomials $G_\mu(\mathbf{y}_\lambda \,|\, \mathbf{y})$ when $\mu \subseteq \lambda$.

Fix two Grassmannian permutations $w \leqslant v$ in S_N with associated partitions $\mu \subseteq \lambda$ with $\ell(\lambda) \leqslant d$ and $\lambda_1 \leqslant N-d$, see, e.g., [36, §2.1]. Let $c^\lambda_{\mu\tau}$ and $K^\lambda_{\mu\tau}$ be the *structure constants* for the Schubert classes in the equivariant cohomology and equivariant K-theory of the Grassmannian, respectively, see, e.g., [24, 26, 18].

Theorem 6.1 (Ikeda–Naruse [24], Kreiman [26])**.** *Fix $d \geqslant 1$. For all $\mu \subset \lambda$ with $\ell(\lambda) \leqslant d$, we have:*

$$c^\lambda_{\mu\lambda} \;=\; \sum_{D\in\mathcal{E}(\lambda/\mu)} \; \prod_{(i,j)\in D} \big(y_{d+j-\lambda'_j} \;-\; y_{\lambda_i+d+1-i}\big).$$

Theorem 6.2 (Graham–Kreiman [18, Theorem 4.5])**.** *Fix $d \geqslant 1$. For all $\mu \subset \lambda$ with $\ell(\lambda) \leqslant d$, we have:*

$$K^\lambda_{\mu\lambda} \;=\; \sum_{D\in\mathcal{D}(\lambda/\mu)} (-1)^{|D|-|\mu|} \prod_{(i,j)\in D} \frac{y_{d+j-\lambda'_j} \;-\; y_{\lambda_i+d+1-i}}{1 \;-\; y_{\lambda_i+d+1-i}}\,.$$

Remark 6.3. To translate from the result in [18, Theorem 4.5] to the one stated here, one needs to do the substitution $y_i \leftarrow \big(1 - e^{\epsilon_i}\big)$, as discussed in [18, §4.3.1, §5.4].

6.2. Multivariate formulas. The following technical lemma gives an evaluation of the factorial Grothendieck polynomials, and provides a bridge to our enumerative problem.

Lemma 6.4. *Fix $d \geqslant 1$. For all $\mu \subset \lambda$ with $\ell(\lambda) \leqslant d$, we have:*

$$G_\mu(\mathbf{y}_\lambda \,|\, \mathbf{y}) \;=\; \sum_{D \in \mathcal{D}(\lambda/\mu)} \beta^{|D|-|\mu|} \prod_{(i,j)\in D} \big(y_{d+j-\lambda'_j} \ominus y_{\lambda_i+d-i+1}\big). \tag{6.1}$$

Proof. We show that both sides of (6.1) satisfy the same identity. First, the factorial Grothendieck polynomials satisfy the Chevalley formula (3.6). Thus, for the LHS of (6.1) we have:

$$G_\mu(\mathbf{y}_\lambda \,|\, \mathbf{y}) \left(\frac{G_1(\mathbf{y}_\lambda|\mathbf{y}) \,-\, G_1(\mathbf{y}_\mu \,|\, \mathbf{y})}{1 + \beta G_1(\mathbf{y}_\mu \,|\, \mathbf{y})} \right) \;=\; \sum_{\nu \supsetneq \mu} \beta^{|\nu/\mu|-1}\, G_\nu(\mathbf{y}_\lambda \,|\, \mathbf{y})\,.$$

By Theorem 6.2, the RHS of (6.1) at $\beta = -1$ equals $K^\lambda_{\mu\lambda}$. On the other hand, Lenart–Postnikov [33, Corollary 8.2] (see also the proof of Proposition 3.1 in [55]) give the following *equivariant K-theory Chevalley formula*:

$$K^\lambda_{\mu\lambda} \left(\frac{K^\lambda_{1\lambda} - 1 + \mathrm{wt}'(\mu)}{\mathrm{wt}'(\mu)} \right) \;=\; \sum_{\nu \mapsto \mu} (-1)^{|\nu/\mu|-1}\, K^\lambda_{\nu\lambda}\,,$$

where

$$\mathrm{wt}'(\mu) \;:=\; \prod_{(i,j)\in\mu} \frac{1 - y_{i+j-1}}{1 - y_{i+j}}.$$

Observe that we have cancellations in the formula for $\mathrm{wt}'(\mu)$, and for each row i of μ only the term $(1 - y_i)/(1 - y_{\mu_i+d-i+1})$ survives in the product. Thus:

$$\mathrm{wt}'(\mu) \;=\; \prod_{i=1}^{d} \frac{1 - y_i}{1 - y_{\mu_i+d-i+1}} \;=\; 1 - G_1\big(\mathbf{y}_\mu \,|\, \mathbf{y}\big)\,|_{\beta=-1}\,,$$

where the second equality follows by (3.7). Therefore, we have:

$$K^\lambda_{\mu\lambda} \left(\frac{K^\lambda_{1\lambda} - G_1(\mathbf{y}_\mu \,|\, \mathbf{y})\,|_{\beta=-1}}{1 - G_1(\mathbf{y}_\mu \,|\, \mathbf{y})\,|_{\beta=-1}} \right) \;=\; \sum_{\nu \supsetneq \mu} (-1)^{|\nu/\mu|-1}\, K^\lambda_{\nu\lambda}\,.$$

This shows that

$$G_\mu(\mathbf{y}_\lambda \,|\, \mathbf{y}) \,|_{\beta=-1} = K^\lambda_{\mu\lambda}.$$

We conclude:

$$G_\mu(\mathbf{y}_\lambda \,|\, \mathbf{y}) \,|_{\beta=-1} = \sum_{D\in\mathcal{D}(\lambda/\mu)} (-1)^{|D|-|\mu|} \prod_{(i,j)\in D} \frac{y_{d+j-\lambda'_j} - y_{\lambda_i+d+1-i}}{1 - y_{\lambda_i+d+1-i}}\,. \tag{6.2}$$

It remains to show that by substituting $y_i \leftarrow (-y_i\beta)$ in (6.2) we get the desired result. Denote the LHS of (6.2) by $F(y_1,\ldots,y_n)$. We easily verify that

$$(-\beta)^{-|\mu|}\, F(-y_1\beta,\ldots,-y_n\beta) = \sum_{D\in\mathcal{D}(\lambda/\mu)} \beta^{|D|-|\mu|} \prod_{(i,j)\in D} \big(y_{d+j-\lambda'_j} \ominus y_{\lambda_i+d-i+1}\big)\,.$$

Finally, for the RHS by (3.1) we have that

$$G_\mu\big(\mathbf{y}_\lambda \,|\, \mathbf{y}\big) \,|_{y_i\leftarrow(-y_i\beta)} = (-\beta)^{|\mu|}\, G_\mu(\mathbf{y}_\lambda \,|\, \mathbf{y})\,, \tag{6.3}$$

as desired. □

Theorem 6.5 (multivariate K-NHLF)**.** *Fix $d \geqslant 1$. For all $\mu \subset \lambda$ with $\ell(\lambda) \leqslant d$, we have:*

$$\begin{aligned}\sum_{T\in\mathrm{SIT}(\lambda/\mu)} \prod_{k=1}^{m(T)} \left(\left[\prod_{i=1}^{d} \frac{1+\beta y_{\nu_i(T_{<k})+d-i+1}}{1+\beta y_{\lambda_i+d-i+1}} \right] - 1 \right)^{-1} \\ = \sum_{D\in\mathcal{D}(\lambda/\mu)} \beta^{|D|-|\lambda|} \prod_{(i,j)\in\lambda\setminus D} \frac{\beta y_{\lambda_i+d-i+1} + 1}{y_{d+j-\lambda'_j} - y_{\lambda_i+d+1-i}}\,.\end{aligned} \tag{6.4}$$

Proof. By Lemma 6.4 and the vanishing property (3.3) of $G_\mu(\mathbf{y}_\mu \,|\, \mathbf{y})$, we have:

$$\frac{G_\mu(\mathbf{y}_\lambda \,|\, \mathbf{y})}{G_\lambda(\mathbf{y}_\lambda \,|\, \mathbf{y})} = \sum_{D\in\mathcal{D}(\lambda/\mu)} \beta^{|D|-|\mu|} \prod_{(i,j)\in\lambda\setminus D} \frac{1}{y_{d+j-\lambda'_j} \ominus y_{\lambda_i+d-i+1}}. \tag{6.5}$$

Alternatively, by iterating (4.1) we obtain:

$$\frac{G_\mu(\mathbf{y}_\lambda|\mathbf{y})}{G_\lambda(\mathbf{y}_\lambda \,|\, \mathbf{y})} = \beta^{|\lambda/\mu|} \sum_{T\in\mathrm{SIT}(\lambda/\mu)} \prod_{k=1}^{m(T)} \left(\left[\prod_{i=1}^{d} \frac{1+\beta y_{\nu_i(T_{<k})+d-i+1}}{1+\beta y_{\lambda_i+d-i+1}} \right] - 1 \right)^{-1}. \tag{6.6}$$

Equating (6.5) and (6.6), we get the result. □

Proof of Theorem 1.4. This follows from Theorem 6.5 by substituting $y_i \leftarrow i$ for all $1 \leqslant i \leqslant d$, and noticing that $y_{d+j-\lambda'_j} - y_{\lambda_i+d-i+1} = -h(i,j)$. □

6.3. The q-analog. By analogy with the straight shape (§4.3), we obtain a q-analog using the substitution $y_i \leftarrow q^i$ for all $i \geqslant 1$.

Theorem 6.6 (q-K-NHLF)**.** *Fix $d \geqslant 1$. For all $\mu \subset \lambda$ with $\ell(\lambda) \leqslant d$, we have:*

$$\begin{aligned} &\sum_{T \in \mathrm{SIT}(\lambda/\mu)} \prod_{k=1}^{m(T)} \left(\left[\prod_{i=1}^{d} \frac{1 + \beta q^{\nu_i(T_{<k})+d-i+1}}{1 + \beta q^{\lambda_i+d-i+1}} \right] - 1 \right)^{-1} \\ &\qquad = \sum_{D \in \mathcal{D}(\lambda/\mu)} \beta^{|D|-|\lambda|} \prod_{(i,j) \in \lambda \setminus D} \frac{\beta q^{\lambda_i+d-i+1} + 1}{q^{d+j-\lambda'_j}(1 - q^{h(i,j)})} \,. \end{aligned} \tag{6.7}$$

We omit the proof as the calculations follow verbatim those in the proof of Theorem 4.7.

Proof of Theorem 1.5. Following the proof of Corollary 1.2, let $\beta \to \infty$ in (6.7). We have:

$$\frac{1 + \beta q^{\nu_i(T_{<k})+d-i+1}}{1 + \beta q^{\lambda_i+d-i+1}} \;\to\; q^{-\lambda_i+\nu_i(T_{<k})} \;=\; q^{-\nu_i(T_{\geqslant k})} \,.$$

Taking the inverse of the product of these terms over all $1 \leqslant i \leqslant d$, we get $q^{a(T)}$. The β terms on the RHS of (6.7) all have exponents zero, which implies the result. □

Finally, as discussed in the introduction (see Remark 1.6), we can now rewrite the RHS of (6.7) in terms of (ordinary) excited diagrams.

Corollary 6.7. *For every $\mu \subset \lambda$, we have:*

$$\begin{aligned} &\sum_{T \in \mathrm{SIT}(\lambda/\mu)} q^{|T|} \prod_{k=1}^{m(T)} \frac{1}{1 - q^{a(T_{\geqslant k})}} \\ &\qquad = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \pi(D)} \frac{1}{1 - q^{h(i,j)}} \prod_{(i,j) \in \lambda \setminus D} \frac{q^{h(i,j)}}{1 - q^{h(i,j)}} \,. \end{aligned} \tag{6.8}$$

Proof. This follows from Theorem 1.5 and the characterization of generalized excited diagrams given in Proposition 5.1. □

6.4. Back to set-valued tableaux. The following *Okounkov–Olshanski formula* (OOF) given in [49] is yet another nonnegative formula for $f^{\lambda/\mu}$. Fix $d \geqslant 1$. For $\mu \subset \lambda$ with $\ell(\lambda) \leqslant d$, we have:

$$f^{\lambda/\mu} \;=\; n! \sum_{T \in \mathrm{SSYT}_d(\mu)} \prod_{(i,j)\in\lambda} \big(\lambda_{d+1-T(i,j)} \;+ i - j\big) \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)}\,, \qquad \text{(OOF)}$$

where $\mathrm{SSYT}_d(\mu)$ denotes the set of SSYTs of shape μ with entries $\leqslant d$. Note that (OOF) is also proved via evaluations of factorial Schur functions, preceding (NHLF) in this approach. The corresponding q-analogs are given in [8, Theorem 1.2] and [45, §1.4], for the summations over $\mathrm{SSYT}(\lambda/\mu)$ and $\mathrm{RPP}(\lambda/\mu)$, respectively.

Here we follow a simple proof in [45, §3.1] via evaluations of factorial Schur functions, to give a (K-OOF) generalization of (OOF) for $\mathrm{SIT}(\lambda/\mu)$ analogous to Theorem 1.4.

Theorem 6.8 (K-OOF)**.** *Fix $d \geqslant 1$. For all $\mu \subset \lambda$ with $\ell(\lambda) \leqslant d$, we have:*

$$\sum_{T\in\mathrm{SIT}(\lambda/\mu)} \prod_{k=1}^{m(T)} \left(\left[\prod_{i=1}^{d} \frac{1+\beta\big(\nu_i(T_{<k}) + d - i + 1\big)}{1 \,+\, \beta\big(\lambda_i + d - i + 1\big)}\right] - 1 \right)^{-1}$$
$$= \prod_{i=1}^{d} \big(1 \,+\, \beta(\lambda_i + d - i + 1)\big)^{\lambda_i}$$
$$\times \sum_{T\in\mathrm{SSVT}_d(\mu)} (-\beta)^{\mathrm{ne}(T)-|\lambda|} \prod_{(i,j)\in\mu, r\in T(i,j)} \frac{\lambda_{d+1-r} + i - j}{1 \,+\, \beta\big(\lambda_{d+1-r} \,+ r\big)} \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)}$$

Proof. We evaluate $G_\mu(\mathbf{y}_\lambda \,|\, \mathbf{y}) \,/\, G_\lambda(\mathbf{y}_\lambda \,|\, \mathbf{y}) \,|_{y_i \leftarrow i}$ in two different ways. First, the LHS is obtained by the substitution $y_i \leftarrow i$ in (6.6). For the RHS we evaluate the numerator and denominator directly. For the denominator we use Proposition 4.3. For the numerator, since $G_\mu(x_1, \ldots, x_d \,|\, \mathbf{y})$ is symmetric in $x_1, \ldots, x_d$ by Proposition 3.3 (i), we have:

$$G_\mu\big(\ominus(\lambda_1 + d),\, \ldots\,,\, \ominus(\lambda_{d-1} + 2),\, \ominus(\lambda_d + 1) \,\big|\, 1, 2, 3, \ldots\big)$$
$$= \; G_\mu\big(\ominus(\lambda_d + 1),\, \ominus(\lambda_{d-1} + 2),\, \ldots\,,\, \ominus(\lambda_1 + d) \,\big|\, 1, 2, 3, \ldots\big).$$

Next, by Definition 3.1 of factorial Grothendieck polynomials, the RHS of the equation above is equal to

$$\sum_{T\in\mathrm{SSVT}_d(\mu)} \beta^{\mathrm{ne}(T)-|\mu|} \prod_{(i,j)\in\mu,\, r\in T(i,j)} \left[\frac{-(\lambda_{d+1-r} \,+\, r)}{1+\beta(\lambda_{d+1-r} \,+\, r)} \oplus (r + j - i)\right].$$

The result then follows by simplifying the power of β and doing the calculation

$$\frac{-(\lambda_{d+1-r} + r)}{1 + \beta(\lambda_{d+1-r} + r)} \oplus (r+j-i) = \frac{-\lambda_{d+1-r} - i + j}{1 + \beta\big(\lambda_{d+1-r} + r\big)}.$$

We omit the details. □

Remark 6.9. Note that the set $\mathrm{SSYT}_d(\mu)$ in (OOF) is finite and plays a role of the set $\mathcal{E}(\lambda/\mu)$ of excited diagrams in (NHLF). This connection is clarified in [45], with reformulations of (OOF) in terms of *puzzles* and *reverse excited diagrams*. Finally, the set $\mathrm{SSVT}_d(\mu)$ plays a role of generalized excited diagrams $\mathcal{D}(\lambda/\mu)$. It would be interesting to reformulate the theorem similarly, in terms of puzzles.

§7. Final remarks and open problems

7.1. The hook-length formula (HLF) has numerous proofs, starting with the original paper [16]. The Littlewood formula (q-HLF) was first given in [34, p. 124]. We refer to [9, §6.2] for an overview of other proofs and generalizations. The Naruse hook-length formula (NHLF) was originally given by Naruse in his talk slides [46]. In our first two papers of this series [41, 42], we give about four proofs of this result, which include both the SSYT and RPP generalizations, see (q-NHLF) and (1.6).

7.2. In [43], we give various enumerative and asymptotic applications of (NHLF). Further applications and comparisons with other tools for estimating $f^{\lambda/\mu} = |\,\mathrm{SYT}(\lambda/\mu)|$ are surveyed in [51]. It would be interesting to find similar applications of the β-deformations presented in this paper. Let us single out Theorem 3.10 in [43] which established a key symmetry via factorial Schur functions, used to obtain a host of product formulas. Note that two elementary proofs of this result are given in [52]; we are especially curious to find its generalization motivated by the factorial Grothendieck polynomials.

7.3. The notation used for the factorial Grothendieck polynomials goes back to the formal group law of *connective K-theory*, and in the context of algebraic combinatorics is explained in [13] as follows.

Let $\mathcal{A}_n^\beta$ be the algebra with generators $u_1, \ldots, u_{n-1}$ satisfying $u_i^2 = \beta u_i$, the exchange and braid relation. Observe that $\mathcal{A}_n^0$ is the *NilCoxeter algebra* and $\mathcal{A}_n^{-1}$ is the *degenerate Hecke algebra*. Then the functions $h_i(t) = e^{tu_i}$ satisfy the *Yang–Baxter equation*:

$$h_i(t)h_{i+1}(t+s)h_i(s) \;=\; h_{i+1}(s)h_i(t+s)h_{i+1}(t).$$

For $h_i(t) = e^{tu_i} = 1 + xu_i$ we have $x = (e^{\beta t} - 1)/\beta$. We can now write this as $x = [t]_\beta$ and note that $[t]_\beta \oplus [s]_\beta = [t+s]_\beta$.

7.4. Our notion of *barely standard Young tableaux* BSYT comes from a similar notion of *barely set-valued tableaux* recently introduced in [57], and is probably the closest relative of SYT that we have. Note that (4.6) can be rewritten as computing the expectation of the repeated entry, similar to [57] (see also [12]), although the resulting formula is more cumbersome.

7.5. The excited diagrams are in bijection with certain *flagged tableaux*:

$$|\mathcal{E}(\lambda/\mu)| = |\,\mathrm{Flag}(\lambda/\mu)|,$$

where $\mathrm{Flag}(\lambda/\mu) \subset \mathrm{SSYT}(\mu)$, see [41, §3.3]. This connection was used in [42, §3.3] to obtain a determinant formula for $|\mathcal{E}(\lambda/\mu)|$. Similarly, the generalized excited diagrams in $\mathcal{D}(\lambda/\mu)$ are in bijection with certain *flagged set-valued tableaux* of shape μ, see an example in Fig. 3. These bijections were obtained by Kreiman [26, §6] and by Knutson–Miller–Yong [28, §5] in the context of Schubert calculus.

7.6. In Theorem 5.9, we gave a determinant formula for the number of generalized excited diagrams of the skew shape δ_{n+2k}/δ_n using the connection between $\mathcal{D}(\lambda/\mu)$ and $\mathcal{P}(\lambda/\mu)$, see Proposition 5.1. A similar determinant formula for $\mathcal{P}(\delta_{n+2k}/\delta_n)$ is proved in [23]. In fact, [23, Corollary 6.4] gives determinant formulas for pleasant diagrams of more general classes of skew shapes called *good* that also include *thick reverse hooks* $(b+c)^{a+c}/b^a$. Using [23, Theorem 6.3], which is an analog of (5.5), one can obtain a determinant formula for generalized excited diagrams of such good skew shapes.

7.7. In [65, Corollary 1.5, Theorem 1.1], Weigandt gave two formulas for double Grothendieck polynomials $\mathfrak{G}_w(\mathbf{x},\mathbf{y})$ in terms of the *bumpless pipe dreams* of w defined by Lam–Lee–Shimozono [30]. When w is vexillary, these formulas reduce to Theorem 5.10 and Corollary 5.11, respectively. Indeed, a bijection between *marked* bumpless pipe dreams of vexillary w and $\mathcal{D}\big(\lambda(w)/\mu(w)\big)$ via the corresponding flagged set-valued tableaux is given in [65, Theorem 1.6]. Similarly, a bijection between vexillary bumpless pipe dreams and ordinary excited diagrams is given in [65, §7.3].

We should mention that bumpless pipe dreams of w behave like (generalized) excited diagrams of shape λ/μ, since the former are connected by certain moves called (K-theoretic) *droop moves* [30, 65]. It would be interesting to further explore this connection.

7.8. There is a large literature on enumeration of increasing tableaux in many special cases based on a trick of adding M_λ implicitly used in (1.3). Notably, for a rectangular shape, the tableaux in $\mathrm{SIT}(a^b)$ are in bijection with certain plane partitions of the same shape, see, e.g., [10, §4] and [20]. This approach fails to give a bijection for general skew shapes λ/μ, except when $\mu = \delta_k$ is a staircase. The latter are characterized by all minimal elements in $M_{\lambda/\mu}$ having the same entries.

7.9. While all our proofs are algebraic, some of our results seem well-positioned to have a direct combinatorial proof. We are especially curious if (K-HLF) has such a proof. Similarly, it would be interesting to use Konvalinka's recursive approach [29] to find a combinatorial proof of our Theorem 1.4.

7.10. The complexity of counting standard increasing tableaux is yet to be understood. In [63, §1.3], the authors give examples of large primes appearing as values, and suggest that the exact formula might not exist. They ask if there are "efficient (possibly randomized or approximate) counting algorithms" for $g^\lambda = |\,\mathrm{SIT}(\lambda)|$ and its refinements.

We conjecture that computing g^λ is $\#\mathsf{P}$-complete. This would partly explain why our hook formulas involve nontrivial β-weights. For the related notion of set-valued tableaux, see a discussion in [40] and the $\#\mathsf{P}$-completeness conjecture in [21, §5.7].

7.11. The LHS of (K-HLF) is equal to the LHS of equation (K-OOF) given in Theorem 6.8. It then follows from the proof of Theorem 6.8 that both can be computed efficiently for a given skew shape λ/μ and $\beta \in \mathbb{Q}$. It would be interesting to see if these have a determinant formula generalizing the *Aitken–Feit determinant formula* for $f^{\lambda/\mu}$ (see, e.g., [59, Corollary 7.16.3] and [51]).

Note that the *Lascoux–Pragacz identity* gives yet another determinant formula for $f^{\lambda/\mu}$, which we used in [42] to give a combinatorial proof of (NHLF). Finally, let us mention that $\mathcal{E}(\lambda/\mu)$ has a determinant formula (see §7.5 above), while Proposition 5.5 is not an equality but gives only a determinant upper bound for $\mathcal{D}(\lambda/\mu)$.

7.12. Following the approach of Stanley [60], we conjecture that for all $\beta \geqslant 0$, there is a limit

$$\lim_{n\to\infty} \frac{\log_2 u(\beta, n)}{n^2}, \qquad \text{where} \qquad u(\beta, n) := \max_{w \in S_n} \Gamma_w(\beta).$$

Using the *Cauchy identity* for Grothendieck polynomials [13, Corollary 5.4], we obtain the following bounds:

$$\frac{1}{4}\log_2(2+\beta) \leqslant \liminf_{n\to\infty} \frac{\log_2 u(\beta, n)}{n^2} \leqslant \limsup_{n\to\infty} \frac{\log_2 u(\beta, n)}{n^2} \leqslant \frac{1}{2}\log_2(2+\beta).$$

In [44], we computed the limit above for $\beta = 0$, when the maximum is restricted to *layered* (231- and 312-avoiding) *permutations*. It would be interesting to see if our analysis can be extended to the case of general $\beta > 0$.

7.13. Dividing both sides of (K-HLF) by $(-1)^n$ and taking $\beta > 0$ gives positive weights in the summation on the LHS over the SITs. Can one efficiently sample from this distribution? Perhaps, there is a deformation of the *NPS algorithm* or the *GNW hook walk*, see [19, 48]? A positive answer to either of these would be remarkable.

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Department of Mathematics
and Statistics,
University of Massachusetts,
Amherst, MA 01003, USA
E-mail: ahmorales@math.umass.edu

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Department of Mathematics,
University of California,
Los Angeles, CA 90095, USA
E-mail: pak@math.ucla.edu

Department of Mathematics,
University of Southern California,
Los Angeles, CA 90089, USA
E-mail: gpanova@usc.edu