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A SIMPLE OBSERVATION ON HEISENBERG-LIKE UNCERTAINTY PRINCIPLES

Abstract. A solution is given to a conjecture proposed recently by Y. Wigderson and A. Wigderson concerning a "Heisenberg-like" uncertainty principle. That conjecture is about the image of the map $f \mapsto \frac{\|f\|_q \|\hat{f}\|_q}{\|f\|_2 \|\hat{f}\|_2}$, $f \in \mathscr{S}(\mathbb{R}) \setminus \{0\}$, where $\mathscr{S}(\mathbb{R})$ stands for the Schwartz class of functions on the real line. Also, a more general question is answered, where the L_2 norm is replaced by the L_p norm in the denominator.

§1. Introduction

The classical Heisenberg uncertainty principle says that

$$\|f\|_2 \|\hat{f}\|_2 \leqslant 4\pi \left(\int |xf(x)|^2 \mathrm{d}x \right)^{1/2} \left(\int |\xi \hat{f}(\xi)|^2 \mathrm{d}\xi \right)^{1/2}$$

for a function f in $\mathscr{S}(\mathbb{R})$, the Schwartz space on real line, where $\hat{f}$ is the Fourier transform defined on $\mathscr{S}(\mathbb{R})$ by $\hat{f}(\xi) = \int\limits_{\mathbb{R}} f(x) \mathrm{e}^{-2\pi \mathrm{i} x\xi} \mathrm{d}x$, and $\|f\|_q$ is the L^q-norm of f, that is, $\|f\|_q^q = \int_{\mathbb{R}} |f|^q \mathrm{d}x$.

In a recent paper A. Wigderson and Y. Wigderson (see [5]) considered a family of "Heisenberg-like" uncertainty principles, and posed the following question.

Question 1.1 (see [5, Conjecture 4.13])**.** For $1 < q \leqslant \infty$ and $q \neq 2$, define the following function $F_q : \mathscr{S}(\mathbb{R}) \setminus \{0\} \to \mathbb{R}_{>0}$,

$$F_q(f) := \frac{\|f\|_q \|\hat{f}\|_q}{\|f\|_2 \|\hat{f}\|_2}.$$

Is F_q surjective?

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In other words, the question is whether the image of the function F_q is all of $\mathbb{R}_{>0}$. Note that, if the image is bounded from below by a constant $c > 0$, then

$$\|f\|_2\|\hat{f}\|_2 \leqslant \frac{1}{c}\|f\|_q\|\hat{f}\|_q,$$

which could be viewed as a variant of the classical uncertainty principle.

In this note (see also [4]) we answer this question affirmatively when $2 < q < \infty$ (the case of $q = \infty$ was proved in [5]), and negatively when $1 < q < 2$. Also, we discuss a more general question.

The author presented these results at the conference "ComPlane: the next generation" on June 17–18, 2021, see [4]. On July 19, 2021, a paper by L. Huang, Z. Liu, J. Wu [3] containing mostly the same results was published on ArXiv.

Our first result is the following.

Theorem 1.2. (i) *If $2 < q < \infty$, then the image of F_q is all of $\mathbb{R}_{>0}$.*
(ii) *If $1 < q < 2$, then the image of F_q is bounded below by 1, i.e., $F_q(f) \geqslant 1$ for any $f \in \mathscr{S}(\mathbb{R}) \setminus \{0\}$.*

We also consider the general case of the function

$$F_{q,p}(f) = \frac{\|f\|_q\|\hat{f}\|_q}{\|f\|_p\|\hat{f}\|_p}, \quad f \in \mathscr{S}(\mathbb{R}) \setminus \{0\},$$

and obtain the following claim.

Theorem 1.3. *Let $1 < q < p < \infty$. Then,*
(i) *if $\frac{1}{p} + \frac{1}{q} \geqslant 1$, then image of $F_{q,p}$ is an infinite subinterval of $[1, \infty)$;*

(ii) *if $\frac{1}{p} + \frac{1}{q} < 1$, then the image of $F_{q,p}$ is $(0, \infty)$.*

Remark 1.4. The case of $1 < p < q < \infty$ follows immediately from Theorem 1.3: it suffices to observe that $F_{p,q}(f) = \frac{1}{F_{q,p}(f)}$.

Note that the bound 1 in Theorem 1.3 (i) is far from optimal. In fact, in the proof the Hausdorff–Young inequality is used:

$$\|\hat{f}\|_{p'} \leqslant \|f\|_p\,, \quad p \in [1, 2],$$

which is weaker than Beckner's theorem [1], which gives

$$\|\hat{f}\|_{p'} \leqslant \sqrt{(p^{1/p})/(p')^{1/p'}}\,\|f\|_p.$$

We did not pursue this direction as it seems that finding the infimum of the image of $F_{q,p}$ is more difficult than determining whether it is surjective.

§2. Preliminaries

Let us recall the topology on $\mathscr{S}(\mathbb{R})$. For $\alpha, \beta \in \mathbb{N}$, define

$$\rho_{\alpha,\beta}(f) = \sup_{x\in\mathbb{R}} \left|x^\alpha \frac{\mathrm{d}^\beta}{\mathrm{d}x^\beta} f(x)\right|.$$

Let $\{\rho_j\}_j$ be an enumeration of $\{\rho_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{N}}$, and define the following metric on $\mathscr{S}(\mathbb{R})$:

$$d(f,g) = \sum_{j=1}^{\infty} 2^{-j} \frac{\rho_j(f-g)}{1+\rho_j(f-g)}.$$

Under this metric, the space $(\mathscr{S}(\mathbb{R}), d)$ is complete, so $\mathscr{S}(\mathbb{R})$ is a Fréchet space. A sequence $\{f_n\} \subset \mathscr{S}(\mathbb{R})$ converges to $f \in \mathscr{S}(\mathbb{R})$ in $\mathscr{S}(\mathbb{R})$ if $d(f_n, f)$ goes to 0 as n tends to infinity.

The following facts are basic properties of $\mathscr{S}(\mathbb{R})$ and the Fourier transform on $\mathscr{S}(\mathbb{R})$ (see, e.g., [2]).

Fact 2.1. First, the Fourier transform $f \mapsto \hat{f}$ is a homeomorphism from $\mathscr{S}$ onto itself. Second, if the functions $\{f_n\}_{n\geqslant 1}$ and f belong to $\mathscr{S}(\mathbb{R})$, and if f_n converges to f in $\mathscr{S}(\mathbb{R})$, then f_n converges to f in L^p for all $1 \leqslant p \leqslant \infty$. Therefore, the mapping $f \mapsto \|f\|_p$ is continuous on $\mathscr{S}(\mathbb{R})$ for all $1 \leqslant p \leqslant \infty$.

Proof. For a proof, see [2], Proposition 2.2.6 and Corollary 2.2.15. $\square$

Combining these facts, we see that the mapping $f \mapsto \|\hat{f}\|_p$ is continuous as a composition of two continuous mappings, so the mapping $f \mapsto F_q(f)$ is continuous from $\mathscr{S}(\mathbb{R}) \setminus \{0\}$ to $\mathbb{R}_{>0}$. As a metric space, $\mathscr{S}(\mathbb{R}) \setminus \{0\}$ is connected, because $\mathscr{S}(\mathbb{R})$ is a Fréchet space with $\dim \mathscr{S}(\mathbb{R}) = \infty$. Recall that between two metric spaces, the image of a connected set by a continuous map is also connected. Now the connected sets in $\mathbb{R}$ are intervals, and we conclude that

Proposition 2.2. *The image of F_q is an interval on $\mathbb{R}_{>0}$.*

Our goal now is to study the endpoints of Im F_q. But first, we show what can be obtained by calculating $F_q(f)$ for a family of simple functions f that was already used in [5]. For $a > 1$, define functions f_a by

$$f_a(x) = \mathrm{e}^{-\pi(a^2-1)x^2} \mathrm{e}^{-2\pi \mathrm{i} a x^2}.$$

Clearly, the function f_a belongs to $\mathscr{S}(\mathbb{R})$, and one can calculate its Fourier transform:

$$\hat{f}_a(\xi) = \frac{1}{a+\mathrm{i}} \exp\left(\frac{-\pi\xi^2(a^2-1)}{(a^2+1)^2}\right) \exp\left(-2\pi\mathrm{i}\xi^2 \frac{a}{(a^2+1)^2}\right).$$

Since $\int_{\mathbb{R}} \mathrm{e}^{-\pi x^2} \mathrm{d}x = 1$, we get, for any $1 \leqslant q < \infty$,

$$||f_a||_{L^q} = \left(\frac{1}{\sqrt{q(a^2-1)}}\right)^{\frac{1}{q}}, \quad ||\hat{f}_a||_{L^q} = \frac{1}{\sqrt{a^2+1}}\left(\frac{a^2+1}{\sqrt{q(a^2-1)}}\right)^{\frac{1}{q}}.$$

So, we get

$$\begin{aligned} F_q(f_a) &= \frac{\|f_a\|_q \|\hat{f}_a\|_q}{\|f_a\|^2} = \sqrt{2}\left(\frac{1}{q}\right)^{1/q}\left(\frac{a^2+1}{a^2-1}\right)^{1/q-1/2} \\ &= \sqrt{2}\left(\frac{1}{q}\right)^{1/q}\left(\frac{t+1}{t-1}\right)^{1/q-1/2}, \quad a^2 = t > 1. \end{aligned}$$

Notice that the mapping $t \mapsto \frac{t+1}{t-1}$ is monotone decreasing on $(1, \infty)$, so if $1/q > 1/2$, the image of the mapping $t \mapsto (\frac{t+1}{t-1})^{1/q-1/2}$, $t > 1$, is $(1, \infty)$. Similarly, if $1/q < 1/2$, then the image of the above mapping is $(0, 1)$. In conclusion, by testing the functions f_a on F_q, we get the following statement.

Proposition 2.3. (i) *If* $1 < q < 2$, *then at least* $\big(\sqrt{2}(\frac{1}{q})^{\frac{1}{q}}, \infty\big) \subset \mathrm{Im}\ F_q$.
(ii) *If* $2 < q < \infty$, *then at least* $\big(0, \sqrt{2}(\frac{1}{q})^{\frac{1}{q}}\big) \subset \mathrm{Im}\ F_q$.

§3. The case of $2 < q < \infty$ for F_q

In this section, we always assume that $2 < q < \infty$, and we will prove that $\mathrm{Im}\ F_q = (0, \infty)$. By Proposition 2.3, we only need to construct a sequence $\{f_n\}_n \subset \mathscr{S}(\mathbb{R})$ so that $F_q(f_n) \to \infty$ as n goes to infinity. We use a construction from [5]. Let $c > 0$, and define the function

$$g_c(x) = \frac{1}{\sqrt{c}}\ \mathrm{e}^{-\pi \frac{x^2}{c^2}} + \sqrt{c}\ \mathrm{e}^{-\pi c^2 x^2}.$$

Clearly, $g_c \in \mathscr{S}(\mathbb{R})$, and $\hat{g_c} = g_c$ for all $c > 0$, therefore $F_q(g_c) = \big(\frac{\|g_c\|_q}{\|g_c\|_2}\big)^2$. A direct calculation shows that

$$\|g_c\|_2^2 = \sqrt{2} + \frac{2c}{\sqrt{c^4+1}}, \quad c > 0. \tag{1}$$

Now, we estimate $\|g_c\|_q^2$. By definition, one has

$$\|g_c\|_q^2 = \left\{ \int \left(\frac{1}{c}\, \mathrm{e}^{-2\pi \frac{x^2}{c^2}} + c\, \mathrm{e}^{-2\pi c^2 x^2} + 2\mathrm{e}^{-\pi(c^2+1/c^2)x^2} \right)^{\frac{q}{2}} \mathrm{d}x \right\}^{\frac{2}{q}}.$$

Notice that $\left(\sum_i a_i\right)^{q/2} \geqslant \sum_i a_i^{q/2}$ when $q/2 > 1$, therefore

$$\begin{aligned}
\|g_c\|_q^2 \geqslant& \left\{ \int \left(\frac{1}{c}\right)^{q/2} \mathrm{e}^{-\pi q \frac{x^2}{c^2}} + c^{q/2}\, \mathrm{e}^{-\pi q c^2 x^2} + 2^{q/2} \mathrm{e}^{\frac{-\pi q}{2}(c^2+1/c^2)x^2} \mathrm{d}x \right\}^{\frac{2}{q}} \\
=& \left\{ \left(\frac{1}{c}\right)^{q/2} \frac{1}{\sqrt{q/c^2}} + c^{q/2} \frac{1}{\sqrt{qc^2}} + 2^{q/2} \frac{1}{\sqrt{q(c^2+1/c^2)/2}} \right\}^{\frac{2}{q}}. \qquad (2)
\end{aligned}$$

In conclusion, for all $c > 0$, we have

$$\begin{aligned}
\|g_c\|_q^2 \geqslant& \left\{ \frac{1}{\sqrt{q}} c^{1-\frac{q}{2}} + \frac{1}{\sqrt{q}} c^{\frac{q}{2}-1} + \frac{2^{(q+1)/2}}{\sqrt{q}} \frac{c}{\sqrt{c^4+1}} \right\}^{\frac{2}{q}} \geqslant \left(\frac{1}{\sqrt{q}} c^{\frac{q}{2}-1} \right)^{2/q} \\
=& \left(\frac{1}{q}\right)^{1/q} c^{1-\frac{2}{q}}. \qquad (3)
\end{aligned}$$

Combining (1) and (3), we get

$$F_q(g_c) \geqslant \left(\frac{1}{q}\right)^{1/q} \frac{c^{1-\frac{2}{q}}}{\sqrt{2} + \frac{2c}{\sqrt{c^4+1}}}.$$

Finally, we have $F_q(g_c) \to \infty$ as $c \to \infty$.

§4. The case of $1 < q < 2$ for F_q

In this section, we always assume that $1 < q < 2$. In fact, the case of $q < 2$ is easier. Let q' be the exponent conjugate to q. Hölder's inequality implies

$$\|f\|_2 \leqslant \|f\|_q^{1/2} \|f\|_{q'}^{1/2}.$$

The Hausdorff–Young inequality states that, for $1 \leqslant q \leqslant 2$, we have

$$\|\hat{f}\|_{q'} \leqslant \|f\|_q\,, \quad f \in \mathscr{S}.$$

Therefore, by choosing $g = \hat{f} \in \mathscr{S}$ we get $\|\hat{g}\|_{q'} \leqslant \|g\|_q$, i.e.,

$$\|f\|_{q'}^{1/2} \leqslant \|\hat{f}\|_q^{1/2}.$$

Combining these inequalities, we get

$$\|f\|_2 \leqslant \|f\|_q^{1/2} \|\hat{f}\|_q^{1/2},$$

which implies that $\frac{\|f\|_q \|\hat{f}\|_q}{\|f\|_2^2} \geqslant 1$ for any $f \in \mathscr{S}(\mathbb{R}) \setminus \{0\}$.

§5. The general case of $F_{p,q}$

In this section, we discuss the general case of the function

$$F_{q,p}(f) = \frac{\|f\|_q \|\hat{f}\|_q}{\|f\|_p \|\hat{f}\|_p}, \quad 1 < q < p < \infty.$$

First, we see what we could get by using the functions f_a from §2:

$$f_a(x) = \mathrm{e}^{-\pi(a^2-1)x^2} \mathrm{e}^{-2\pi \mathrm{i} a x^2}, \quad a > 1.$$

A direct calculation shows that for each $1 < q < \infty$,

$$\|f_a\|_q = \left(\frac{1}{q(a^2-1)}\right)^{\frac{1}{2q}}, \quad \|\hat{f}_a\|_q = (a^2+1)^{-1/2} \left(\frac{a^2+1}{\sqrt{q(a^2-1)}}\right)^{\frac{1}{q}}.$$

Replacing q by p, we obtain

$$F_{q,p}(f_a) = \frac{(1/q)^{1/q}}{(1/p)^{1/p}} \left(\frac{a^2+1}{a^2-1}\right)^{\frac{1}{q}-\frac{1}{p}} = \frac{(1/q)^{1/q}}{(1/p)^{1/p}} \left(\frac{t+1}{t-1}\right)^{\frac{1}{q}-\frac{1}{p}}, \quad t = a^2 > 1.$$

So we get $\left(\frac{(1/q)^{1/q}}{(1/p)^{1/p}}, \infty\right) \subset \operatorname{Im} F_{q,p}$.

5.1. Theorem 1.3, case (i). We prove (i) of Theorem 1.3: $\frac{1}{p} + \frac{1}{q} \geqslant 1$. Note that $q < 2$ in this case. We divide it into two subcases: $q < p < 2$ and $q < 2 \leqslant p$. These two subcases are a little bit different, although the results are the same.

5.1.1. *The case of* $q < p < 2$. First, we write p as a convex combination of q and 2:

$$p = \lambda q + (1-\lambda)2, \quad 0 < \lambda < 1.$$

Then, $\int |f|^p = \int |f|^{\lambda q} |f|^{(1-\lambda)2}$. Notice that $\frac{1}{\lambda} \in (1, \infty)$. By Hölder's inequality ($\lambda + (1-\lambda) = 1$), we have

$$\|f\|_p \leqslant \|f\|_q^{\frac{1/p-1/2}{1/q-1/2}} \|f\|_2^{\frac{1/q-1/p}{1/q-1/2}}. \tag{4}$$

Inequality (4) is also fulfilled for $\hat{f}$,

$$\|\hat{f}\|_p \leqslant \|\hat{f}\|_q^{\frac{1/p-1/2}{1/q-1/2}} \|\hat{f}\|_2^{\frac{1/q-1/p}{1/q-1/2}}. \tag{5}$$

Combining the above two inequalities and the definition of $F_{q,p}$, we get

$$F_{q,p}(f) \geqslant \left(\frac{\|f\|_q \|\hat{f}\|_q}{\|f\|_2 \|\hat{f}\|_2} \right)^{\frac{1/q-1/p}{1/q-1/2}} = \big(F_q(f)\big)^{\frac{1/q-1/p}{1/q-1/2}}.$$

Finally, notice that $\frac{1/q-1/p}{1/q-1/2} > 0$, so if $1 < q < p < 2$, then we come back to the case of $1 < q < 2,\ p = 2$ in §4, which asserts that the image of F_q is bounded below by 1. Therefore, the image of $F_{q,p}$ is bounded below by 1.

5.1.2. *The case of* $q < 2 \leqslant p$. Since $1 < p' \leqslant 2$, $\|\hat{f}\|_p \leqslant \|f\|_{p'}$ by the Hausdorff–Young inequality, which shows that

$$\frac{\|f\|_q \|\hat{f}\|_q}{\|f\|_p \|\hat{f}\|_p} \geqslant \frac{\|f\|_q \|\hat{f}\|_q}{\|\hat{f}\|_{p'} \|f\|_{p'}} = F_{q,p'}(f).$$

But this case was treated in Subsection 5.1.1, as $1 < q \leqslant p' < 2$.

5.2. Theorem 1.3, case (ii). Now we discuss the case of $\frac{1}{p} + \frac{1}{q} < 1$. Recall that $q < p$, therefore $p > 2$. Here we use the function g_c from §3. Recall that for $c > 0$,

$$g_c(x) = \frac{1}{\sqrt{c}}\, \mathrm{e}^{-\pi \frac{x^2}{c^2}} + \sqrt{c}\, \mathrm{e}^{-\pi c^2 x^2}, \quad g_c = \hat{g}_c.$$

Inequality (2) implies that ($p > 2$)

$$\|g_c\|_p^2 \geqslant \left\{ \frac{1}{\sqrt{p}} c^{1-\frac{p}{2}} + \frac{1}{\sqrt{p}} c^{\frac{p}{2}-1} + \frac{2^{(p+1)/2}}{\sqrt{p}} \frac{c}{\sqrt{c^4+1}} \right\}^{\frac{2}{p}} \sim \left(\frac{c^{\frac{p}{2}-1}}{\sqrt{p}} \right)^{2/p}, \quad \text{as } c \to \infty.$$

Similarly, for $q > 1$, we have

$$(a_1 + a_2 + a_3)^{\frac{q}{2}} \leqslant \max\{1, 3^{q/2-1}\} \left(a_1^{\frac{q}{2}} + a_2^{\frac{q}{2}} + a_3^{\frac{q}{2}} \right),$$

so we merely reverse inequality (2) and get

$$\|g_c\|_q^2 \leqslant \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} \frac{c}{\sqrt{c^4+1}} \leqslant 4, \quad \text{when } q = 2.$$

For $q < 2$ and $c \to \infty$ we get

$$\|g_c\|_q^2 \leqslant \left\{ \frac{1}{\sqrt{q}} c^{1-\frac{q}{2}} + \frac{1}{\sqrt{q}} c^{\frac{q}{2}-1} + \frac{2^{(q+1)/2}}{\sqrt{q}} \frac{c}{\sqrt{c^4+1}} \right\}^{\frac{2}{q}} \sim \left(\frac{c^{1-\frac{q}{2}}}{\sqrt{q}} \right)^{2/q},$$

and, finally, for $q > 2$ and $c \to \infty$ we have

$$\|g_c\|_q^2 \leqslant 3^{1-\frac{2}{q}} \left\{ \frac{1}{\sqrt{q}} c^{1-\frac{q}{2}} + \frac{1}{\sqrt{q}} c^{\frac{q}{2}-1} + \frac{2^{(q+1)/2}}{\sqrt{q}} \frac{c}{\sqrt{c^4+1}} \right\}^{\frac{2}{q}} \sim 3^{1-\frac{2}{q}} \left(\frac{c^{\frac{q}{2}-1}}{\sqrt{q}} \right)^{2/q}.$$

Consequently, as c goes to infinity, we have

$$\|g_c\|_q^2 / \|g_c\|_p^2 \leqslant C_{p,q}\ c^{2(1/q+1/p-1)}$$

when $q \leqslant 2$, and $\|g_c\|_q^2/\|g_c\|_p^2 \leqslant C_{p,q}\ c^{2(1/p-1/q)}$ when $q > 2$, where $C_{p,q}$ depends only on p and q. Finally, note simply that $1/q + 1/p - 1 < 0$ and $1/p - 1/q < 0$, so $\lim_{c\to\infty} \|g_c\|_q^2/\|g_c\|_p^2 = 0$.

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