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M. Ahsanullah, V. B. Nevzorov, A. V. Stepanov

DISTRIBUTIONS AND CHARACTERIZATIONS ASSOCIATED WITH A RANDOM WALK

Abstract. In the present paper, we discuss some properties of the distribution and density of the random direction walk after n steps. We further calculate moments of the random walk and propose some characterizations of its distribution. We illustrate our results by tables and graphs.

§1. Introduction

An n step Pearson's random walk is a walk in the plane that starts at the origin 0 and consists of n steps of length 1 each taken into a uniformly random direction. Pearson (1905) proposed this problem. Let X be the distance traveled in n steps. Kluyver (1905) gave the probability density function (pdf) $p_n(x)$ of the distance X after n steps of unit length. We denote $\mathrm{PRW}(n,x)$ as the distribution of the distance from the origin at the n-th step.The pdf $p_n(x)$ of $\mathrm{PRW}(n,x)$ as given by Kluyver(1905) is as follows

$$p_n(x) = \int_0^\infty xtJ_0(xt)(J_0(t))^n\,dt, \quad 0 \leqslant x \leqslant n, \tag{1.1}$$

where $J_0(.+96)$ is the Bessel function of first kind and zero-th order. For $n = 2,3,4$, the pdf $p_n(x)$ has the following forms

$$p_2(x) = \frac{2}{\pi}(4-x^2)^{-1/2}, \quad 0 \leqslant x \leqslant 2,$$

$$p_3(x) = \frac{2\sqrt{3}}{\pi}\frac{x}{3+x^2}\,{}_2F_1\left({\frac13, \frac23 \atop 1}\middle|\frac{x^2(9-x^2)^2}{(3+x^2)^3}\right), \quad 0 < x \leqslant 3,$$

$$p_4(x) = \frac{2}{\pi^2}\frac{\sqrt{16-x^2}}{x}\,{}_3F_2\left({\frac12, \frac12, \frac12 \atop \frac56, \frac76}\middle|\frac{(16-x^2)^3}{108x^4}\right), \quad 0 < x \leqslant 4,$$

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where

$$ {}_pF_q\left(\begin{smallmatrix}\alpha_1,\alpha_2,\ldots,\alpha_p\\ \beta_1,\beta_2,\ldots,\beta_q\end{smallmatrix}\Big| x\right) = 1 + \sum_{k=0}^{\infty} \frac{(\alpha_1)_k,(\alpha_2)_k,\ldots,(\alpha_p)_k}{k!(\beta_1)_k,(\beta_2)_k,\ldots,(\beta_q)_k} x^k $$

with $(a)_k = a(a+1)\ldots(a+k-1)$ is the hypergeometric series.

Rayleigh (1905) showed that for $n \geqslant 5, P_n(x)$ is close to the pdf $p_{n,l}(x)$, where

$$ p_{n,l}(x) = \frac{2x}{n}\ e^{-\frac{x^2}{n}}, \quad x \geqslant 0. \tag{1.2} $$

We will denote this distribution as PRWL(n,x). For some basic properties of the pdf $p_n(x)$ see Borwein et al (2012).

In this paper several distributional properties and characterizations of PRW(n,x) and PRWL(n,x) for some values of n will be given.

§2. Basic Properties

The cumulative distribution function (cdf) $P_n(x)$ of PRW(n,x) is given by

$$ P_n(x) = \int_0^{\infty} xJ_1(xt)(J_0(t))^n\, dt, \quad x \geqslant 0, \tag{2.1} $$

where $J_1(x)$ is the Bessel function of first order.

The cdf $P_{n,l}(x)$ of PRWL(n,x) is as follows

$$ P_{n,l}(x) = \int_0^{x} \frac{2u}{n}\ e^{-\frac{u^2}{n}}\, du = 1 - e^{-\frac{x^2}{n}}, \quad x \geqslant 0. \tag{2.2} $$

Using the property $\frac{d}{dx}J_0(x) = -J_1(x)$, we obtain

$$ P_n(1) = \int_0^{\infty} \big(J_1(t)J_0(t)\big)^n\, dt = \frac{-(J_0(t))^{n+1}}{n+1}\Big|_{\infty}^{\in} = \frac{1}{n+1} $$

and

$$ P_{n,l}(1) = 1 - e^{-\frac{1}{n}}. $$

Table 2.1 gives $P_n(1)$ and $P_{n,l}(1)$ for some values of n.

Table 2.1. $P_n(1)$ and $P_{n,l}(1)$

n	5	6	7	8	9	10	25
$P_n(1)$	0.166 67	0.142 86	0.125	0.111 11	0.1	0.0909	0.03846 2
$P_{nl}(1)$	0.181 27	0.153 52	0.133 12	0.117 5	0.105 16	0.09516 3	0.039211

Let $\mu_n(m)$ be the m-th moment of PRW(n,x), then we have

$$\mu_n(m) = \int_o^n x^m p_n(x)\,dx. \tag{2.3}$$

An alternative expression of $\mu_n(m)$ is

$$\mu_n(m) = \int_0^1 \int_0^1 \cdots \int_0^1 \left|1+e^{2\pi i x_1}+\cdots+e^{2\pi i x_{n-1}}\right|^m dx_{n-1},\ldots,dx_2 dx_1. \tag{2.4}$$

From (2.4), we obtain for $n=2$

$$\mu_2(m) = \int_0^1 \left|1+e^{2\pi i x}\right|^m dx = 2^{m+1}\int_0^{1/2} \cos^m(\pi t)\,dt = \frac{\Gamma(m+1)}{\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\frac{m}{2}+1\right)}.$$

Using (2.4), we obtain the following values of $\mu_n(m)$

$$\begin{aligned} \mu_3(m) &=_3 F_2 \left({\tfrac{1}{2}, \tfrac{-m}{2}. \tfrac{-m}{2} \atop 1,1}\middle|4\right) \\ \mu_4(m) &= \sum_{j=0} \left(\frac{\Gamma(\frac{m}{2}+1}{\Gamma(l+1)\Gamma(\frac{m}{2}-j+1}\right)^2 \ {}_3F_2\left({\tfrac{1}{2}, \tfrac{-m}{2}+j. \tfrac{-m}{2}+j \atop 1,1}\middle|4\right) \\ \mu_n(2m) &= \sum_{m_1+m_2+\ldots\_m_n=m} \left(\frac{m!}{m_1!m_1!\ldots m_n!}\right)^2, \end{aligned} \tag{2.5}$$

where $0 \leqslant m_1, m_2, \ldots m_n \leqslant m$.

On simplifications, we obtain from (2.5) for even m.

$$\begin{aligned} \mu_n(2) &= n \\ \mu_n(4) &= 2n^2 - n \\ \mu_n(6) &= 6n^3 - 9n^2 + 4n \\ \mu_n(8) &= 24n^4 - 72n^3 + 96n^2 - 33n \\ \mu_n(10) &= 120n^5 - 600n^4 + 1250n^3 - 1225n^2 + 456n \end{aligned}$$

Let $\mu_{n,l}(m)$ be the m-th moment of PRWL(n,x), then

$$\mu_{n,l}(m) = \int_0^\infty \frac{2x^{m+1}}{n} e^{-\frac{x^2}{n}}\,dx = \int_0^\infty (ny)^{m/2} e^{-y}\,dy = n^{m/2}\,\Gamma\Big(\frac{m}{2}+1\Big). \tag{2.6}$$

The table 2.2 gives for $n=1$ to 6 and $m=1$ to 10, $\mu_n(m)$, $\mu_{5,l}(m)$, $\mu_{6,l}(m)$.

Table 2.2. $n = 1$ to 6, $m = 1$ to 10, $\mu_n(m)$, $\mu_{5,l}(m)$, $\mu_{6,l}(m)$.

$n \setminus m$	1	2	3	4	5	6	7	8	9	10
2	1.27324	2	3.39531	6	10.8650	20	37.2514	70	132.449	252
3	1.57460	3	6.45168	15	36.7052	93	241,544	639	1714.62	4653
4	1.79909	4	10.1207	28	82.6515	256	822.273	2716	9169.62	31504
5	2.00816	5	14.2896	45	152.316	545	2037.14	7885	31393.1	127905
6	2.19380	6	18.9133	66	248.759	996	4186.19	18308	82718.9	384156

μ_{5l}	1.9817	5.0	14.862	50	185.78
μ_{6l}	2.170 8	6.0	19.537	72.0	293.06
μ_{5l}	750.0	3251.2	15000.0	73151.	3.75×10^5
μ_{6l}	1296.0	6154, 2	11104.0	$1.661\,6 \times 10^5$	$9.331\,2 \times 10^5$

It seems that the higher moments using Rayleigh approximating distributions are quite different from the values of $\mathrm{PRW}(n, x)$ for $n = 5$ and 6.

The hazard rate $\lambda(x)$ of a random variable X with pdf $f(x)$ and cdf $F(x)$ is defined as $\lambda(x) = \frac{f(x)}{1-F(x)}$ for $F(x) \neq 1$. It is difficult to find the hazard rate $\lambda_n(x)$ of the random variable X is from $\mathrm{PRW}(n, x)$ for $n > 2$. The hazard rates $\lambda_2(x)$ of $\mathrm{PRW}(2, x)$ and $\lambda_{n,l}(x)$ of PRWL are respectively

$$\lambda_2(x) = \frac{\frac{2}{\pi}(4 - x^2)^{-1/2}}{1 - \frac{2}{\pi} \arcsin \frac{1}{2} x} \quad \text{and} \quad \lambda_{n,l}(x) = \frac{2x}{n}.$$

The figure 2.2. gives the hazard rates of $\lambda_2(x)$ and $\lambda_{5,l}(x)$.

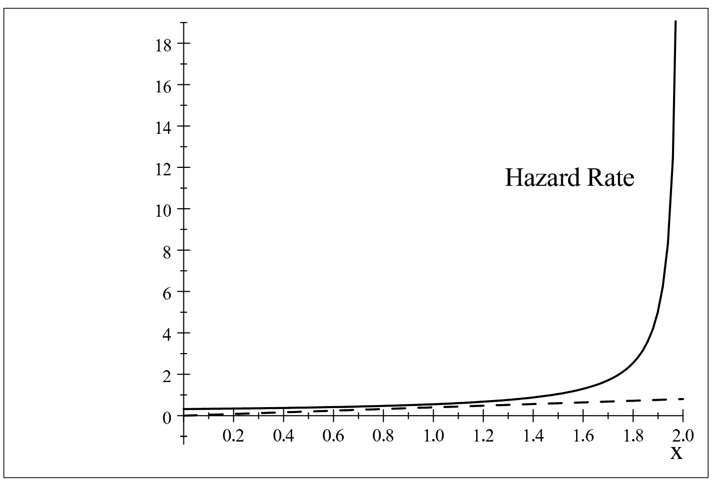

Figure 2.2. Hazard Rates-$\lambda_2(x)$-solid and $\lambda_{5,l})x)$-Dash.

Suppose that $X_1, X_2, \ldots X_m$ are independent copies of the random variable X. Let $M_m = \max(X_1, X_2, \ldots, X_m)$ and $M_{(m)} = \min(X_1, X_2, \ldots, X_m)$.

If the random variable X has PRWL(n, x) distribution, then (See Ahsanullah and Nevzorov (2001)) M_m belongs to the domain of attraction of Type 1 extreme value distribution of the maximum and $M_{(m)}$ belongs to the domain of attraction of type III distribution of the minimum. However it is difficult to determine the domains of attractions M_m and $M_{(m)}$ if the random variable X has PRW(n, x) for $n > 2$ distribution. Here the domain of attraction of M_m and $M_{(m)}$ when X has the distribution PRW(2) will be given.

We have

$$\lim_{m\to\infty} \frac{2\sin\frac{\pi}{2}(\frac{1}{m}) - 2\sin\frac{\pi}{2}(\frac{2}{m})}{2\sin\frac{\pi}{2}(\frac{2}{m}) - 2\sin\frac{\pi}{m}(\frac{4}{n})} = 2^{-1}$$

and

$$\lim_{m\to\infty} \frac{2\sin\frac{\pi}{2}(1-\frac{1}{m}) - 2\sin\frac{\pi}{2}(1-\frac{2}{m})}{2\sin\frac{\pi}{2}(1-\frac{2}{m}) - 2\sin\frac{\pi}{m}(1-\frac{4}{n})} = 2^{-2}.$$

It follows from Theorems 2.1.5 and 2.1.9 of Ahsanullah and Nevzorov (2001) that M_m belongs to the domain of attraction of type III distribution of the maximum with $F(x) = e^{-x}$, $x < 0$ and $M_{(m)}$ belongs to the domain of attraction of type III distribution of minimum with cdf $F(x) = 1 - e^{-x^2}$, $x > 0$.

It is difficult to calculate the Shannon entropy of PRW(n, x) for $n > 2$. The Shannon entropy of PRW$(2, x)$ and PRWL(n, x) will be given here.

Let EN$(2, x)$ be the Shannon entropy of PRW$(2, x)$, then

$$-EN(2,x) = \int_0^2 \ln\left(\frac{2}{\pi}\left(4-x^2\right)^{-1/2}\right)\frac{2}{\pi}\left(4-x^2\right)^{-1/2}dx$$
$$= \ln\left(\frac{2}{\pi}\right) - \frac{1}{\pi}\int_0^2 \ln\left(4-x^2\right)\left(4-x^2\right)^{-1/2}dx.$$

Let $x = 2\sin\theta$, then

$$-EN(2,x) = \ln\left(\frac{2}{\pi}\right) - \frac{1}{\pi}\int_0^{\pi/2} \ln\left(4\cos^2\theta\right)\frac{2\cos\theta}{2\cos\theta}\,d\theta$$
$$= \ln\left(\frac{2}{\pi}\right) - \frac{1}{\pi}\int_0^{\pi/2} \left(2\ln 2 + 2\ln\cos\theta\right)d\theta$$

$$= \ln 2 - \ln \pi - \ln 2 - \frac{2}{\pi} \int\limits_0^{\pi/2} \ln \cos \theta \, d\theta$$

$$= -\ln \pi + \frac{2}{\pi} \frac{\pi}{2} \ln 2 = -\ln \pi + \ln 2.$$

If ENL(n, x) be the Shannon entropy of NRWL(n, x), then

$$-ENL(n,x) = \int\limits_0^{\infty} \ln \left(\frac{2x}{n} e^{-\frac{x^2}{n}} \right) \frac{2x}{n} e^{-\frac{x^2}{n}} \, dx$$

$$= \ln \left(\frac{2}{n} \right) + \int\limits_0^{\infty} \left(\ln x - \frac{x^2}{n} \right) \frac{2x}{n} \; e^{-\frac{x^2}{n}} \, dx$$

$$= \ln 2 - \ln n + \frac{1}{2} \ln n - \frac{1}{2}\gamma - 1,$$

where γ is Euler's constant. Thus

$$ENL(n, x) = \; -\ln 2 + \frac{1}{2} \ln n + \frac{1}{2}\gamma + 1.$$

§3. Characterizations

The following two theorems give the characterizations of the PRW$(2, x)$ by the truncated $m(> 0)$-th moment.

Theorem 3.1. *Suppose that the random variable X is absolutely continuous with cdf $F(x)$ with $F(0) = 0$, $F(x) > 0$, for $0 < x < 2$, $F(x) = 1$ for $x \geqslant 2$ and pdf $f(x)$. Assume that $\mathbf{E}(X^m)$ exists for $m \geqslant 1$. Then*

$$\mathbf{E}(X^m | X \leqslant x) = g(x)\tau(x),$$

where

$$\tau(x) = \frac{f(x)}{F(x)}, \quad g(x) = \frac{p(x)}{2} \left(4 - x^2\right)^{1/2},$$

$$p(x) = \frac{x^{m+1}}{(m+1)\pi} + \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(2k-1)!!x^{n+2k+1}}{(m+2k+1)2^{3k}k!}$$

if and only if $f(x) = \frac{2}{\pi}\left(4 - x^2\right)^{-1/2}$, $0 \leqslant x \leqslant 2$.

Proof. If $f(x) = \frac{2}{\pi}\left(4 - x^2\right)^{-1/2}$, then

$$f(x)g(x) = \int\limits_0^x \frac{2u^m}{\pi}\left(4 - u^2\right)^{-1/2} du = \frac{2}{\pi}\int\limits_0^x (2x)^m \left(1 - x^2\right)^{-1/2} dx$$

$$= \frac{2^{m+1}}{\pi}\left(\int\limits_0^x x^m \left(1 + \sum_{k=1}^{\infty} \frac{(2k-1)!!x^{2k}}{2^k k!}\right) du\right),$$

with $(2k-1)!! = 1 \cdot 3 \cdot 5 \ldots (2k-1)$

$$= \frac{2^{m+1}}{\pi}\left(\frac{x^{m+1}}{m+1} + \sum_{k=0}^{\infty} \frac{(2k-1)!!x^{n+2k+1}}{(m+2k+1)2^k k!}\right)$$

$$= \frac{1}{\pi} p(x), say.$$

Thus $g(x) = \frac{p(x)}{2}\left(4 - x^2\right)^{1/2}$.

Suppose $g(x) = \frac{p(x)}{2}\left(4 - x^2\right)^{1/2}$, then

$$g'(x) = x^m - \frac{p(x)}{2}\left(4 - x^2\right)^{1/2}\left(\frac{x}{4 - x^2}\right) = x^m - g(x)\left(\frac{x}{4 - x^2}\right)$$

and

$$\frac{x^m - g'(x)}{g(x)} = \frac{x}{4 - x^2}.$$

We have $\frac{x^m - g'(x)}{g(x)} = \frac{f'(x)}{f(x)}$.

Thus

$$\frac{f'(x)}{f(x)} = \frac{x}{4 - x^2}.$$

Integrating both sides of the above equation with respect to x, we obtain $f(x) = c\left(4 - x^2\right)^{-1/2}$ where c is a constant.

Using the condition $\int\limits_0^2 f(x)\,dx = 1$, we obtain

$$f(x) = \frac{2}{\pi}\left(4 - x^2\right)^{-1/2}, \quad 0 \leqslant x \leqslant 2. \qquad \square$$

Theorem 3.2. *Suppose that the random variable* X *is absolutely continuous with cdf* $F(x)$ *with* $F(0) = 0$, $F(x) > 0$, *for* $0 < x < 2$ *and* $F(x) = 1$ *for* $x \geqslant 2$ *and the pdf of* X *is* $f(x)$. *Assume that* $\mathbf{E}(X^m)$ *exists for* $m \geqslant 1$. *Then* $\mathbf{E}(X^m | X \geqslant x) = h(x)r(x)$, *where* $r(x) = \frac{f(x)}{1-F(x)}$,

$h(x) = \frac{\pi q(x)}{2}\left(4-x^2\right)^{1/2}$, $q(x) = \mathbf{E}(X) - \frac{1}{\pi}p(x)$ *if and only if* $f(x) = \frac{2}{\pi}\left(4-x^2\right)^{-1/2}$, $0 \leqslant x \leqslant 2$.

Proof. If $f(x) = \frac{2}{\pi}\left(4-x^2\right)^{-1/2}$ then

$$\begin{aligned}
f(x)h(x) &= \int_x^2 \frac{2u^m}{\pi}\left(4-u^2\right)^{-1/2} du \\
&= \mathbf{E}(X^m) - \int_0^x \frac{2u^m}{\pi}\left(4-u^2\right)^{-1/2} du \\
&= \mathbf{E}(X^m) - \frac{1}{\pi}p(x) = q(x), \textit{say}. \\
\mathbf{E}(\ X^m) &= \int_0^2 \frac{2u^m}{\pi}\left(4-u^2\right)^{-1/2} du \\
&= \int_0^{\pi/2} \frac{2^{m+1}}{\pi}\sin^m\theta\, d\theta \\
&= \frac{2^{2n+1}}{\pi}\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)}\frac{\pi}{2} \\
&= 2^{2n}\frac{(2n-1)!!}{(2n)!!} \quad \text{if } m = 2n \\
&= \frac{2^{2n+2}}{\pi}\frac{1\cdot 3\cdot 5\ldots(2n)}{1\cdot 3\cdot 5\ldots(2n+1)} \\
&= \frac{2^{2n+2}}{\pi}\frac{(2n)!!}{(2n+1)!!} \quad \text{if } m = 2n+1.
\end{aligned}$$

where $(2n)!! = 2\cdot 4\cdot 6\ldots(2n)$ and $(2n+1)!! = 1\cdot 3\cdot 5\ldots(2n+1)$.
Thus

$$\begin{aligned}
h(x) &= \frac{\pi q(x)}{2\left(4-x^2\right)^{-1/2}} \\
h'(x) &= -\ x^m - \frac{\pi q(x)}{2\left(4-x^2\right)^{-1/2}}\left(\frac{x}{4-x^2}\right)
\end{aligned}$$

$$= -x^m - h(x)\left(\frac{x}{4-x^2}\right)$$

and

$$-\frac{x^m + h'(x)}{h(x)} = \frac{x}{4-x^2}.$$

We have $-\frac{x^m+h'(x)}{h(x)} = \frac{f'(x)}{f(x)}$.

Thus

$$\frac{f'(x)}{f(x)} = \frac{x}{4-x^2}.$$

Integrating both sides of the above equation with respect to x, we obtain $f(x) = c\left(4-x^2\right)^{-1/2}$ where c is a constant.

Using the condition $\int\limits_0^2 f(x)dx = 1$, we obtain

$$f(x) = \frac{2}{\pi}\left(4-x^2\right)^{-1/2}, \quad 0 \leqslant x \leqslant 2. \qquad \square$$

The following two theorems give the characterizations of PRWL(n,x) based on the truncated m-th moment.

Theorem 3.3. *Suppose that the random variable X is absolutely continuous with cdf $F(x)$ with $F(0) = 0$, $F(x) > 0$, for all $x > 0$ and pdf $f(x)$. Assume that $\mathbf{E}(X^m)$, $m > 0$ exists. Then*

$$\mathbf{E}(X^m | X \leqslant x) = \alpha(x)\tau(x), \quad \textit{where} \quad \alpha(x) = \frac{n^{m/2+1}\Gamma_{\frac{x^2}{n}}(m+1)}{2xe^{-\frac{x^2}{n}}}$$

and

$$\tau(x) - \frac{f(x)}{F(x)}, \quad \textit{if and only if} \quad f(x) = \frac{2x}{n}e^{-\frac{x^2}{n}}, \quad x \geqslant 0.$$

Proof. Suppose $f(x) = \frac{2x}{n}e^{-\frac{x^2}{n}}$, then

$$f(x)\alpha(x) = \int\limits_0^x \frac{2u^{m+1}}{n}e^{-\frac{u^2}{n}}\,du$$

Let $u^2/n = t$, then

$$f(x)\alpha(x) - \int\limits_0^{x^2/n} (tn)^{m/2}e^{-t}\,dt$$

$$= n^{m/2}\Gamma_{\frac{x^2}{n}}\left(\frac{m}{2}+1\right) = u(x), say,$$

where

$$\Gamma_a(b) = \int\limits_0^a x^{b-1}e^{-t}\,dt.$$

Thus

$$\alpha(x) = \frac{nu(x)}{2xe^{-\frac{x^2}{n}}}.$$

Suppose

$$\alpha(x) = \frac{nu(x)}{2xe^{-\frac{x^2}{n}}},$$

then

$$\alpha'(x) = x^m - \frac{nu(x)}{2xe^{-x^2}}\left(\frac{1}{x}-\frac{2x}{n}\right) = x^m - \alpha(x)\left(\frac{1}{x}-\frac{2x}{n}\right).$$

Thus

$$\frac{x^m-\alpha'(x)}{\alpha(x)} = \left(\frac{1}{x}-\frac{2x}{n}\right).$$

We have

$$\frac{x^m-\alpha'(x)}{\alpha(x)} = \frac{f'(x)}{f(x)},$$

thus

$$\frac{f'(x)}{f(x)} = \left(\frac{1}{x}-\frac{2x}{n}\right).$$

Integrating both sides of the above equation with respect to x and using the condition $\int\limits_0^\infty f(x)\,dx = 1$, we obtain

$$f(x) = \frac{2x}{n}e^{-\frac{x^2}{n}}, \quad x \geqslant 0. \qquad \square$$

Theorem 3.4. *Suppose the random variable X is absolutely continuous with cdf $F(x)$ with $F(0) = 0$, $F(x) > 0$, for all $x > 0$ and pdf $f(x)$. Assume that $\mathbf{E}(X^m)$, $m > 0$ exists. Then $\mathbf{E}(X^m|X \geqslant x) = \beta(x)r(x)$, where $r(x) - \frac{f(x)}{1-F(x)}$*

$\beta(x) = \frac{n(\mathbf{E}(X^m)-u(x))}{2xe^{-\frac{x^2}{n}}}$, and $\mathbf{E}(X^m) = (n)^{m/2}\Gamma\left(\frac{m}{2}+1\right)$ if and only if

$f(x) = \frac{2x}{n}e^{-\frac{x^2}{n}}, x \geqslant 0.$

Proof. Suppose $f(x) = \frac{2x}{n} e^{-\frac{x^2}{n}}$, then

$$f(x)\beta(x) = \int_x^\infty \frac{2u^{m+1} e^{-u^2\,du}}{n}$$

$$= \mathbf{E}(X^m) - \int_0^x \frac{2u^{m+1} e^{-u^2\,du}}{n} = \mathbf{E}(X^m) - u(x),$$

with $\mathbf{E}(X^m) = (n)^{m/2}\Gamma\big(\frac{m}{2}+1\big)$.

Thus

$$\beta(x) = \frac{n(\mathbf{E}(X^m) \quad - u(x))}{2xe^{-\frac{x^2}{n}}},$$

$$\beta'(x) = -x^m - \frac{n(\mathbf{E}(X^m) \quad - u(x))}{2xe^{-\frac{x^2}{n}}}\left(\frac{1}{x} - \frac{2x}{n}\right),$$

$$= -x^m - \beta(x)\left(\frac{1}{x} - \frac{2x}{n}\right),$$

and

$$-\frac{x^m + \beta'(x)}{\beta(x)} = \left(\frac{1}{x} - \frac{2x}{n}\right).$$

Since

$$-\frac{x^m + \beta'(x)}{\beta(x)} = \frac{f'(x)}{f(x)},$$

we will have

$$\frac{f'(x)}{f(x)} = \left(\frac{1}{x} - \frac{2x}{n}\right).$$

Integrating both sides of the above equation with respect to x and using the condition $\int_0^\infty f(x)\,dx = 1$, we obtain

$$f(x) = \frac{2x}{n} e^{-\frac{x^2}{n}}, \quad x \geqslant 0. \qquad \square$$

Remark 1. For some characterizations of Pearson's two unequal step random walk see Ahsanullah (2020).

Remark 2. It will be interesting to use the ideas presented by Volkova, Karakulov, and Nikitin (2017) to test the goodness of fit of PRW$(2, x)$ using the characterization Theorems 3.1 and 3.2.

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Department of Management
Sciences, Rider University,
2083 Lawrenceville Road,
Lawrenceville, NJ, 08648 USA
E-mail: ahsan@rider.edu

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Department of Mathematics and Mechanics,
St.Petersburg State University,
Universitetskaya nab., 7/9,
199034 St.Petersburg, Russia
E-mail: valnev@mail.ru

Institute of Physics, Mathematics and
Information Technology,
Immanuel Kant Baltic Federal University,
A. Nevskogo 14, 236041 Kaliningrad, Russia
E-mail: alexeistep45@mail.ru