Записки научных
семинаров ПОМИ
Том 507, 2021 г.

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A NOTE ON A LOCAL COMBINATORIAL FORMULA FOR THE EULER CLASS OF A PL SPHERICAL FIBER BUNDLE

Abstract. We present a local combinatorial formula for the Euler class of an n-dimensional PL spherical fiber bundle as a rational number e_{CH} associated to a chain of $n+1$ abstract subdivisions of abstract n-spherical PL cell complexes. The number e_{CH} is a combinatorial (or matrix) Hodge-theoretic twisting cochain in Guy Hirsch's homology model of the bundle associated with the PL combinatorics of the bundle.

§1. Introduction

Local combinatorial formulas for characteristic classes of *combinatorial manifolds* is a rich, fruitful, intriguing subject which is far from being finished, see [16] for the history and an overview. We switch the attention to the conceptually (but not immediately) related problem of rational local combinatorial formulas for characteristic classes of combinatorial *spherical fiber bundles* in the case of the Euler class. The Euler class of a closed oriented combinatorial manifold is its Euler characteristic. In this situation, a local combinatorial formula is well known. The formula is the "Levitt curvature" [24], a discrete analog (see [12]) of the Gauss–Bonnet–Chern curvature. A combinatorial manifold has nothing like the spherical bundle associated to the tangent bundle, therefore, bundle local formulas and manifold local formulas are not immediately related. For the case of spherical bundles, only circle bundles (bundles with fiber S^1) are investigated. We have simple local combinatorial formulas [21, Secs. 1, 2], [28] in this situation based on Kontsevich's cyclic connection form on metric polygons. A brief sketch of another local combinatorial formula for circle bundles is presented around Proposition 2 in [17]. The sketch constructs the formula as a local obstruction to constructing a chain-level section representing the transgression differential in the Serre spectral sequence of the bundle.

Key words and phrases: combinatorics of bundles, Euler class.
Research is supported by the Russian Science Foundation grant 19-71-30002.

The Gelfand–MacPherson sketch for circle bundles has a natural generalization for arbitrary S^n-bundles, and our work emerged from an attempt to organize and develop this idea. As a result, we obtained a canonical construction with a canonically looking formula (33) representing the local combinatorial twisting cochain for the Hirsch homology model of the bundle. Formula (33) and the Levitt curvature should have some mutual relation in the case of combinatorial manifolds. But it is a challenge to find this relation.

1. The problem of finding a local combinatorial formula for the Euler class of a spherical fiber bundle in the PL category. The PL category is defined using triangulations. A spherical PL fiber bundle $S^n \to E \xrightarrow{p} B$ has a triangulation given by a map of simplicial complexes $\boldsymbol{E} \xrightarrow{\boldsymbol{p}} \boldsymbol{B}$. We call the stalk of the triangulation over a base simplex $\sigma^k \in \boldsymbol{B}(k)$ an elementary simplicial spherical bundle. It triangulates the trivial bundle $S^n \times \Delta^k \xrightarrow{\pi} \Delta^k$ in such a way that any simplex in the total space is mapped onto a face of the base simplex (see Fig. 1). The triangulation $\boldsymbol{E} \xrightarrow{\boldsymbol{p}} \boldsymbol{B}$ is

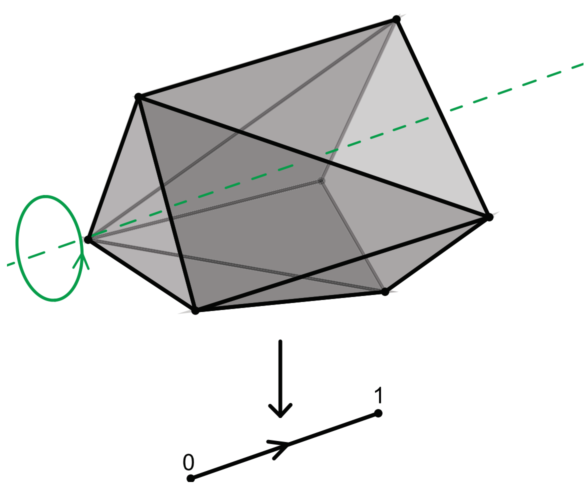

Fig. 1. An elementary simplicial circle bundle.

assembled from elementary simplicial spherical bundles using boundary maps which are combinatorial automorphisms of the elementary bundles. Suppose that the base simplicial complex is locally ordered. Then it has a complex of ordered cochains, which computes the singular cohomology of the base. We wish to find a universal rational function of the combinatorial isomorphism class of elementary triangulated oriented S^n-bundles over an $(n+1)$-simplex such that this value being assigned to the base $(n+1)$-simplex is a rational cocycle representing the Euler class of the bundle. Thus, it is supposed to be independent of the boundary combinatorial automorphisms composing the bundle from elementary bundles. Such a formula will be called a *simplicial local combinatorial formula for the Euler class.*

Since the Euler class is an integer characteristic class, a rational formula should have integer periods, now in the combinatorial setting. That is, its evaluation at integer $(n+1)$-simplicial cycles in the base are integers depending on the isomorphism class of the bundle and the homology class of the cycle, and independent of the triangulation. In particular, if we triangulate a differentiable S^n-bundle over a differentiable closed oriented $(n+1)$-base, we should obtain the same Euler number of the bundle from combinatorial and from differential considerations. Therefore, the arithmetics of the formula is highly nontrivial.

2. There is an alternative to triangulations and a somewhat dual combinatorics of PL fiber bundles investigated in [27].

By a *local system* $\mathcal{G}$ on $\boldsymbol{B}$ with values in some category $\boldsymbol{G}$ we mean a map associating to any vertex v of $\boldsymbol{B}$ an object $G(v)$ of $\boldsymbol{G}$, and to any oriented edge (v_0, v_1) a $\boldsymbol{G}$-morphism $G(v_0) \xrightarrow{G(v_0,v_1)} G(v_1)$, in such a way that any 2-simplex (v_0, v_1, v_2) goes to the composition $G(v_0, v_2) = G(v_1, v_2)G(v_0, v_1)$. Alternatively, we may say that we have a simplicial map $\boldsymbol{B} \xrightarrow{\mathcal{G}} \mathscr{N}\boldsymbol{G}$, where $\mathscr{N}$ denotes the nerve of the category.

One may encode a spherical PL fiber bundle p by a local system on the base simplicial complex $\boldsymbol{B}$ with values in the category $\boldsymbol{S}^n$ of abstract regular spherical PL cell complexes and corresponding aggregations (this is explained in Sec. 5). Any triangulation has such a local system canonically associated with it. For this combinatorics, a local formula for the Euler class of a PL S^n-bundle is a universal Euler $(n+1)$-cocycle, which is a function of a chain of $n+1$ subdivisions (or aggregations) of n-spherical cell complexes, e.g., a chain of subdivisions of convex polytopes. This function measures certain combinatorial asymmetry in the chain. We call it the *aggregation local combinatorial formula for the Euler class*, and it is defined in §5, Subsec. 13.

3. This note is devoted to a simple observation: the combinatorial model of a spherical PL fiber bundle as a local system of abstract aggregations (or subdivisions) of PL spherical cell complexes on a triangulated base ([27]) smoothly and naturally fits into the very classical theory of Hirsch homology models of fibrations, since it produces exactly the "bigraded model of a fibration" and its Serre spectral sequence is the bicomplex spectral sequence. Thus, we immediately produce a local combinatorial formula for the Euler class as a "twisting cochain." In fact, we substitute the combinatorics of cellular local systems from [27] into the deformation theory of local systems of spherical chain complexes as expressed in [22, Corollary 2.5] with an explicit formula [22, formula (3)] for the Euler cocycle. The simplicial local combinatorial formula in this setting is derived from the formula for local systems. The resulting aggregation local formula (Theorem 8) is composed from combinatorial Hodge-theoretic retractions of chain cellular spheres in homology. This is expected to be an interesting subject of combinatorics, statistics, thermodynamics, etc. of "higher Kirchhoff theorems" [7–9, 14, 26].

Unless stated otherwise, we assume that our coefficients are in some characteristic 0 field A, since our goal is a rational formula.

§2. The Euler class of an oriented spherical fiber bundle

4. The Gysin homomorphism and trangression differential. An oriented spherical fiber bundle

$$S^n \to E \xrightarrow{p} B \tag{1}$$

has an integer Euler characteristic class $\mathscr{E}(p) \in H^{n+1}(B;\mathbb{Z})$. In [10, 32, 33], the Euler class of a fiber bundle (1) was identified via the Gysin homomorphism $G = \frown\mathscr{E}(p)$ of the homological Gysin exact sequence of the bundle

$$\begin{aligned}\cdots \xrightarrow{p} H_{n+k+1}(B;\mathbb{Z}) \xrightarrow{\frown\mathscr{E}(p)} H_k(B;\mathbb{Z}) \xrightarrow{j}\\ H_{n+k}(E;\mathbb{Z}) \xrightarrow{p} H_{n+k}(B;\mathbb{Z}) \xrightarrow{\frown\mathscr{E}(p)} \cdots\\ \cdots \xrightarrow{\frown\mathscr{E}(p)} H_1(B;\mathbb{Z}) \xrightarrow{j} H_{n+1}(E;\mathbb{Z})\\ \xrightarrow{p} H_{n+1}(B;\mathbb{Z}) \xrightarrow{\frown\mathscr{E}(p)} H_0(B;\mathbb{Z}) \to 0.\end{aligned} \tag{2}$$

The Gysin homomorphism is the differential on the $(n+1)$th page of the Leray–Serre spectral sequence of the bundle

$$H_{n+k+1}(B; \underset{\approx\mathbb{Z}}{H_0(S^n;\mathbb{Z})}) \approx E^{n+1}_{n+k+1,0}(p) \xrightarrow{d_{n+1}=\frown\mathscr{E}(p)} E^{n+1}_{k,n}(p) \approx H_k(B, \underset{\approx\mathbb{Z}}{H_n(S^n;\mathbb{Z})}).$$

And thus the first one, the transgression differential

$$H_{n+1}(B; \underset{\approx\mathbb{Z}}{H_0(S^n;\mathbb{Z})}) \approx E^{n+1}_{n+1,0}(p) \xrightarrow{\frown\mathscr{E}(p)} E^{n+1}_{0,n}(p) \approx H_0(B, \underset{\approx\mathbb{Z}}{H_n(S^n;\mathbb{Z})}), \quad (3)$$

can be regarded as the Euler class itself if the base is connected.

§3. Guy Hirsch's model of a fibration

Guy Hirsch in [20] introduced "Hirsch homology models of fibrations." For certain fibrations $F \to E \xrightarrow{g} B$, he detected a subcomplex of the chain complex $C_\bullet(E)$ of the form $C_\bullet(B) \otimes H_\bullet(F)$ that has the same homology as E. In [6], E. Brown defined the "twisted tensor product" and recognized Hirsch's model in the form of the twisted Eilenberg–Zilber theorem, $C_\bullet(E) \approx C_\bullet(B) \otimes_{\tau(g)} H_\bullet(F)$, where $\tau(g)$ is a "twisting cochain."

Twisting cochains for Hirsch models allow one to obtain a chain-level understanding of the differentials in the Serre spectral sequence of a Serre fibration. This was investigated in detail by the Georgian school [3, 4, 23]. The modern setup for Brown's twisting cochain has the form of A_∞ local systems [22, Secs. 1, 2].

There are two steps in the construction of the Hirsch model of a fibration $F \to E \xrightarrow{g} B$: an algebraic step and a topological step.

In the case of a base polyhedron B ([3, 22, 23]), the algebraic step investigates a local system $\mathcal{L}$ of chain complexes with fixed homology $H_\bullet(F) = \sum_{k=0}^{n} H_k(F)$ on a locally ordered triangulation of the base $\boldsymbol{B}$, $|\boldsymbol{B}| = B$. Let $\operatorname{Tot}\mathcal{L}$ be the naturally filtered total complex of the bicomplex $C_\bullet(\boldsymbol{B}; \mathcal{L}_\bullet)$ of $\mathcal{L}$. The Hirsch–Brown model of $\mathcal{L}$ is a deformation of the differential in the complex $C_\bullet(\boldsymbol{B}) \otimes H_\bullet(F) \rightsquigarrow C_\bullet(\boldsymbol{B}) \otimes_{\tau(\mathcal{L})} H_\bullet(F)$ using a *twisting cochain* $\tau(\mathcal{L})$ such that $\operatorname{Tot}\mathcal{L}$ is filtered homotopy equivalent to $C_\bullet(\boldsymbol{B}) \otimes_{\tau(\mathcal{L})} H_\bullet(F)$ and produces an equivalent spectral sequence with differentials readable at the chain level.

At the next topological step, we try to replace the fibration g over the polyhedron $|\boldsymbol{B}|$ by a local system of *spaces* $\mathcal{W}$ over $\boldsymbol{B}$ in such a way that certain chain complexes $C_\bullet(\mathcal{W}_v)$ of the local system give rise to the “bigraded model of the fibration,” a bicomplex with total complex equivalent to the total singular complex of the fibration ([4,13,15]). For this bigraded model, the algebraic step becomes applicable and we obtain $C_\bullet(\boldsymbol{B}) \otimes_{\tau(\mathcal{W}(g))} H_\bullet(F)$ as the Hirsch model of g, and thus we can see chain-level formulas for differentials in the Serre spectral sequence of g up to irrationalities in the construction of the bigraded model.

§4. Local systems of spherical chain complexes on a simplicial base, the twisting cochain, and a formal Euler cocycle

We present, in a form suitable for us, the classical construction ([3, 22, 23]) of the twisting cochain for a local system of spherical chain complexes on a simplicial base. Our coefficiens are in a characteristic 0 field A.

5. Local systems of spherical chain complexes. Let $\mathrm{Ch}(S^n)$ be the category of oriented spherical chain complexes. Objects are length $n+1$ chain complexes of finite-dimensional vector spaces over A, where we denote the differentials by γ:

$$K_\bullet = (0 \to K_n \xrightarrow{\gamma_n} K_{n-1} \xrightarrow{\gamma_{n-1}} \cdots \xrightarrow{\gamma_1} K_0 \to 0).$$

We suppose that $H_i(K) = A$ if $i = 0, n$ and $H_i(K) = 0$ otherwise. Also, we suppose that K has a fixed augmentation denoted by $K_0 \xrightarrow{p_0} A$ and a fixed “orientation” i_n making the sequence exact:

$$0 \to A \xrightarrow{i_n} K_n \xrightarrow{\gamma_n} K_{n-1}.$$

The orientation fixes the “fundamental class” $i(1) \in K_n$ regarded as a generator in $H_n(K)$. Morphisms in $\mathrm{Ch}(S^n)$ are degree 0 chain maps commuting with the augmentation and orientation. We have a special homology spherical complex with zero differentials

$$H_\bullet(S^n) = (0 \to \underset{0}{A} \to 0 \to \cdots \to 0 \to \underset{n}{A} \to 0). \tag{4}$$

Let $\boldsymbol{B}$ be a locally ordered simplicial complex equipped with a $\mathrm{Ch}(S^n)^{\mathrm{op}}$-valued local system $\mathcal{L}$ on $\boldsymbol{B}$ defined by a simplicial map $\boldsymbol{B} \xrightarrow{\mathcal{L}} \mathscr{N}\,\mathrm{Ch}(\mathrm{S^n};\mathrm{A})^{\mathrm{op}}$.

Associating to a simplex $\sigma_p = (v_0, \dots, v_p) \in \boldsymbol{B}(p)$ the complex $L(v_p)$ sitting over its *last vertex*, and to the ith face inclusion $d_i\sigma_p \to \sigma_p$ the identity map of complexes if $i = 0, \dots, p-1$ and the map $L(v_{p-1}, v_p)$ if $i = p$, we obtain from the simplicial local system a simplicial constructible sheaf on $\boldsymbol{B}$.

6. The Leray–Gysin spectral sequence of a spherical local system. We can consider the complex $C_\bullet(\boldsymbol{B}; \mathcal{L}_\bullet)$ of simplicial chains on $\boldsymbol{B}$ with coefficients in $\mathcal{L}$. It is a bigraded module with two anticommuting differentials. The module $C_p(\boldsymbol{B}, \mathcal{L}_q)$ is formed by (p, q) chains $\sigma_p c_q$ assigning to a simplex $\sigma_p = (v_0, \dots, v_p) \in \boldsymbol{B}(p)$ an element $c_q \in L_q(v_p)$, i.e., a q-element of the complex over the last vertex v_p. We have two anticommuting differentials: the simplicial horizontal differential

$$\begin{gathered} C_p(\boldsymbol{B}, \mathcal{L}_q) \xrightarrow{\partial} C_{p-1}(\boldsymbol{B}, \mathcal{L}_q), \\ \partial(\sigma_p c_q) = \sum_{i=0}^{p-1} (-1)^i (d_i\sigma_p) c_q + (-1)^p (d_p\sigma_p) L(v_{p-1}, v_p)(c_q), \end{gathered} \tag{5}$$

where a nontrivial transition map appears only in the last summand, and the vertical differential

$$\begin{gathered} C_p(\boldsymbol{B}, \mathcal{L}_q) \xrightarrow{\tilde{\gamma}} C_p(\boldsymbol{B}, \mathcal{L}_{q-1}), \\ \tilde{\gamma}(\sigma_p c_q) = (-1)^p \sigma_p \gamma_{\sigma_p}(c_q), \end{gathered} \tag{6}$$

induced by the differential in $\mathrm{Ch}(S^n)$. Thus, we obtain the total complex $\mathrm{Tot}(\boldsymbol{B}, \mathcal{L})$ with total differential

$$\mathrm{Tot} = \partial + \tilde{\gamma} \tag{7}$$

and horizontal filtration

$$F_p(\mathrm{Tot}\, C_\bullet(\boldsymbol{B}; \mathcal{L}_\bullet))_d = \bigoplus_{\substack{k_1 + k_2 = d \\ k_1 \leqslant p}} C_{k_1}(\boldsymbol{B}; \mathcal{L}_{k_2}).$$

Its first quadrant spectral sequence $E^{\bullet}_{\bullet,\bullet}$ starting from page zero converges to the chain homology of the Tot-complex. On page 0 we have

$$\begin{array}{c} \mathsf{E}^0_{p,q} = \mathsf{C}_p(\mathbf{B};\mathcal{L}_q) \\ \Big\downarrow {\scriptstyle d_0=\tilde{\gamma}} \\ \mathsf{E}^0_{p,q} = \mathsf{C}_p(\mathbf{B};\mathcal{L}_{q-1}) \end{array} .$$

On $E^1_{\bullet\bullet}$, the 1-differential is the horizontal differential ∂:

$$C_{p-1}(\boldsymbol{B}; H_q(S^n)) = E^1_{p-1,q} \xleftarrow{d_1=\partial} E^1_{p,q} = C_p(\boldsymbol{B}; H_q(S^n)).$$

That is, $E^1_{p,q} \approx C_p(\boldsymbol{B}; H_q(S^n))$. Thus, $E^2_{p,q} = H_p(\boldsymbol{B}; H_q(S^n))$. On page $n+1$, we get the Gysin–Leray transgression differential (3) as the *formal* Gysin homomorphism. “Formal,” because a priori the local system $\mathcal{L}$ is not related to any spherical Serre fibration on $|\boldsymbol{B}|$.

7. Brown’s twisting cochain as a formal Euler cocycle. We need some objects and notions. The complex $C_\bullet(\boldsymbol{B})$ has an Alexander–Whitney coalgebra structure. Let $H = H_\bullet = \sum_i H_i$ be some graded module. Then $\mathrm{Hom}(H, H)$ has an algebra structure with respect to composition, and we can consider the DGA $C^\bullet(\boldsymbol{B}; \mathrm{Hom}(H, H))$ with the Alexander–Whitney product. A *twisting cochain* τ is a cochain

$$\tau = \tau^1 + \tau^2 + \dots, \tau^i \in C^i(\boldsymbol{B}; \mathrm{Hom}^{i-1}(H, H)) \tag{8}$$

satisfying the condition

$$\delta\tau = -\tau \smile \tau. \tag{9}$$

There is a pairing $H \otimes \mathrm{Hom}(H, H) \to H$ sending $e_q \in H_q$ and $f_i \in \mathrm{Hom}_i(H, H)$ to an element $e_q f_i = f_i(e_q) \in H_{q+i}$. We have the cap product $C_k(\boldsymbol{B}; H) \otimes C^p(\boldsymbol{B}; \mathrm{Hom}(H, H) \xrightarrow{\frown} C_{p-k}(\boldsymbol{B}, H)$. Having a twisting cochain τ, we can deform the differential as

$$\partial \rightsquigarrow \partial + \frown \tau \tag{10}$$

on $C_\bullet(\boldsymbol{B}) \otimes H$, thus obtaining a new filtered DG module $C_\bullet(\boldsymbol{B}) \otimes_\tau H$. The new differential respects the horizontal filtration on $C_\bullet(\boldsymbol{B}) \otimes H$, therefore, it just adds some nontrivial differentials to the trivial bicomplex spectral sequence in the simplest possible way.

Now let $H = H(S^n)$. Assume that the twisting cochain has $\tau^1 = 0$. In this situation, a twisting cochain τ on $C_\bullet(\boldsymbol{B}) \otimes H(S^n)$ is just a *formal Euler cocycle*. Indeed, from dimension considerations Eq. (9) becomes the cocycle condition $\delta\tau = 0$, thus τ has a single nonzero element $\tau^{n+1} \in C^{n+1}(\boldsymbol{B}; \mathrm{Hom}^n(H(S^n), H(S^n)))$. The spectral sequence of the horizontal filtration becomes the Gysin–Leray sequence with transgression differential $\frown \tau$ on page $n+1$. So, $\tau(\sigma_{n+1})(1)$ is nothing but an A-valued "Euler" simplicial $(n+1)$-cocycle on the base.

8. Strong deformational retractions, the basic homology perturbation lemma. Recall the notions of strong deformational retraction and homology perturbation. Let C, K be chain complexes of modules over a commutative ring with unit. A strong deformation retraction (SDR) (see [19]) of C on K is the data $\langle F, i, p\rangle$ of a diagram of chain maps

$$\begin{tikzcd} C \arrow[loop left, "F"] \arrow[r, shift left, "p"] & K \arrow[l, shift left, "i"] \end{tikzcd}. \tag{11}$$

Here the retraction operator $C \xrightarrow{F} C[1]$ shifts the dimension by one, and the following conditions hold:

$$pi = \mathrm{Id}, \quad dF + Fd = \mathrm{Id} - ip, \tag{12}$$

$$Fi = 0, \quad pF = 0, \quad F^2 = 0. \tag{13}$$

The annihilation conditions (13) can be satisfied if (12) holds by perturbing F. In particular, i splits the exact sequence

$$0 \to \mathrm{Ker}\, p \to C \xrightarrow{p} K \to 0$$

with the projection to the kernel given by $dF + Fd$, and thus represents C as the direct sum of K and a contractible $\mathrm{Ker}\, p$.

Let $H(C)$ be the homology complex of C regarded as a complex with zero differentials. The complex C is said to be *homology split* if there exists a *homology splitting* (see, e.g., [18, §1]) that is an SDR of C on $H(C)$,

$$\begin{tikzcd} C \arrow[loop left, "F"] \arrow[r, shift left, "p"] & H(C) \arrow[l, shift left, "i"] \end{tikzcd},$$

representing C as the direct sum of its homology module and a trivial chain complex. If the complex and its homology are free, then a homology splitting exists. In particular, if C has trivial homology, then its homology splitting is a degree 1 contraction $C \xrightarrow{F} C[1]$ such that $F^2 = 0$ and $dF + Fd = \mathrm{Id}$.

If complexes C, K are filtered and the SDR data preserves filtrations, then the SDR is *filtered*.

A perturbation of an SDR data is a homomorphism $C \xrightarrow{\psi} C[-1]$ of degree -1 such that $(d_C + \psi)^2 = 0$, i.e., $d_C + \psi$ is also a differential. The fundamental tool for obtaining otherwise unavailable formulas is the following lemma.

Lemma 1 (basic perturbation lemma [19]). *Let $\langle F, i, p\rangle$ be a filtered SDR* (11) *and ψ be a perturbation of this SDR. Then*

$$\begin{tikzcd} C_{d_C+\psi} \arrow[loop left, "F_\psi"] \arrow[r, shift left, "p_\psi"] & K_{d_K+d_\psi} \arrow[l, shift left, "i_\psi"] \end{tikzcd}$$

is a filtered SDR, where

$$d_\psi = p\psi\Sigma^\psi i, \tag{14}$$

$$p_\psi = p(1 - \psi\Sigma_i^\psi F), \tag{15}$$

$$i_\psi = \Sigma^\psi i, \tag{16}$$

$$F_\psi = \Sigma^\psi F. \tag{17}$$

Here

$$\Sigma^\psi = \sum_{j\geqslant 0}(-1)^i(F\psi)^j = 1 - F\psi + F\psi F\psi - F\psi F\psi F\psi \cdots .$$

9. The Hirsch model of a $\mathrm{Ch}(S^n)$-local system with retractions on homology, its twisting cochain, and a formal Euler cocycle. Here we present a deformation of a $\mathrm{Ch}(S^n)^{\mathrm{op}}$-local system $\mathcal{L}$ onto the trivial local system $C_\bullet(\boldsymbol{B}; H(S^n))$ by a locally defined formal Euler cocycle. The deformation and the cocycle are determined by a certain free extra structure: retractions onto the homology of the complexes $L_v, v \in \boldsymbol{B}(0)$. Suppose that for any spherical complex L_v in the local system, we have fixed a strong deformational retraction on the homology $H(S^n)$. This will be the extra data $\mathcal{F}$:

$$\begin{tikzcd} L_v \arrow[loop left, "F_v"] \arrow[r, shift left, "p_v"] & H(S^n), v \in \mathbf{B}(0) \arrow[l, shift left, "i_v"] \end{tikzcd} . \tag{18}$$

Proposition 2 (Corollary 2.5 and formula (3) in [22]). *There is a filtering preserving the strong deformational retraction of* $\mathrm{Tot}(\boldsymbol{B};\mathcal{L})$ *onto* $C_\bullet(\boldsymbol{B}) \otimes_{\mathscr{E}(\mathcal{F})} H(S^n)$ *where*

$$\begin{aligned}\mathscr{E}(\mathcal{F})(v_0,\dots,v_{n+1}) = p_n L_{0,1} F_1 L_{1,2},\dots,F_n L_{n,n+1} i_0 \\ \in Z^{n+1}(\boldsymbol{B};\mathrm{Hom}^n(H(S^n),H(S^n)))\end{aligned} \tag{19}$$

is the Euler cocycle for the local system of spherical chain complexes $\mathcal{L}$ *endowed with the retractions on homology* $\mathcal{F}$.

Proof. 1. This is a typical application of the basic homology perturbation lemma 1. Let us forget about the horizontal differential ∂ in $\mathrm{Tot}(\boldsymbol{B};\mathcal{L})$ and obtain the complex $\mathrm{Tot}^{\tilde\gamma}(\boldsymbol{B};\mathcal{L})$. The family of retractions $\mathcal{F}$ (18) provides a filtered SDR of $\mathrm{Tot}^{\tilde\gamma}(\boldsymbol{B},\mathcal{L})$ on the filtered module $C^0_\bullet(\boldsymbol{B}) \otimes H(S^n)$ with zero differentials. Now we perturb the differential $\tilde\gamma \rightsquigarrow \tilde\gamma + \partial$ restoring the initial differential in $\mathrm{Tot}(\boldsymbol{B};\mathcal{L})$ and see what happens by the lemma identities. We compute the new differential $d_\partial = p\partial\Sigma^\partial i$ on the element $\sigma^k h \in C^0_k(\boldsymbol{B}) \otimes H(S^n)$. Directly applying the annihilation conditions (13), we get the expression

$$d_\partial(\sigma^k h) = \partial\sigma^k h + \sigma^k \frown \mathscr{E}(\mathcal{F})(v_0,\dots,v_{n+1})(h).$$

2. Now we check that $\mathscr{E}(\mathcal{F})(v_0,\dots,v_{n+1})$ is a cocycle. Consider the bigraded module of simplicial cochains $C^\bullet(\boldsymbol{B};\mathcal{L}_\bullet)$ with values in the local system. Put $c^p_q(\sigma_p) \in \mathcal{L}_q(v_0\sigma_p)$; we assign an element over the *first* vertex. It has the anticommuting horizontal codifferential

$$\delta(x(\sigma_p)) = L(v_0,v_1)(x(d_0\sigma_p)) + \sum_i^p (-1)^i x(d_i\sigma_p) \in L(v_0(\sigma_p)) \tag{20}$$

and the vertical differential $\tilde\gamma$. We introduce a new local system $\mathrm{Hom}_\bullet(H;L_v)$ on $\boldsymbol{B}$. Then we have a system of cochains

$$\begin{array}{rl} i \in & C^0(\boldsymbol{B};\mathrm{Hom}_0(H;L_v)), \\ \delta i \in & C^1(\boldsymbol{B};\mathrm{Hom}_0(H;L_v)), \\ F\delta i \in & C^1(\boldsymbol{B};\mathrm{Hom}_1(H;L_v)), \\ \delta F\delta i \in & C^2(\boldsymbol{B};\mathrm{Hom}_2(H;L_v)), \\ \dots & \dots \\ (F\delta)^n i \in & C^n(\boldsymbol{B};\mathrm{Hom}_n(H;L_v)), \\ \delta(F\delta)^n i \in & C^{n+1}(\boldsymbol{B};\mathrm{Hom}_n(H;L_v)). \end{array} \tag{21}$$

The differentials $\tilde{\gamma}$ and δ anticommute. We can prove by induction on k that the cochain $\delta(F\delta)^k i$ is a cycle for $\tilde{\gamma}$. The inductive step is

$$\tilde{\gamma}\delta(F\delta)^k i = -\delta\tilde{\gamma}F\underbrace{\delta(F\delta)^{k-1}i}_{\tilde{\gamma}\text{-cycle}} = -\delta\delta(F\delta)^{k-1} = 0,$$

since by (12) if $\tilde{\gamma}x = 0$ then $\tilde{\gamma}Fx = x$. Hence, $\delta(F\delta)^n i$ is a $\tilde{\gamma}$-cycle, and, therefore, it is proportional to the fundamental class $i_n(1)$. Simultaneously, it is a δ-cocycle being a δ-coboundary. Therefore, $p_n\delta(F\delta)^n i_0 \in Z^{n+1}(\boldsymbol{B}; \mathrm{Hom}(H;H))$ is a δ-cocycle.

Now we compute (21) using (20):

$$\delta i(v_0, v_1) = L(v_0, v_1)i(v_1) - i(v_0) \in L_{v_0}(0),$$

$$F\delta i = F_{v_0}L(v_0, v_1)i(v_1) - \underbrace{F_{v_0}i(v_0)}_{=0 \text{ since } Fi=0},$$

$$\begin{aligned}&\delta F\delta i(v_0, v_1, v_2)\\ &\qquad = L(v_0, v_1)F(v_1)L(v_1, v_2)i(v_2) - F(0)(L(0,1)i(1) + L(0,2)i(2)),\end{aligned}$$

$$(F\delta)^k i(0, 1, \dots, k) = F(0)L(0,1)F(1)\dots L(k, k-1)i(k) + \underbrace{F^2(0)(...)}_{=0},$$

$$\begin{aligned}&\delta(F\delta)^k i(0, \dots, k+1)\\ &= L(0,1)F(1)L(1,2)F(2)\dots L(k+1, k)i(k+1) + F(0)\sum_{i=1}^{k+1}(-1)^i(\dots).\end{aligned}$$

Finally, since $p(0)F(0) = 0$, we get

$$\begin{aligned}&p(F\delta)^n i(0, \dots, n+1)\\ &\qquad = p(0)L(0,1)F(1)L(1,2)\dots F(n)L(n, n+1)i(n+1)\\ &\qquad\qquad = \mathscr{E}(\mathcal{F})(v_0, \dots, v_{n+1}),\end{aligned}$$

and, therefore, $\mathscr{E}(\mathcal{F})(v_0, \dots, v_{n+1})$ is a cocycle and an Euler twisting cochain for the Hirsch model of the spherical local system $\mathcal{L}$. $\square$

§5. The PL combinatorics of a spherical fiber bundle

10. Geometric and abstract regular spherical PL cell complexes. Let S^n be the n-dimensional sphere in the PL category. A *regular geometric cell complex* structure B on S^n is a covering of S^n by a collection of closed embedded PL balls B such that the interiors of the balls form a partition of S^n and the boundary of a ball is a union of balls. The face complex of a convex polytope is the most obvious example. The other names for these objects in the literature are "ball complexes" ([31, Appendix to Chap. 2, formula (5)]) or "regular CW complexes" ([25, Chap. III, Secs. 1, 2]). Partially ordered by inclusion, the set $P(B)$ of balls in B defines B up to a PL homeomorphism (see [5]). Thus, we can define an *abstract regular spherical PL cell complex* as a finite poset P for which the simplicial order complex ΔP is PL homeomorphic to S^n and all the principal lower ideals are PL homeomorphic to balls.

Unless specified otherwise, all our cell complexes are regular and PL.

If we have a poset P, then P^{op} is the poset with the order reversed. It is a special feature of the PL category that if a poset P is an abstract spherical cell complex, then P^{op} is also an abstract spherical cell complex; we say that it is *dual* to P. This follows from the "PL invariance of a star" theorem. For simplicial triangulations of manifolds, it is often called the *Poincaré dual complex*, which appears in combinatorial proofs of the Poincaré duality.

11. Geometric and abstract subdivisions, cellular local systems. We follow [27]. A geometric spherical cell complex B_0 is a *subdivision* of B_1 (or B_1 is an aggregation of B_0) if the relative interior of any ball from B_0 is contained in the relative interior of a ball from B_1. We denote this by $B_0 \trianglelefteq B_1$. A geometric aggregation creates a poset map $P(B_0) \to P(B_1)$. A poset map of abstract spherical cell complexes is called an *abstract aggregation* if up to a PL homeomorphism it can be represented by a geometric aggregation. With the direction of arrows reversed, we call such a morphism an *abstract subdivision*. Thus, we get the category $\boldsymbol{S}^n$ of abstract regular n-spherical PL cell complexes and abstract aggregations. We suppose that abstract spherical cell complexes are *oriented*, i.e., for any complex S a fundamental class $[S]$ is chosen and an aggregation map aggregates the fundamental class to the fundamental class.

An *abstract spherical cellular local system* $\mathcal{S}$ on $\boldsymbol{B}$ is a local system with values in $\boldsymbol{S}^n$. With $\mathcal{S}$ we can associate a PL cellular spherical fiber bundle $\mathscr{T}\mathrm{ot}\,\mathcal{S} \to \boldsymbol{B}$ using iterated cellular cylinders of subdivision maps (this cellular bundle is called "prismatic" in [27]). This goes as follows. We can realize any chain of abstract aggregations over a simplex as a chain of geometric aggregations of the geometric realization of the first complex in the chain. Then we can construct the geometric cellular cylinders of the corresponding subdivisions over the simplex (see Fig. 2). These geometric prismatic trivial bundles constructed separately over each base simplex can be glued together fiberwise using for PL transition maps the parametric Alexander trick. The result is a PL spherical cellular fiber bundle $\mathscr{T}\mathrm{ot}\,\mathcal{S}$.

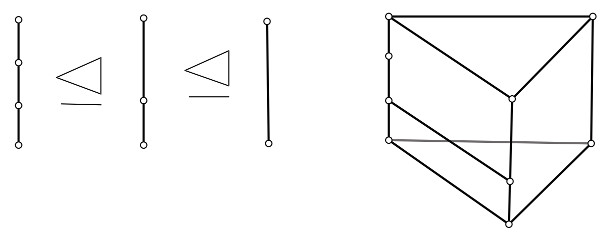

Fig. 2

12. Bundle triangulations versus cellular local systems. The PL category is defined using triangulations. A spherical PL fiber bundle $S^n \to E \xrightarrow{p} B$ has a triangulation by a map of simplicial complexes $\boldsymbol{E} \xrightarrow{\boldsymbol{p}} \boldsymbol{B}$. We call the stalk of the triangulation over a base simplex $\sigma^k \in \boldsymbol{B}(k)$ an elementary simplicial spherical bundle. It triangulates the trivial bundle $S^n \times \Delta^k \xrightarrow{\pi} \Delta^k$ in such a way that any simplex in the total space is mapped onto a face of the base simplex. A triangulation $\boldsymbol{E} \xrightarrow{\boldsymbol{p}} \boldsymbol{B}$ is assembled from elementary simplicial spherical bundles using boundary maps which are combinatorial automorphisms of elementary bundles. Let us fix the following statement.

Proposition 3. *A triangulation of a spherical PL fiber bundle* $\boldsymbol{E} \xrightarrow{\boldsymbol{p}} \boldsymbol{B}$ *has a canonically associated spherical cellular local system* $\mathcal{S}(\boldsymbol{p})$ *on the* *first derived subdivision* $\operatorname{Sd}\boldsymbol{B}$ *of* $\boldsymbol{B}$ *such that the bundle* $\mathscr{T}\mathrm{ot}\,\mathcal{S}(\boldsymbol{p})$ *is PL isomorphic to* $\boldsymbol{p}$.

Proof. This specifically PL topology statement is based on M. Cohen's theory of transverse cellular maps and corresponding cylinders [11]. Consider an elementary triangulated S^n-bundle over a simplex $\boldsymbol{R} \xrightarrow{\boldsymbol{q}} \Delta^k$. The simplex Δ^k has ordered vertices $v_0, \ldots, v_k$, Take an interior point $x \in \mathrm{int}\Delta^k$. Take the simplicial 0-face $\Delta^{k-1} \xrightarrow{\delta_0} \Delta^k$ and a point $x_0 \in \mathrm{int}\Delta^{k-1}$ in the 0-face. The fiber $q^{-1}(x)$ has the structure of an abstract regular spherical PL cell complex $P(x)$ induced from the triangulation $\boldsymbol{R}$. It is a "multi-simplicial complex." Its balls are simplicial prisms (see [1]). The prisms are products of simplices. This comes from the fact that the general fiber of a simplicial projection of a simplex onto a simplex is a product of simplices which are the fibers of the projection over the vertices in the base. When we move the point x in the base to the point x_0 in the 0-face, all the factors of the prisms in the fiber coming from $p^{-1}(v_0)$ shrink to points. This creates multi-simplicial boundary degeneration maps which we regard

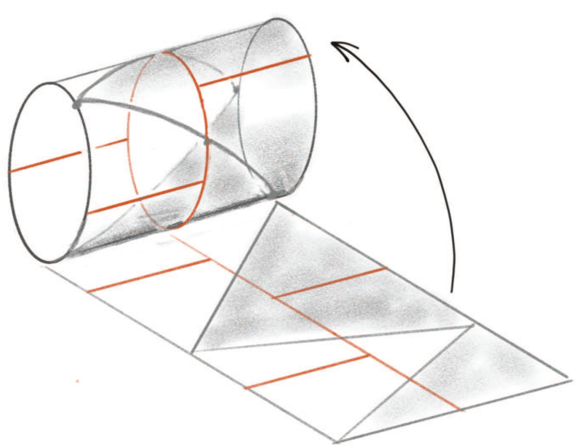

Fig. 3. An elementary triangulated circle bundle over the interval and the dual pattern of circle subdivisions.

as poset maps $P(x) \xrightarrow{\delta_0^*} P(x_0)$. So, we see over the first derived subdivision of Δ^k iterated cylinders of those maps. The key fact is that these boundary degeneration poset maps are exactly Cohen's *transverse cellular maps* ([11, Theorem 8.1]), the poset maps of abstract PL spherical cell complexes which are *dual to aggregation maps*. Thus, $P^{\mathrm{op}}(x) \xrightarrow{(\delta_0^*)^{\mathrm{op}}} P^{\mathrm{op}}(x_0)$ is an aggregation morphism. Therefore, over the first derived subdivision $\operatorname{Sd}\Delta^k$ of the base simplex Δ^k we canonically obtain a diagram $\mathcal{S}(\boldsymbol{q})$ of aggregations of abstract PL spherical cell complexes (see Fig. 3). Now we may mention that, applying the Kan derived subdivision functor $\operatorname{Sd}\boldsymbol{R} \xrightarrow{\operatorname{Sd}\boldsymbol{q}} \operatorname{Sd}\Delta^k$, we obtain a simplicial spherical bundle over $\operatorname{Sd}\Delta^k$ triangulating both the elementary bundle $\boldsymbol{q}$ and the cellular bundle $\mathscr{T}\mathrm{ot}\,\mathcal{S}(\boldsymbol{q})$. The construction commutes with assembling $\boldsymbol{p}$ from elementary bundles. $\square$

13. A local formula for the Euler class of a spherical bundle represented by an $\boldsymbol{S}^n$-local system. By Proposition 3, we can functorially replace a triangulated spherical bundle on $\boldsymbol{B}$ by an $\boldsymbol{S}^n$-local system on $\operatorname{Sd}\boldsymbol{B}$. In the language of the combinatorics of an $\boldsymbol{S}^n$-local system $\mathcal{S}$ on the base $\boldsymbol{B}$, a rational local formula for the Euler class is a rational number associated to the combinatorics of the stalk of the local system $\mathcal{S}$ over an $(n+1)$-simplex of the base and representing the simplicial Euler cocycle for the bundle $|\mathscr{T}\mathrm{ot}\,\mathcal{S}| \to |\boldsymbol{B}|$. This stalk is just a chain of $n+1$ aggregations of abstract cellular spheres, which can always be realized as a chain of geometric aggregations (subdivisions). The local system $\mathcal{S}$ is assembled from stalks over simplices using boundary combinatorial automorphisms of stalks. Therefore, a rational local formula for the Euler class in this setting is a rational function of a chain of aggregations of spherical cell complexes depending only on the combinatorics of the chain and invariant under automorphisms of boundary subchains. We call such a formula an *aggregation local combinatorial formula for the Euler class*. From an aggregation formula we can obtain a simplicial local formula (see Subsec. 1) by Proposition 3, integrating the aggregation simplicial cocycle over the derived subdivision of the base of the elementary simplicial bundle.

§6. A local system of abstract spherical cell complexes as a bigraded model of a spherical fiber bundle

Assume that we have an abstract $\boldsymbol{S}^n$-spherical cellular local system $\boldsymbol{B} \xrightarrow{\mathcal{S}} \mathscr{N}\boldsymbol{S}^n$ on a simplicial locally ordered complex $\boldsymbol{B}$. Let us associate to

$\mathcal{S}$ a local system of $\operatorname{Ch} S^n$-chain complexes

$$\boldsymbol{B} \xrightarrow{\mathcal{R}(\mathcal{S})} \mathscr{N}(\operatorname{Ch} S^n)^{\mathrm{op}}.$$

Pick arbitrary orientations of the cells of any complex S_v, $v \in \boldsymbol{B}(0)$, and form cellular chain complexes $R_v = C_\bullet(S_v)$. Now let $S_0 \xrightarrow{S(0,1)} S_1$ be an aggregation morphism. By definition, it is representable by an orientation-preserving homeomorphism $|S_0| \xrightarrow{f} |S_1|$. For any closed k-ball $B \in |S_1|$, $g^{-1}(B)$ is a union of closed k-balls from $|S_0|$. We associate to $S(0,1)$ the subdivision chain map

$$R(S_1) \xrightarrow{R(S(0,1))} R(S_0) \tag{22}$$

sending a k-cell from $R(S_1)$ to the sum of k-cells which it aggregates with the relative orientations. By the acyclic carriers argument, these maps are quasi-isomorphisms, and they obviously commute with compositions. The maps on 0-chains commute with augmentations. The fact that we are in the oriented situation means that fundamental classes $[S] \in Z_n(S)$ are fixed and the chain map $R(S(0,1))$ sends the fundamental class to the fundamental class.

The key (albeit trivial) statement of this paper is as follows.

Proposition 4. *The cellular chain complex $C_\bullet(\mathscr{T}\mathrm{ot}\,\mathcal{S})$ is isomorphic to $\operatorname{Tot}(\boldsymbol{B}, \mathcal{R}(\mathcal{S}))$.*

Proof. This is an immediate corollary of the construction of the "prismatic bundle" $\mathscr{T}\mathrm{ot}\,\mathcal{S}$ (Subsec. 11, [27]). The cellular differential of the prismatic bundle decomposes in a natural way into the sum of vertical and horizontal differentials. □

But the corollary is that we get an explicit "bigraded model of the fibration."

Corollary 5. *The algebraic bicomplex spectral sequence of the local system $\mathcal{R}(\mathcal{S})$ on $\boldsymbol{B}$ is the Leray–Serre spectral sequence of the PL spherical fiber bundle $|\mathscr{T}\mathrm{ot}\,\mathcal{S}| \to |\boldsymbol{B}|$.*

What follows by Proposition 2 is the following corollary.

Corollary 6. *If we endow the local system $\mathcal{R}(\mathcal{S})$ on $\boldsymbol{B}$ with a system $\mathcal{F}$ of SDRs on $H(S^n)$, then* (19) *is an expression for the simplicial Euler cocycle on the base of $\boldsymbol{B}$ of the fiber bundle $|\mathscr{T}\mathrm{ot}\,\mathcal{S}| \to |\boldsymbol{B}|$ represented by the local system $\mathcal{S}$ of aggregations of abstract spherical cell complexes.*

§7. The combinatorial Hodge-theoretic twisting cochain and a local combinatorial formula for the Euler class of a PL spherical fiber bundle

Now we may choose an SDR of $R(S_v)$ on homology and see what happens. There is a freedom of interesting choices, but the simplest one is combinatorial Hodge-theoretic (or Moore–Penrose) matrix retractions.

14. A rational matrix homology splitting of a cellular sphere. Now our coefficients are rational numbers. Let S be the n-dimensional PL cellular sphere. The cells of S have fixed orientations and are linearly ordered in every dimension. Let S have w ordered vertices and f ordered top cells.

We can form a based integer cellular chain complex $C_\bullet(S)$ with differentials γ_i represented by matrices with entries $0, 1, -1$. Set $R_\bullet = C_\bullet(S) \otimes \mathbb{Q}$. We have the special trivial complex

$$H(S^n; \mathbb{Q}) = (0 \to \mathbb{Q} \to 0 \to \ldots \to 0 \to \mathbb{Q} \to 0).$$

It has $\mathbb{Q}$ as the 0th and nth terms, 0 as all the other terms, and the zero differential. We denote by $\mathbb{Q}_0, \mathbb{Q}_n$ the two nontrivial modules of $H(S^n; \mathbb{Q})$. We suppose that $H(S^n; \mathbb{Q})$ has fixed bases as modules over $\mathbb{Q}$ identified as units in $\mathbb{Q}_0, \mathbb{Q}_n$. The complex $R_\bullet$ has a fixed augmentation

$$R_0 \xrightarrow{p_0} \mathbb{Q}: \quad p_0(\beta_1, \ldots, \beta_w) = \sum_{i=1}^{v} \beta_i, \quad p_0\gamma_1 = 0. \tag{23}$$

Fix the cellular fundamental class $[S] \in R_n$. Fixing the fundamental class allows us to define

$$\mathbb{Q} \xrightarrow{i_n} R_n: \quad i_n(\alpha) = \alpha[A], \quad \gamma_n i_n = 0. \tag{24}$$

Thus, we have identified $R_\bullet$ as an object of $\mathrm{Ch}(S^n; \mathbb{Q})$.

Our aim is to find a matrix representation for the SDR data of $R_\bullet$ onto $H(S^n; \mathbb{Q})$.

Unwinding conditions (11)–(13) for the special case of a retraction $R_\bullet$ onto its homology $H(S^n;\mathbb{Q})$, we get the diagram

$$
\begin{tikzcd}
0 \arrow[d, dotted, shift left] & 0 \arrow[d, shift left] \\
R_n \arrow[u, dotted, shift left] \arrow[r, shift left, "p_n"] \arrow[d, shift left, "\gamma_n"] & \mathbb{Q}_n \arrow[l, shift left, "i_n"] \arrow[u, shift left] \\
R_{n-1} \arrow[u, shift left, "F_n"] \arrow[d, shift left, "\gamma_{n-1}"] & 0 \\
\vdots \arrow[u, shift left, "F_{n-1}"] \arrow[d, shift left, "\gamma_2"] & \vdots \\
R_1 \arrow[u, shift left, "F_2"] \arrow[d, shift left, "\gamma_1"] & 0 \\
R_0 \arrow[u, shift left, "F_1"] \arrow[r, shift left, "p_0"] \arrow[d, dotted, shift left] & \mathbb{Q}_0 \arrow[l, shift left, "i_0"] \arrow[d, shift left] \\
0 \arrow[u, dotted, shift left] & 0\,. \arrow[u, shift left]
\end{tikzcd}
\tag{25}
$$

We can translate axioms (12), (13) into a symmetric form: the two complexes

$$
\begin{array}{c}
0 \to \mathbb{Q}_n \xrightarrow{i_n} R_n \xrightarrow{\gamma_n} R_{n-1} \xrightarrow{\gamma_{n-1}} \dots \xrightarrow{\gamma_1} R_0 \xrightarrow{p_0} \mathbb{Q}_0 \to 0, \\
0 \leftarrow \mathbb{Q}_n \xleftarrow{p_n} R_n \xleftarrow{F_n} R_{n-1} \xleftarrow{F_{n-1}} \dots \xleftarrow{F_1} R_0 \xleftarrow{i_0} \mathbb{Q}_0 \leftarrow 0.
\end{array}
\tag{26}
$$

are acyclic chain and cochain complexes, respectively, on the same graded free module over $\mathbb{Q}$, the operators of the second one are SDR null homotopy operators (i.e., SDRs onto zero) for the first one. This means that we have the following identities:

$$
\begin{array}{rcll}
\gamma_1 F_1 + i_0 p_0 & = & \mathrm{Id}, & \\
i_n p_n + F_n \gamma_n & = & \mathrm{Id}, & \\
\gamma_j F_j + F_{j-1}\gamma_{j-1} & = & \mathrm{Id} & \text{for } j \neq 1, n,
\end{array}
\tag{27}
$$

$$
\gamma_j\gamma_{j-1} = 0,\ \ \gamma_n i_n = 0,\ \ p_0\gamma_1 = 0,\ \ p_n F_n = 0,\ \ F_1 i_0 = 0,\ \ F_{j-1}F_j = 0.
$$

For the matrix SDR of the chain sphere $R_\bullet$ on homology, we have fixed all the data of the first row in (26). To get a retraction of $R_\bullet$ onto $H(S^n;\mathbb{Q})$, we need to find the data in the second row of (26) satisfying conditions (27) together with the data in the first row.

For a matrix M over $\mathbb{Q}$, denote by $M^\dagger$ its Moore–Penrose inverse matrix.

Lemma 7. *A matrix homology splitting of $R_\bullet$ is provided by the data $\langle F, i, p\rangle$ where $F_i = \gamma_i^\dagger$, $i_0 = p_0^\dagger$, $p_n = i_n^\dagger$.*

Proof. The lemma follows from the combinatorial Hodge theory for a free based rational chain complex with fixed bases, which provides a strong deformational retraction $\langle \gamma^\dagger, i, p\rangle$ of $R_\bullet$ onto the homology $H(S^n;\mathbb{Q})$ (see, for example, [29, § A.1]). Here $\gamma_i^\dagger$ is the Moore–Penrose inverse of the matrix γ_i, $i_0 = p_0^\dagger$, $p_n = i_n^\dagger$. □

We will present matrix formulas. Our complex $R_\bullet = C_\bullet(S;\mathbb{Q})$ is based. We have canonical scalar products making the bases in $R_\bullet$ orthonormal, and, therefore, we have a combinatorial Hodge theory. Let

$$R_{i-1} \xrightarrow{\gamma_i^\intercal} R_i$$

be the metric adjoint differential for γ_i, which is represented simply by the transposed matrix. Let

$$\Delta_i = \gamma_i^\intercal \gamma_i + \gamma_{i+1}\gamma_{i+1}^\intercal \tag{28}$$

be the combinatorial Laplace matrix operator. Our cellular sphere has w vertices and f top cells. For a top cell j, we denote by $o(j,[A])$ its orientation relative to the fundamental class. Then the matrix formulas for the combinatorial Hodge theory (or Moore–Penrose) homology splitting of $R_\bullet$ are as follows. Define matrices

$$i_0, i_n, p_0, p_n$$

by the formulas

$$\begin{aligned}
\mathbb{Q}_0 \xrightarrow{i_0} R_0 : &\quad & i_0(\beta) &= \tfrac{1}{w}\underbrace{(\beta,\dots,\beta)}_{w},\\
\mathbb{Q}_n \xrightarrow{i_n} R_n : &\quad & i_n(\alpha) &= \alpha[A],\\
R_0 \xrightarrow{p_0} \mathbb{Q}_0 : &\quad & p_0(\beta_1,\dots,\beta_w) &= \sum_{j=1}^{w} \beta_j,\\
R_n \xrightarrow{p_n} \mathbb{Q}_n : &\quad & p_n(\alpha_1,\dots,\alpha_f) &= \tfrac{1}{f}\sum_{j=1}^{n} o(j,[A])\alpha_j.
\end{aligned} \tag{29}$$

Set

$$G_j = \begin{cases} (\Delta_j + i_j p_j)^{-1}, & j = 0, 1, \\ \Delta_j^{-1}, & j \neq 0, n, \end{cases} \tag{30}$$

and set

$$\gamma_j^\dagger = \gamma_j^\intercal G_{j-1} = G_j \gamma_{j-1}^\intercal. \tag{31}$$

15. A Hodge-theoretic local combinatorial formula for the Euler class of a PL spherical fiber bundle. Now we can insert Hodge-theoretic matrix homology splittings of cellular spheres into formula (19) for the formal Euler cocycle and get a theorem.

Assume that we have a chain of spherical aggregations in $\boldsymbol{S}^n$:

$$\mathcal{S} = (S_0 \to S_1 \to \ldots \to S_{n+1}).$$

Assume that we have the corresponding chain of based chain complexes in $\mathrm{Ch}(S^n, \mathbb{Q})$ (see §6):

$$\mathcal{R}(\mathcal{S}) = (R_\bullet^0 \xleftarrow{R(0,1)} R_\bullet^1 \xleftarrow{R(1,2)}) \ldots \xleftarrow{R(n,n+1)} R_\bullet^{n+1}),$$

where $R(i, i+1)$ are subdivision chain quasi-isomorphisms (22). The differential in $R_\bullet^i$ is denoted by γ. For every $R_\bullet^i$, we have a Hodge-theoretic strong deformational retraction onto $H(S^n; \mathbb{Q})$:

$$\begin{tikzcd} R_\bullet^i \arrow[loop left, "\gamma^\dagger"] \arrow[r, shift left, "p"] & H(S^n;\mathbb{Q}) \arrow[l, shift left, "i"] \end{tikzcd}. \tag{32}$$

The matrices of all the operators are defined by formulas (29)–(31).

Theorem 8. *For a chain of spherical aggregations* $\mathcal{S}$*, the rational number obtained as the matrix product*

$$e_{CH}(S) = p_n R(0,1)\gamma^\dagger R(1,2) \ldots \gamma^\dagger R(n, n+1) i_0 \tag{33}$$

is an aggregation local combinatorial formula for the Euler class of PL S^n*-fiber bundles (as in Subsec.* 13*).*

Proof. This is the matrix formula from Proposition 2 for the twisting cochain in the bigraded model of the PL spherical fiber bundle defined by a local system of aggregations (Corollary 6). It is invariant under all choices involved and invariant under automorphisms of $R_\bullet^i$, because all the involved Laplace and Green operators are. It depends up to sign only on the bundle orientation. □

§8. Notes

We do not know much about the behavior of the local formula (33). To our deep shame, we do not know why the cocycle is a coboundary if n is even. It should be, since the Euler class of an even-dimensional spherical bundle is zero. We suspect that the absolute value of the number (33) has a small absolute upper bound, but we do not know it. Meanwhile, we can compare the formula in an interesting way with the formulas for circle bundles [21, 28]. We postpone this for a later publication. Formula (33) should have an interesting interpretation in terms of cellular combinatorial physics and statistics. It is composed from Moore–Penrose inverses of differentials, which have a very interesting description [8, Theorem 5.3] based on Lothar Berg's theorem [2], which have no analog yet in the differential Hodge theory.

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Поступило 13 сентября 2021 г.