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ELEMENTARY COVERING NUMBERS IN ODD-DIMENSIONAL UNITARY GROUPS

Abstract. Let (K,Δ) be a Hermitian form field and $n \geqslant 3$. We prove that if $\sigma \in \mathrm{U}_{2n+1}(K,\Delta)$ is a unitary matrix of level (K,Δ), then any short root transvection $T_{ij}(x)$ is a product of 4 elementary unitary conjugates of σ and σ^{-1}. Moreover, the bound 4 is sharp. We also show that any extra short root transvection $T_i(x,y)$ is a product of 12 elementary unitary conjugates of σ and σ^{-1}. If the level of σ is $(0,K\times 0)$, then any $(0,K\times 0)$-elementary extra short root transvection $T_i(x,0)$ is a product of 2 elementary unitary conjugates of σ and σ^{-1}.

§1. Introduction

The investigation of products of conjugacy classes in different types of groups is a popular topic in group theory during the last 30–40 years. Many papers were devoted to this theme, for example [1, 4–14, 17, 18]. A lot of these works are concerned with the computation of covering numbers.

Let G be a group and and $S \subseteq G$ a subset. For any subset $X \subseteq G$ define $\mathrm{cn}_X(S)$ as the least positive integer m such that $S \subseteq X^m = \{x_1 \dots x_m \mid x_1,\dots,x_m \in X\}$. If no such m exists, then $\mathrm{cn}_X(S) := \infty$. We call $\mathrm{cn}_X(S)$ the *covering number of S with respect to X*. For any set $\mathcal{X}$ of subsets of G, define $\mathrm{cn}_{\mathcal{X}}(S)$ as the supremum of all covering numbers $\mathrm{cn}_X(S)$, where $X \in \mathcal{X}$. We call $\mathrm{cn}_{\mathcal{X}}(S)$ the *covering number of S with respect to $\mathcal{X}$*. Note that if $\mathcal{X}$ is the set of all conjugacy classes in G that are not contained in a proper normal subgroup, then $\mathrm{cn}_{\mathcal{X}}(G)$ is the usual covering number $\mathrm{cn}(G)$ as defined in [8]. We call $\mathrm{scn}_X(S) := \mathrm{cn}_{X\cup X^{-1}}(S)$ the *symmetric covering number of S with respect to X* and $\mathrm{scn}_{\mathcal{X}}(S) := \sup\{\mathrm{scn}_X(S) \mid X \in \mathcal{X}\}$ the *symmetric covering number of S with respect to $\mathcal{X}$*.

The hyperbolic unitary groups $\mathrm{U}_{2n}(R,\Lambda)$ were defined by A. Bak in 1969 [2]. They embrace the classical Chevalley groups of type C_m and D_m, namely the even-dimensional symplectic and orthogonal groups $\mathrm{Sp}_{2n}(R)$

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and $O_{2n}(R)$. In 2018, A. Bak and the author defined odd-dimensional unitary groups $U_{2n+1}(R,\Delta)$ [3]. These groups generalise the even-dimensional unitary groups $U_{2n}(R,\Lambda)$ and embrace all classical Chevalley groups. The groups $U_{2n+1}(R,\Delta)$ are in turn embraced by V. Petrov's odd unitary groups, which were introduced in [15].

Let (K,Δ) be a Hermitian form field and $n \geqslant 3$. Denote the odd unitary group $U_{2n+1}(K,\Delta)$ by G and its elementary subgroup $EU_{2n+1}(K,\Delta)$ by E. It follows from the Sandwich Classification Theorem 28 that if H is a subgroup of G normalised by E, then there is a unique odd form ideal (I,Ω) of (K,Δ) such that

$$EU_{2n+1}((K,\Delta),(I,\Omega)) \subseteq H \subseteq CU_{2n+1}((K,\Delta),(I,\Omega)),$$

where $EU_{2n+1}((K,\Delta),(I,\Omega))$ denotes the relative elementary subgroup of level (I,Ω) and $CU_{2n+1}((K,\Delta),(I,\Omega))$ the full congruence subgroup of level (I,Ω). The odd form ideal (I,Ω) is called the *level* of H. The *level* of a conjugacy class C in G is the level of the subgroup of G generated by C.

Let $\mathcal{C}$ denote the set of all conjugacy classes of level (K,Δ), S_{short} the set of all nontrivial short root transvections, and S_{extra} the set of all nontrivial extra short root transvections. In this paper we prove that $\text{scn}_{\mathcal{C}}(S_{\text{short}}) \leqslant 4$ and $\text{scn}_{\mathcal{C}}(S_{\text{extra}}) \leqslant 12$. Moreover, we show that the bound $\text{scn}_{\mathcal{C}}(S_{\text{short}}) \leqslant 4$ is sharp, i.e., there is no better bound valid for all Hermitian form fields (K,Δ) and $n \geqslant 3$.

If the Hermitian form B is degenerate and $K \times 0 \subseteq \Delta$, then there is a second nonzero odd form ideal, namely, $(0, K \times 0)$. Let $\mathcal{D}$ denote the set of all conjugacy classes of level $(0, K \times 0)$ and T the set of all nontrivial $(0, K \times 0)$-elementary extra short root transvections. We prove that $\text{scn}_{\mathcal{D}}(T) = 1$ if $K = \mathbb{F}_2$ and $(0,1) \in \Delta$, and $\text{scn}_{\mathcal{D}}(T) = 2$ otherwise.

The rest of the paper is organised as follows. In Sec. 2 we recall some standard notation which is used throughout the paper. In Sec. 3, we recall the definitions of the groups $U_{2n+1}(R,\Delta)$ and some important subgroups. In Sec. 4, we prove our main results, namely, Theorems 33, 35, 36, and 39. The results of Sec. 4 are still valid if one replaces all occurrences of "conjugacy class" by "E-class" or more generally by "H-class," where $E \leqslant H \leqslant G$ is a fixed intermediate group.

§2. Notation

$\mathbb{N}$ denotes the set of all positive integers. If G is a group and $g,h \in G$, we let $g^h := h^{-1}gh$, ${}^hg := hgh^{-1}$, and $[g,h] := ghg^{-1}h^{-1}$. By a ring we mean an associative ring with 1 such that $1 \neq 0$. By an ideal we mean a two-sided ideal. If $m, n \in \mathbb{N}$ and R is a ring, then the set of all $m \times n$ matrices over R is denoted by $\mathrm{M}_{m\times n}(R)$. Instead of $\mathrm{M}_{n\times n}(R)$ we may write $\mathrm{M}_n(R)$. If $\sigma \in M_{m\times n}(R)$, we denote the transpose of σ by σ^t, the entry of σ at position (i,j) by σ_{ij}, the ith row of σ by σ_{i*} and the jth column of σ by σ_{*j}. The group of all invertible matrices in $\mathrm{M}_n(R)$ is denoted by $\mathrm{GL}_n(R)$ and the identity element of $\mathrm{GL}_n(R)$ by e or $e_{n\times n}$. If $\sigma \in \mathrm{GL}_n(R)$, then the entry of σ^{-1} at position (i,j) is denoted by σ'_{ij}, the ith row of σ^{-1} by σ'_{i*} and the jth column of σ^{-1} by σ'_{*j}. Furthermore, we denote by ${}^nR$ the set of all row vectors of length n with entries in R and by R^n the set of all column vectors of length n with entries in R. We consider ${}^nR$ as left R-module and R^n as right R-module.

§3. Odd-dimensional unitary groups

We describe Hermitian form rings (R,Δ) and odd form ideals (I,Ω) first, then the odd-dimensional unitary group $\mathrm{U}_{2n+1}(R,\Delta)$ and its elementary subgroup $\mathrm{EU}_{2n+1}(R,\Delta)$ over a Hermitian form ring (R,Δ). For an odd form ideal (I,Ω), we recall the definitions of the following subgroups of $\mathrm{U}_{2n+1}(R,\Delta)$: the preelementary subgroup $\mathrm{EU}_{2n+1}(I,\Omega)$ of level (I,Ω), the elementary subgroup $\mathrm{EU}_{2n+1}((R,\Delta),(I,\Omega))$ of level (I,Ω), the principal congruence subgroup $\mathrm{U}_{2n+1}((R,\Delta),(I,\Omega))$ of level (I,Ω), the normalised principal congruence subgroup $\mathrm{NU}_{2n+1}((R,\Delta),(I,\Omega))$ of level (I,Ω), and the full congruence subgroup $\mathrm{CU}_{2n+1}((R,\Delta),(I,\Omega))$ of level (I,Ω).

3.1. Hermitian form rings and odd form ideals. First we recall the definitions of a ring with involution with symmetry and a Hermitian ring.

Definition 1. Let R be a ring and

$$\begin{aligned} \bar{\ }\colon R &\to R \\ x &\mapsto \bar{x} \end{aligned}$$

an antiisomorphism of R (i.e., $\bar{\ }$ is bijective, $\overline{x+y} = \bar{x}+\bar{y}$, $\overline{xy} = \bar{y}\bar{x}$ for any $x, y \in R$ and $\bar{1} = 1$). Furthermore, let $\lambda \in R$ such that $\bar{\bar{x}} = \lambda x \bar{\lambda}$ for any $x \in R$. Then λ is called a *symmetry* for $\bar{\ }$, the pair $(\bar{\ },\lambda)$ an *involution with symmetry* and the triple $(R,\bar{\ },\lambda)$ a *ring with involution with symmetry*. A subset $A \subseteq R$ is called *involution invariant* if and only if $\bar{x} \in A$ for any $x \in A$. A *Hermitian ring* is a quadruple $(R,\bar{\ },\lambda,\mu)$, where $(R,\bar{\ },\lambda)$ is a ring with involution with symmetry and $\mu \in R$ is a ring element such that $\mu = \bar{\mu}\lambda$.

Remark 2. Let $(R, ^-, \lambda, \mu)$ be a Hermitian ring.

(a) It is easy to show that $\bar{\lambda} = \lambda^{-1}$.

(b) The map

$$\_ : R \to R,$$
$$x \mapsto \underline{x} := \bar{\lambda}\bar{x}\lambda$$

is the inverse map of $^-$. One checks easily that $(R, \_ , \underline{\lambda}, \underline{\mu})$ is a Hermitian ring.

Next we recall the definition of an $R^\bullet$-module.

Definition 3. If R is a ring, let $R^\bullet$ denote the underlying set of the ring equipped with the multiplication of the ring, but not the addition of the ring. A *(right) $R^\bullet$-module* is a not necessarily Abelian group $(G, \dot{+})$ equipped with a map

$$\circ : G \times R^\bullet \to G,$$
$$(a, x) \mapsto a \circ x$$

such that the following holds:

(i) $a \circ 0 = 0$ for any $a \in G$,

(ii) $a \circ 1 = a$ for any $a \in G$,

(iii) $(a \circ x) \circ y = a \circ (xy)$ for any $a \in G$ and $x, y \in R$, and

(iv) $(a \dot{+} b) \circ x = (a \circ x) \dot{+} (b \circ x)$ for any $a, b \in G$ and $x \in R$.

Let G and G' be $R^\bullet$-modules. A group homomorphism $f : G \to G'$ satisfying $f(a \circ x) = f(a) \circ x$ for any $a \in G$ and $x \in R$ is called a *homomorphism of $R^\bullet$-modules*. A subgroup H of G which is $\circ$-stable (i.e., $a \circ x \in H$ for any $a \in H$ and $x \in R$) is called an *$R^\bullet$-submodule*. Moreover, if $A \subseteq G$ and $B \subseteq R$, we denote by $A \circ B$ the subgroup of G generated by $\{a \circ b \mid a \in A, b \in B\}$. We treat $\circ$ as an operator with higher priority than $\dot{+}$.

An important example of an $R^\bullet$-module is the Heisenberg group, which we define next.

Definition 4. Let $(R, ^-, \lambda, \mu)$ be a Hermitian ring. Define the map

$$\dot{+} : (R \times R) \times (R \times R) \to R \times R,$$
$$((x_1, y_1), (x_2, y_2)) \mapsto (x_1, y_1) \dot{+} (x_2, y_2) := (x_1 + x_2, y_1 + y_2 - \bar{x}_1 \mu x_2).$$

Then $(R\times R,\dot{+})$ is a group, which we call the *Heisenberg group* and denote by $\mathfrak{H}$. Equipped with the map

$$\circ : (R \times R) \times R^{\bullet} \to R \times R,$$
$$((x,y),a) \mapsto (x,y)\circ a := (xa, \bar{a}ya),$$

$\mathfrak{H}$ becomes an $R^{\bullet}$-module.

Remark 5. We denote the inverse of an element $(x,y) \in \mathfrak{H}$ by $\dot{-}(x,y)$. One checks easily that $\dot{-}(x,y) = (-x, -y - \bar{x}\mu x)$ for any $(x,y) \in \mathfrak{H}$.

In order to define the odd-dimensional unitary groups we need the notion of a Hermitian form ring.

Definition 6. Let $(R, {}^{-}, \lambda, \mu)$ be a Hermitian ring. Let $(R,+)$ have the $R^{\bullet}$-module structure defined by $x \circ a = \bar{a}xa$. Define the *trace map*

$$\mathrm{tr}\colon \mathfrak{H} \to R,$$
$$(x,y) \mapsto \bar{x}\mu x + y + \bar{y}\lambda.$$

One checks easily that tr is a homomorphism of $R^{\bullet}$-modules. Set

$$\Delta_{\min} := \{(0, x - \overline{x}\lambda) \mid x \in R\}$$

and

$$\Delta_{\max} := \ker(\mathrm{tr}).$$

An $R^{\bullet}$-submodule Δ of $\mathfrak{H}$ lying between $\Delta_{\min}$ and $\Delta_{\max}$ is called an *odd form parameter* for $(R, {}^{-}, \lambda, \mu)$. Since $\Delta_{\min}$ and $\Delta_{\max}$ are $R^{\bullet}$-submodules of $\mathfrak{H}$, they are respectively the smallest and largest odd form parameters. A pair $((R, {}^{-}, \lambda, \mu), \Delta)$ is called a *Hermitian form ring.* We shall usually abbreviate it by (R, Δ).

Next we define an odd form ideal of a Hermitian form ring.

Definition 7. Let (R, Δ) be a Hermitian form ring and I an involution invariant ideal of R. Set $J(\Delta) := \{y \in R \mid \exists z \in R : (y,z) \in \Delta\}$ and $\widetilde{I} := \{x \in R \mid \overline{J(\Delta)}\mu x \subseteq I\}$. Obviously $\widetilde{I}$ and $J(\Delta)$ are right ideals of R and $I \subseteq \widetilde{I}$. Moreover, set

$$\Omega^I_{\min} := \{(0, x - \bar{x}\lambda) \mid x \in I\} \dot{+} \Delta \circ I$$

and

$$\Omega^I_{\max} := \Delta \cap (\widetilde{I} \times I).$$

An $R^{\bullet}$-submodule Ω of $\mathfrak{H}$ lying between $\Omega^I_{\min}$ and $\Omega^I_{\max}$ is called a *relative odd form parameter of level I*. Since $\Omega^I_{\min}$ and $\Omega^I_{\max}$ are $R^{\bullet}$-submodules of $\mathfrak{H}$, they are respectively the smallest and the largest relative odd form parameters of level I. If Ω is a relative odd form parameter of level I, then (I, Ω) is called an *odd form ideal* of (R, Δ).

3.2. The odd-dimensional unitary group. Let (R, Δ) be a Hermitian form ring and $n \in \mathbb{N}$. Set $M := R^{2n+1}$. We use the following indexing for the elements of the standard basis of M: $(e_1, \dots, e_n, e_0, e_{-n}, \dots, e_{-1})$. That means that e_i is the column whose ith coordinate is one and all the other coordinates are zero if $1 \leqslant i \leqslant n$, the column whose $(n+1)$th coordinate is one and all the other coordinates are zero if $i = 0$, and the column whose $(2n+2+i)$th coordinate is one and all the other coordinates are zero if $-n \leqslant i \leqslant -1$. If $u \in M$, then we call $(u_1, \dots, u_n, u_{-n}, \dots, u_{-1})^t \in R^{2n}$ the *hyperbolic part* of u and denote it by u_{hb}. We set $u^* := \bar{u}^t$ and $u^*_{\text{hb}} := \bar{u}^t_{\text{hb}}$. Moreover, we define the maps

$$B : M \times M \to R,$$

$$(u, v) \mapsto u^* \begin{pmatrix} 0 & 0 & \pi \\ 0 & \mu & 0 \\ \pi\lambda & 0 & 0 \end{pmatrix} v = \sum_{i=1}^{n} \bar{u}_i v_{-i} + \bar{u}_0 \mu v_0 + \sum_{i=-n}^{-1} \bar{u}_i \lambda v_{-i}$$

and

$$Q : M \to \mathfrak{H},$$

$$u \mapsto (Q_1(u), Q_2(u)) := (u_0, u^*_{\text{hb}} \begin{pmatrix} 0 & \pi \\ 0 & 0 \end{pmatrix} u_{\text{hb}}) = (u_0, \sum_{i=1}^{n} \bar{u}_i u_{-i}),$$

where $\pi \in M_n(R)$ denotes the matrix with ones on the skew diagonal and zeros elsewhere.

Lemma 8 ([3, Lemma 12]). (i) *B is a λ-Hermitian form, i.e., B is biadditive,*

$$B(ux, vy) = \bar{x} B(u, v) y \ \forall u, v \in M, x, y \in R$$

and $B(u, v) = \overline{B(v, u)}\lambda \ \forall u, v \in M$.

(ii) $Q(ux) = Q(u) \circ x \ \forall u \in M, x \in R$, $Q(u + v) \equiv Q(u) \dot{+} Q(v) \dot{+} (0, B(u, v)) \bmod \Delta_{\min} \ \forall u, v \in M$ *and* $\operatorname{tr}(Q(u)) = B(u, u) \ \forall u \in M$.

Definition 9. The group

$$\begin{aligned} \mathrm{U}_{2n+1}(R, \Delta) := \{\sigma \in \mathrm{GL}_{2n+1}(R) \mid B(\sigma u, \sigma v) \\ = B(u, v) \wedge Q(\sigma u) \equiv Q(u) \bmod \Delta \ \forall u, v \in M\} \end{aligned}$$

is called the *odd-dimensional unitary group*.

Remark 10. The groups $\mathrm{U}_{2n+1}(R,\Delta)$ include as special cases the even-dimensional unitary groups $\mathrm{U}_{2n}(R,\Lambda)$ and all classical Chevalley groups. On the other hand, the groups $\mathrm{U}_{2n+1}(R,\Delta)$ are embraced by Petrov's odd unitary groups $\mathrm{U}_{2l}(R,\mathfrak{L})$. For details see [3, Remark 14(c) and Example 15].

Definition 11. We define the sets $\Theta_+ := \{1,\dots,n\}$, $\Theta_- := \{-n,\dots,-1\}$, $\Theta := \Theta_+ \cup \Theta_- \cup \{0\}$, and $\Theta_{\mathrm{hb}} := \Theta \setminus \{0\}$. Moreover, we define the map

$$\begin{aligned}\epsilon : \Theta_{\mathrm{hb}} &\to \{\pm 1\},\\ i &\mapsto \begin{cases} 1, & \text{if } i \in \Theta_+,\\ -1, & \text{if } i \in \Theta_-.\end{cases}\end{aligned}$$

Lemma 12 ([3, Lemma 17])**.** *Let $\sigma \in \mathrm{GL}_{2n+1}(R)$. Then $\sigma \in \mathrm{U}_{2n+1}(R,\Delta)$ if and only if Conditions (i) and (ii) below are satisfied.*

(i)

$$\begin{aligned}\sigma'_{ij} &= \lambda^{-(\epsilon(i)+1)/2}\bar{\sigma}_{-j,-i}\lambda^{(\epsilon(j)+1)/2}\ \forall i,j \in \Theta_{\mathrm{hb}},\\ \mu\sigma'_{0j} &= \bar{\sigma}_{-j,0}\lambda^{(\epsilon(j)+1)/2}\ \forall j \in \Theta_{\mathrm{hb}},\\ \sigma'_{i0} &= \lambda^{-(\epsilon(i)+1)/2}\bar{\sigma}_{0,-i}\mu\ \forall i \in \Theta_{\mathrm{hb}},\ \textit{and}\\ \mu\sigma'_{00} &= \bar{\sigma}_{00}\mu.\end{aligned}$$

(ii)

$$Q(\sigma_{*j}) \equiv (\delta_{0j},0) \bmod \Delta\ \forall j \in \Theta.$$

Lemma 13. *Let $\sigma \in \mathrm{U}_{2n+1}(R,\Delta)$. If $\sigma_{*j} = e_k x$ for some $j,k \in \Theta_{\mathrm{hb}}$ and invertible $x \in R$, then $\sigma_{-k,*} = (e_{-j}\hat{x})^t$, where $\hat{x} = \lambda^{(\epsilon(k)-1)/2}\bar{x}^{-1}\lambda^{(1-\epsilon(j))/2}$.*

Proof. Since $e = \sigma^{-1}\sigma$, we have

$$\delta_{ij} = (\sigma^{-1}\sigma)_{ij} = \sigma'_{i*}\sigma_{*j} = \sigma'_{i*}e_k x = \sigma'_{ik}x$$

for any $i \in \Theta$. It follows from the previous lemma that $\sigma_{-k,*} = (e_{-j}\hat{x})^t$. □

3.3. The polarity map.

Definition 14. The map

$$\begin{aligned}\tilde{}: M &\longrightarrow M^*,\\ u &\longmapsto \begin{pmatrix}\bar{u}_{-1}\lambda & \dots & \bar{u}_{-n}\lambda & \bar{u}_0\mu & \bar{u}_n & \dots & \bar{u}_1\end{pmatrix},\end{aligned}$$

where $M^* = {}^{2n+1}R$, is called the *polarity map*. Clearly $\tilde{}$ is *involutary linear*, i.e., $\widetilde{u+v} = \tilde{u} + \tilde{v}$ and $\widetilde{ux} = \bar{x}\tilde{u}$ for any $u,v \in M$ and $x \in R$.

Lemma 15 ([16, Lemma 16]). *If* $\sigma \in \mathrm{U}_{2n+1}(R,\Delta)$ *and* $u \in M$*, then* $\widetilde{\sigma u} = \widetilde{u}\sigma^{-1}$.

3.4. The elementary subgroup. We introduce the following notation. Let $(R, \underline{\ }, \underline{\lambda}, \underline{\mu})$ be the Hermitian ring defined in Remark 2(b) and $\mathfrak{H}^{-1}$ the corresponding Heisenberg group. Note that the underlying set of both $\mathfrak{H}$ and $\mathfrak{H}^{-1}$ is $R \times R$. We denote the group operation (respectively, scalar multiplication) of $\mathfrak{H}$ by $\dot{+}_1$ (respectively, $\circ_1$) and the group operation (respectively, scalar multiplication) of $\mathfrak{H}^{-1}$ by $\dot{+}_{-1}$ (respectively, $\circ_{-1}$). Furthermore, we set $\Delta^1 := \Delta$ and $\Delta^{-1} := \{(x,y) \in R \times R \mid (x,\bar{y}) \in \Delta\}$. One checks easily that $((R, \underline{\ }, \underline{\lambda}, \underline{\mu}), \Delta^{-1})$ is a Hermitian form ring. Analogously, if (I,Ω) is an odd form ideal of (R,Δ), we set $\Omega^1 := \Omega$ and $\Omega^{-1} := \{(x,y) \in R \times R \mid (x,\bar{y}) \in \Omega\}$. One checks easily that (I,Ω^{-1}) is an odd form ideal of (R,Δ^{-1}).

If $i,j \in \Theta$, let e^{ij} denote the matrix in $M_{2n+1}(R)$ with 1 in the (i,j)th position and 0 in all other positions.

Definition 16. If $i,j \in \Theta_{\mathrm{hb}}$, $i \neq \pm j$ and $x \in R$, the element

$$T_{ij}(x) := e + xe^{ij} - \lambda^{(\epsilon(j)-1)/2}\bar{x}\lambda^{(1-\epsilon(i))/2}e^{-j,-i}$$

of $\mathrm{U}_{2n+1}(R,\Delta)$ is called an (*elementary*) *short root transvection.* If $i \in \Theta_{\mathrm{hb}}$ and $(x,y) \in \Delta^{-\epsilon(i)}$, the element

$$T_i(x,y) := e + xe^{0,-i} - \lambda^{-(1+\epsilon(i))/2}\bar{x}\mu e^{i0} + ye^{i,-i}$$

of $\mathrm{U}_{2n+1}(R,\Delta)$ is called an (*elementary*) *extra short root transvection.* The extra short root transvections of the kind

$$T_i(0,y) = e + ye^{i,-i}$$

are called (*elementary*) *long root transvections.* If an element of $\mathrm{U}_{2n+1}(R,\Delta)$ is a short or extra short root transvection, then it is called an *elementary transvection.* The subgroup of $\mathrm{U}_{2n+1}(R,\Delta)$ generated by all elementary transvections is called the *elementary subgroup* and is denoted by $\mathrm{EU}_{2n+1}(R,\Delta)$.

Lemma 17 ([3, Lemma 20])**.** *The following relations hold for the elementary transvections*:

$$T_{ij}(x) = T_{-j,-i}(-\lambda^{(\epsilon(j)-1)/2}\bar{x}\lambda^{(1-\epsilon(i))/2}), \tag{S1}$$
$$T_{ij}(x)T_{ij}(y) = T_{ij}(x+y), \tag{S2}$$
$$[T_{ij}(x), T_{kl}(y)] = e \text{ if } k \neq j, -i \text{ and } l \neq i, -j, \tag{S3}$$
$$[T_{ij}(x), T_{jk}(y)] = T_{ik}(xy) \text{ if } i \neq \pm k, \tag{S4}$$
$$[T_{ij}(x), T_{j,-i}(y)] = T_i(0, xy - \lambda^{(-1-\epsilon(i))/2}\bar{y}\bar{x}\lambda^{(1-\epsilon(i))/2}), \tag{S5}$$
$$T_i(x_1,y_1)T_i(x_2,y_2) = T_i((x_1,y_1) \dot{+}_{-\epsilon(i)} (x_2,y_2)), \tag{E1}$$
$$[T_i(x_1,y_1), T_j(x_2,y_2)] = T_{i,-j}(-\lambda^{-(1+\epsilon(i))/2}\bar{x}_1\mu x_2) \text{ if } i \neq \pm j, \tag{E2}$$
$$[T_i(x_1,y_1), T_i(x_2,y_2)] = T_i(0, -\lambda^{-(1+\epsilon(i))/2}(\bar{x}_1\mu x_2 - \bar{x}_2\mu x_1)), \tag{E3}$$
$$[T_{ij}(x), T_k(y,z)] = e \text{ if } k \neq j, -i, \text{ and} \tag{SE1}$$
$$\begin{aligned}[T_{ij}(x), T_j(y,z)] = {} & T_{j,-i}(z\lambda^{(\epsilon(j)-1)/2}\bar{x}\lambda^{(1-\epsilon(i))/2}) \\ & \cdot T_i(y\lambda^{(\epsilon(j)-1)/2}\bar{x}\lambda^{(1-\epsilon(i))/2}, xz\lambda^{(\epsilon(j)-1)/2}\bar{x}\lambda^{(1-\epsilon(i))/2}).\end{aligned} \tag{SE2}$$

Definition 18. Let $x \in R$ be invertible and $i, j \in \Theta_{\text{hb}}$ be such that $i \neq \pm j$. Define

$$\begin{aligned} D_{ij}(x) := {} & e + (x-1)e^{ii} + (x^{-1}-1)e^{jj} \\ & + (\lambda^{(\epsilon(i)-1)/2}\bar{x}^{-1}\lambda^{-(\epsilon(i)-1)/2} - 1)e^{-i,-i} \\ & + (\lambda^{(\epsilon(j)-1)/2}\bar{x}\lambda^{-(\epsilon(j)-1)/2} - 1)e^{-j,-j} \\ = {} & T_{ij}(x-1)T_{ji}(1)T_{ij}(x^{-1}-1)T_{ji}(-x) \in \text{EU}_{2n+1}(R,\Delta). \end{aligned}$$

Clearly, $(D_{ij}(x))^{-1} = D_{ij}(x^{-1})$.

Definition 19. Let $i, j \in \Theta_{\text{hb}}$ be such that $i \neq \pm j$. Define

$$\begin{aligned} P_{ij} := {} & e - e^{ii} - e^{jj} - e^{-i,-i} - e^{-j,-j} + e^{ij} - e^{ji} + \lambda^{(\epsilon(i)-\epsilon(j))/2}e^{-i,-j} \\ & - \lambda^{(\epsilon(j)-\epsilon(i))/2}e^{-j,-i} \\ = {} & T_{ij}(1)T_{ji}(-1)T_{ij}(1) \in \text{EU}_{2n+1}(R,\Delta). \end{aligned}$$

Clearly, $(P_{ij})^{-1} = P_{ji}$.

The two lemmas below are easy to check.

Lemma 20. *Let $i, j, k \in \Theta_{\text{hb}}$ be such that $i \neq \pm j$ and $k \neq \pm i, \pm j$. Let $a \in R$ be invertible, $x \in R$, and $(y,z) \in \Delta^{-\epsilon(i)}$. Then*

(1) ${}^{D_{ik}(a)}T_{ij}(x) = T_{ij}(ax)$,
(2) ${}^{D_{kj}(a)}T_{ij}(x) = T_{ij}(xa)$, *and*
(3) ${}^{D_{-i,k}(a^{-1})}T_i(y, z) = T_i(ya, \lambda^{-(\epsilon(i)+1)/2}\bar{a}\lambda^{(\epsilon(i)+1)/2}za)$.

Lemma 21 ([3, Lemma 23])**.** *Let $i, j, k \in \Theta_{\text{hb}}$ be such that $i \neq \pm j$ and $k \neq \pm i, \pm j$. Let $x \in R$ and $(y, z) \in \Delta^{-\epsilon(i)}$. Then*
(i) ${}^{P_{ki}}T_{ij}(x) = T_{kj}(x)$,
(ii) ${}^{P_{kj}}T_{ij}(x) = T_{ik}(x)$, *and*
(iii) ${}^{P_{-k,-i}}T_i(y, z) = T_k(y, \lambda^{(\epsilon(i)-\epsilon(k))/2}z)$.

3.5. Relative subgroups. In this subsection, (I, Ω) denotes an odd form ideal of (R, Δ).

Definition 22. A short root transvection $T_{ij}(x)$ is called (I, Ω)-*elementary* if $x \in I$. An extra short root transvection $T_i(x, y)$ is called (I, Ω)-*elementary* if $(x, y) \in \Omega^{-\epsilon(i)}$. The subgroup $\mathrm{EU}_{2n+1}(I, \Omega)$ of $\mathrm{EU}_{2n+1}(R, \Delta)$ generated by the (I, Ω)-elementary transvections is called the *preelementary subgroup of level* (I, Ω). Its normal closure $\mathrm{EU}_{2n+1}((R, \Delta), (I, \Omega))$ in $\mathrm{EU}_{2n+1}(R, \Delta)$ is called the *elementary subgroup of level* (I, Ω).

If $\sigma \in M_{2n+1}(R)$, we call the matrix $(\sigma_{ij})_{i,j\in\Theta_{\text{hb}}} \in M_{2n}(R)$ the *hyperbolic part* of σ and denote it by σ_{hb}. Furthermore, we define the submodule $M(R, \Delta) := \{u \in M \mid u_0 \in J(\Delta)\}$ of M.

Definition 23. The subgroup

$$\mathrm{U}_{2n+1}((R, \Delta), (I, \Omega)) := \{\sigma \in \mathrm{U}_{2n+1}(R, \Delta) \mid \sigma_{\text{hb}} \equiv e_{\text{hb}} \bmod I \text{ and } Q(\sigma u) \equiv Q(u) \bmod \Omega \ \forall u \in M(R, \Delta)\}$$

of $\mathrm{U}_{2n+1}(R, \Delta)$ is called *the principal congruence subgroup of level* (I, Ω).

Lemma 24 ([3, Lemma 28])**.** *Let $\sigma \in \mathrm{U}_{2n+1}(R, \Delta)$. Then*

$$\sigma \in \mathrm{U}_{2n+1}((R, \Delta), (I, \Omega))$$

if and only if Conditions (i) *and* (ii) *below are satisfied.*
(i) $\sigma_{\text{hb}} \equiv e_{\text{hb}} \bmod I$.
(ii) $Q(\sigma_{*j}) \in \Omega \ \forall j \in \Theta_{\text{hb}}$ *and* $(Q(\sigma_{*0}) \dot{-} (1, 0)) \circ a \in \Omega \ \forall a \in J(\Delta)$.

Definition 25. The subgroup $\mathrm{NU}_{2n+1}((R, \Delta), (I, \Omega)) :=$

$$\mathrm{Normaliser}_{\mathrm{U}_{2n+1}(R,\Delta)}(\mathrm{U}_{2n+1}((R, \Delta), (I, \Omega)))$$

of $\mathrm{U}_{2n+1}(R, \Delta)$ is called the *normalised principal congruence subgroup of level* (I, Ω).

Definition 26. The subgroup $\mathrm{CU}_{2n+1}((R,\Delta),(I,\Omega)) :=$

$\{\sigma \in \mathrm{NU}_{2n+1}((R,\Delta),(I,\Omega)) \mid [\sigma, \mathrm{EU}_{2n+1}(R,\Delta)] \leqslant \mathrm{U}_{2n+1}((R,\Delta),(I,\Omega))\}$

of $\mathrm{U}_{2n+1}(R,\Delta)$ is called the *full congruence subgroup of level* (I,Ω).

3.6. The standard commutator formulas and the sandwich classification theorem. We call the ring R *quasifinite*, if it is a direct limit of subrings R_i $(i \in \Phi)$ which are almost commutative (i.e., finitely generated as modules over their centers), involution invariant, and contain λ and μ.

Theorem 27 ([3, Theorem 39])**.** *Suppose that R is quasifinite and $n \geqslant 3$. Then $\mathrm{EU}_{2n+1}((R,\Delta),(I,\Omega))$ is a normal subgroup of*

$$\mathrm{NU}_{2n+1}((R,\Delta),(I,\Omega))$$

and the standard commutator formulas

$$\begin{aligned} &[\mathrm{CU}_{2n+1}((R,\Delta),(I,\Omega)), \mathrm{EU}_{2n+1}(R,\Delta)] \\ =&[\mathrm{EU}_{2n+1}((R,\Delta),(I,\Omega)), \mathrm{EU}_{2n+1}(R,\Delta)] \\ =&\,\mathrm{EU}_{2n+1}((R,\Delta),(I,\Omega)) \end{aligned}$$

hold. In particular, from the absolute case $(I,\Omega) = (R,\Delta)$, it follows that $\mathrm{EU}_{2n+1}(R,\Delta)$ is perfect and normal in $\mathrm{U}_{2n+1}(R,\Delta)$.

Theorem 28 ([3, Theorem 80])**.** *Suppose that R is quasifinite and $n \geqslant 3$. Let H be a subgroup of $\mathrm{U}_{2n+1}(R,\Delta)$. Then H is normalised by*

$$\mathrm{EU}_{2n+1}(R,\Delta)$$

if and only if there is an odd form ideal (I,Ω) of (R,Δ) such that

$$\mathrm{EU}_{2n+1}((R,\Delta),(I,\Omega)) \subseteq H \subseteq \mathrm{CU}_{2n+1}((R,\Delta),(I,\Omega)).$$

Moreover, (I,Ω) is uniquely determined.

Recall that if H is a subgroup of $\mathrm{U}_{2n+1}(R,\Delta)$ normalised by

$$\mathrm{EU}_{2n+1}(R,\Delta),$$

then the uniquely determined odd form ideal (I,Ω) in Theorem 28 is called the level of H.

§4. Elementary covering numbers in $\mathrm{U}_{2n+1}(K,\Delta)$

In this section, $n \geqslant 3$ denotes an integer and (K,Δ) a Hermitian form field (i.e., (K,Δ) is a Hermitian form ring and K a field). We denote the odd unitary group $\mathrm{U}_{2n+1}(K,\Delta)$ by G and its elementary subgroup $\mathrm{EU}_{2n+1}(K,\Delta)$ by E.

If $\mu \neq 0$ or $K \times 0 \not\subseteq \Delta$, then there are only two odd form ideals in (K,Δ), namely, $(0,0)$ and (K,Δ). If $\mu = 0$ and $K \times 0 \subseteq \Delta$, then there is a third odd form ideal, namely, $(0, K\times 0)$. It follows from [3, Remark 26] that $\mathrm{NU}_{2n+1}((K,\Delta),(I,\Omega)) = \mathrm{U}_{2n+1}(K,\Delta)$ for any odd form ideal (I,Ω). Hence,

$$\mathrm{EU}_{2n+1}((K,\Delta),(I,\Omega)), \mathrm{U}_{2n+1}((K,\Delta),(I,\Omega))$$

and $\mathrm{CU}_{2n+1}((K,\Delta),(I,\Omega))$ are normal subgroups of $\mathrm{U}_{2n+1}(K,\Delta)$.

Recall that the level of a conjugacy class C in G is the level of the subgroup generated by C. In Subsec. 4.1 we investigate covering numbers with respect to conjugacy classes of level (K,Δ). In Subsec. 4.2 we investigate covering numbers with respect to conjugacy classes of level $(0, K\times 0)$.

Lemma 29. *Let C be a conjugacy class in G and σ an element of C. Set*

$$Y := \{\sigma_{ij}, \sigma_{ii} - \sigma_{jj}, \sigma_{i0} J(\Delta), \overline{J(\Delta)}\mu\sigma_{0j}, \overline{J(\Delta)}\mu(\sigma_{00} - \sigma_{jj})J(\Delta) \mid i,j \in \Theta_{\mathrm{hb}}, i \neq j\}$$

and

$$Z := \{Q(\sigma_{*j}), (Q(\sigma_{*0}) \dot{-} (1,0)) \circ y \dot{+} (y,z) \dot{-} (y,z) \circ \sigma_{ii} \mid i,j \in \Theta_{\mathrm{hb}}, (y,z) \in \Delta\}.$$

Let I be the involution invariant ideal generated by Y and set $\Omega := \Omega^I_{\min} \dot{+} Z \circ K$. Then (I,Ω) is the level of C.

Proof. If (J,Σ) and (J',Σ') are odd form ideals, then we call (J,Σ) *smaller* than (J',Σ') if $J \subseteq J'$ and $\Sigma \subseteq \Sigma'$. It order to prove the assertion of the lemma, it suffices to show that (I,Ω) is the smallest odd form ideal such that $\langle C\rangle \subseteq \mathrm{CU}_{2n+1}((R,\Delta),(I,\Omega))$. But that follows from [16, Lemma 31]. □

4.1. Elementary covering numbers with respect to conjugacy classes of level (K,Δ). We denote by $\mathcal{C}$ the set of all conjugacy classes of level (K,Δ), by S_{short} the set of all nontrivial short root transvections, and by S_{extra} the set of all nontrivial extra short root transvections. We will prove that $\mathrm{scn}_{\mathcal{C}}(S_{\text{short}}) \leqslant 4$ and $\mathrm{scn}_{\mathcal{C}}(S_{\text{extra}}) \leqslant 12$. In order to do that we need three lemmas.

Lemma 30. *Any two elements of S_{short} are conjugated.*

Proof. The lemma follows from Lemmas 20 and 21. □

In the following lemma, we drop the assumption that $n \geqslant 3$.

Lemma 31. *Let $n \geqslant 1$ and $\sigma \in \mathrm{U}_{2n+1}(K,\Delta)$. Then either $({}^{\tau}\sigma)_{*1} = e_{-1}x$ for some $\tau \in \mathrm{EU}_{2n+1}(K,\Delta)$ and $x \in K$, or $({}^{\tau}\sigma)_{*1} = e_2x$ for some $\tau \in \mathrm{EU}_{2n+1}(K,\Delta)$ and $x \in K$, or $\sigma_{*1} = e_1x + e_0y$ for some $x, y \in K$.*

Proof. First suppose that $\sigma_{-1,1} \neq 0$. Then $({}^{\tau}\sigma)_{*1} = e_{-1}x$ for some $x \in K$, where $\tau = (\prod_{i\neq 0,\pm 1} T_{i,-1}(*))T_1(*)$. Now suppose that $\sigma_{-1,1} = 0$ and $\sigma_{j1} \neq 0$ for some $j \neq 0, \pm 1$. We may assume that $j = 2$ (conjugate σ by a product of P_{kl}'s). Clearly, $({}^{\tau}\sigma)_{*1} = e_2x$ for some $x \in K$ where $\tau = (\prod_{i\neq 0,-1,\pm 2} T_{i2}(*))T_{-2}(*)$. The assertion of the lemma follows. □

Lemma 32. *Let $\sigma \in G$ be such that $\sigma_{*1} = e_2x$ for some $x \in K$. Then there is a $\tau \in E$ such that $({}^{\tau}\sigma)_{*1} = e_2x$ and $({}^{\tau}\sigma)_{i,-2} = 0$ for some $i \in \{\pm 3\}$.*

Proof. We may asume that $\sigma_{\pm 3,-2} \neq 0$ (otherwise we can choose $\tau = e$). First suppose that $\sigma_{-1,-2} \neq 0$. Then the assertion of the lemma holds with $\tau = T_{-3,-1}(-\sigma_{-3,-2}\,(\sigma_{-1,-2})^{-1})$. Suppose now that $\sigma_{-1,-2} = 0$ (note that we also have $\sigma_{-2,-2} = 0$ by Lemma 13). Then the assertion of the lemma holds with

$$\tau = (\prod_{i\neq 0,\pm 1,\pm 2,\pm 3} T_{i,-3}(*))T_{-3}(*).$$

□

Theorem 33. *Let $\mathcal{C}$ denote the set of all conjugacy classes of level (K,Δ) and S_{short} the set of all nontrivial short root transvections. Then*

$$\mathrm{scn}_{\mathcal{C}}(S_{\text{short}}) \leqslant 4.$$

Proof. Let $C \in \mathcal{C}$ and $\sigma \in C$. In order to prove the theorem it suffices to show that $\mathrm{scn}_C(S_{\text{short}}) \leqslant 4$.

Case 1. Suppose that $\sigma_{ij} \neq 0$ for some $i, j \in \Theta_{\text{hb}}, i \neq j$. Then there is a product τ of P_{kl}'s such that $({}^{\tau}\sigma)_{t1} \neq 0$ for some $t \neq 0, 1$. The proof of Lemma 31 shows that there is a $\rho \in E$ such that either $({}^{\rho\tau}\sigma)_{*1} = e_{-1}x$ or $({}^{\rho\tau}\sigma)_{*1} = e_2x$ for some $x \in K$. Set $\zeta := {}^{\rho\tau}\sigma$.

Subcase 1.1. Suppose that $\zeta_{*1} = e_{-1}x$. It follows from Lemma 13 that

$$\zeta = \begin{pmatrix} 0 & 0 & \widehat{x} \\ 0 & A & v \\ x & u & z \end{pmatrix}$$

for some $A \in \mathrm{M}_{2n-1}(K)$, $u \in \mathrm{M}_{1\times(2n-1)}(K)$, $v \in \mathrm{M}_{(2n-1)\times 1}(K)$, and $\widehat{x}, z \in K$. Clearly, $A \in \mathrm{U}_{2n-1}(K, \Delta)$ by Lemma 12. By Lemma 31 we may assume that $\zeta_{-3,2}$ (the penultimate entry of the first column of A) equals zero. One checks easily that $[T_{31}(1), [T_{1,-2}(1), \zeta]] = T_{3,-2}(1)$. It follows that $T_{3,-2}(1) \in CC^{-1}CC^{-1}$. Thus, $S_{\text{short}} \subseteq CC^{-1}CC^{-1}$ by Lemma 30.

Suppose that $\zeta_{*1} = e_2 x$. By Lemma 32, we may assume that $\zeta_{i,-2} = 0$ for some $i \in \{\pm 3\}$. By Lemma 13, we have $\zeta_{-2,*} = (e_{-1}\widehat{x})^t$ for some $\widehat{x} \in K$. One checks easily that $[T_{2i}(-1), [T_{12}(1), \zeta]] = T_{1i}(1)$. It follows that $T_{1i}(1) \in CC^{-1}CC^{-1}$. Thus, $S_{\text{short}} \subseteq CC^{-1}CC^{-1}$ by Lemma 30.

Suppose that $\sigma_{ij} = 0$ for any $i, j \in \Theta_{\text{hb}}, i \neq j$ and $\sigma_{kk} \neq \sigma_{ll}$ for some $k, l \in \Theta_{\text{hb}}, k \neq l$. Clearly $({}^{T_{kl}(1)}\sigma)_{kl} = \sigma_{ll} - \sigma_{kk} \neq 0$ and hence one can apply Case 1 to ${}^{T_{kl}(1)}\sigma$.

Suppose that $\sigma_{ij}, \sigma_{jj} - \sigma_{ii} = 0$ for any $i, j \in \Theta_{\text{hb}}, i \neq j$. Then σ has the form

$$\sigma = \left(\begin{array}{ccc|c|ccc} y & & & * & & & \\ & \ddots & & \vdots & & & \\ & & y & * & & & \\ \hline * & \dots & * & * & * & \dots & * \\ \hline & & & * & y & & \\ & & & \vdots & & \ddots & \\ & & & * & & & y \end{array}\right)$$

for some $y \in K$. Since the level of C is (K, Δ), it follows from Lemma 29 that $\mu \neq 0$ (note that if $\mu = 0$, then $\sigma_{i0} = 0$ for any $i \in \Theta_{\text{hb}}$ by Lemma 12).

Subcase 3.1. Suppose that $\sigma_{0j} \neq 0$ for some $j \in \Theta_{\text{hb}}$. By Lemma 12, we have $(\sigma_{0j}, 0) = Q(\sigma_{*j}) \in \Delta$. Choose an $i \in \Theta_{\text{hb}}$ such that $i \neq \pm j$. Then $({}^{T_i(\sigma_{0j},0)}\sigma)_{ij} \neq 0$ and hence one can apply Case 1 to ${}^{T_i(\sigma_{0j},0)}\sigma$.

Suppose that $\sigma_{0j} = 0$ for any $j \in \Theta_{\text{hb}}$. It follows from Lemma 13 that σ is a diagonal matrix. Since the level of C equals (K, Δ), it follows from Lemma 29 that $J(\Delta) = K$ and $\sigma_{00} \neq \sigma_{11}$. Since $J(\Delta) = K$ we can choose a $y \in K$ such that $(1, y) \in \Delta$. Clearly, $({}^{T_{-1}(1,y)}\sigma)_{01} = \sigma_{11} - \sigma_{00} \neq 0$ and hence one can apply Case 1 or Subcase 3.1 to ${}^{T_{-1}(1,y)}\sigma$. $\square$

The corollary below follows from Relation (SE2) in Lemma 17.

Corollary 34. *Let $\mathcal{C}$ denote the set of all conjugacy classes of level* (K,Δ) *and* S_{extra} *the set of all nontrivial extra short root transvections. Then* $\operatorname{scn}_{\mathcal{C}}(S_{\text{extra}}) \leqslant 12$.

Theorem 35. *Suppose that* $\bar{\ } = \mathrm{id}$, $\lambda = -1$, $\mu = 1$, *and* $\Delta = \Delta_{\max} = 0 \times K$ *(hence G is isomorphic to the symplectic group* $\mathrm{Sp}_{2n}(K)$*). Moreover, suppose that K has characteristic* 2. *Then* $\operatorname{scn}_{\mathcal{C}}(S_{\text{short}}) = 3$ *or* $\operatorname{scn}_{\mathcal{C}}(S_{\text{short}}) = 4$.

Proof. In view of Theorem 33 it suffices to find an $C \in \mathcal{C}$ such that $\operatorname{scn}_C(S_{\text{short}}) \geqslant 3$. Let C be the conjugacy class of $T_1(0,1)$. Note that $C = C^{-1}$ since K has characteristic 2. Assume that $T_{12}(1) \in C$. Then there is a $\sigma \in H$ such that ${}^{\sigma}T_1(0,1) = T_{12}(1)$. Let u be the first column of σ. It follows from Lemma 15 that ${}^{\sigma}T_1(0,1) = e + u\widetilde{u}$. Since by assumption ${}^{\sigma}T_1(0,1) = T_{12}(1)$, we obtain $u_1 \neq 0$. But then

$$0 = (T_{12}(1))_{1,-1} = (e + u\widetilde{u})_{1,-1} = u_1^2 \neq 0$$

which is absurd.
Assume now that $T_{12}(1) \in CC$. Then there are $\sigma, \tau \in H$ such that

$$({}^{\sigma}T_1(0,1))({}^{\tau}T_1(0,1)) = T_{12}(1)$$
$$\Leftrightarrow {}^{\sigma}T_1(0,1) = T_{12}(1)({}^{\tau}T_1(0,1)).$$

Let u and v be the first columns of σ and τ, respectively. It follows from Lemma 15 that

$$e + u\widetilde{u} = T_{12}(1)(e + v\widetilde{v}). \tag{1}$$

Let $i \in \Theta \setminus \{1, -2\}$. It follows from Equation (1) that

$$u_i\widetilde{u} = v_i\widetilde{v}. \tag{2}$$

Clearly, either $u_i, v_i \neq 0$ or $u_i, v_i = 0$. Assume $u_i, v_i \neq 0$. Then $\widetilde{v} = v_i^{-1}u_i\widetilde{u}$ which implies that $v = uk$ for some nonzero $k \in K$. Choose a $j \in \Theta$ such that $(\widetilde{u})_j \neq 0$. It follows from Equation (2) that $u_i(\widetilde{u})_j = v_i(\widetilde{v})_j = k^2u_i(\widetilde{u})_j$ whence $k^2 = 1$. Hence, $v\widetilde{v} = uk^2\widetilde{u} = u\widetilde{u}$ which leads to a contradiction (consider the first two rows of the matrices in Equation (1)). Hence, we have shown that $u_i, v_i = 0$ for any $i \in \Theta \setminus \{1, -2\}$. By considering the entries of the matrices in Equation (1) at positions $(1,2)$, $(1,-1)$, and $(-2,2)$, we obtain $u_1u_{-2} = v_1v_{-2} + 1$, $u_1^2 = v_1^2$, and $u_{-2}^2 = v_{-2}^2$. It follows that $(u_1u_{-2})^2 = (v_1v_{-2})^2 + 1 = (u_1u_{-2})^2 + 1$ which is absurd.
We have shown that neither $S_{\text{short}} \subseteq C$ nor $S_{\text{short}} \subseteq CC$. It follows that $\operatorname{scn}_C(S_{\text{short}}) \geqslant 3$. Thus, $\operatorname{scn}_{\mathcal{C}}(S_{\text{short}}) = 3$ or $\operatorname{scn}_{\mathcal{C}}(S_{\text{short}}) = 4$ by Theorem 33. □

Theorem 36. *Suppose that* $^{-}=\mathrm{id}$, $\lambda=-1$, $\mu=0$, *and* $\Delta=\Delta_{\max}=K\times K$ (*hence* G *is Proctor's odd symplectic group* $\mathrm{Sp}_{2n+1}(K)$, *see* [3, *Example* 15(4)]). *Moreover, suppose that* K *has characteristic* 2 *and contains an element of order* $\geqslant 4$. *Then* $\mathrm{scn}_{\mathcal{C}}(S_{\mathrm{short}})=4$.

Proof. In view of Theorem 33, it suffices to find an $C\in\mathcal{C}$ such that $\mathrm{scn}_C(S_{\mathrm{short}})\geqslant 4$. Choose an $x\in K$ of order $\geqslant 4$ and set

$$\alpha:=\mathrm{diag}(1,\dots,1,x,1,\dots,1)\in G,$$

where x is at position $(0,0)$. Let C be the conjugacy class of

$$\beta:=\alpha T_1(0,1)=e+e^{1,-1}+(x-1)e^{00}.$$

Assume that $T_{12}(1)\in C^{i_1}\dots C^{i_m}$ for some $m\in\{1,2,3\}$ and $i_1,\dots,i_m\in\{\pm1\}$. Since $\det(\beta)=x$ has order $\geqslant 4$, it follows that $m=2$ and $p_1=-p_2$. We only consider the case $p_1=1$ and $p_2=-1$ and leave the case $p_1=-1$ and $p_2=1$ to the reader. So assume that $T_{12}(1)\in CC^{-1}$. Then there are $\sigma,\tau\in H$ such that

$${}^{\sigma}\beta({}^{\tau}\beta^{-1})=T_{12}(1)\ \Leftrightarrow\ {}^{\sigma}\beta=T_{12}(1)({}^{\tau}\beta).$$

Let u and v be the first columns of σ and τ, respectively. It follows from Lemmas 12 and 15 that

$$e+u_{\mathrm{hb}}\widetilde{u}_{\mathrm{hb}}+e_0w=T_{12}(1)(e+v_{\mathrm{hb}}\widetilde{v}_{\mathrm{hb}}+e_0w') \tag{3}$$

for some $w,w'\in\mathrm{M}_{1\times(2n+1)}(K)$. Let $i\in\Theta_{hb}\setminus\{1,-2\}$. It follows from Equation (3) that

$$u_i\widetilde{u}_{\mathrm{hb}}=v_i\widetilde{v}_{\mathrm{hb}}. \tag{4}$$

Clearly, either $u_i,v_i\neq 0$ or $u_i,v_i=0$. Assume $u_i,v_i\neq 0$. Then $\widetilde{v}_{\mathrm{hb}}=v_i^{-1}u_i\widetilde{u}_{\mathrm{hb}}$ which implies that $v_{\mathrm{hb}}=u_{\mathrm{hb}}k$ for some nonzero $k\in K$. Choose a $j\in\Theta_{\mathrm{hb}}$ such that $(\widetilde{u})_j\neq 0$. It follows from Equation (4) that $u_i(\widetilde{u})_j=v_i(\widetilde{v})_j=u_ik^2(\widetilde{u})_j$ whence $k^2=1$. Hence, $v_{\mathrm{hb}}\widetilde{v}_{\mathrm{hb}}=u_{\mathrm{hb}}\widetilde{u}_{\mathrm{hb}}$ which leads to a contradiction (consider the first two rows of the matrices in Equation (3)). Hence we have shown that $u_i,v_i=0$ for any $i\in\Theta_{hb}\setminus\{1,-2\}$. By considering the entries of the matrices in Equation (3) at positions $(1,2)$, $(1,-1)$, and $(-2,2)$, we obtain $u_1u_{-2}=v_1v_{-2}+1$, $u_1^2=v_1^2$ and $u_{-2}^2=v_{-2}^2$. It follows that $(u_1u_{-2})^2=(v_1v_{-2})^2+1=(u_1u_{-2})^2+1$ which is absurd.
We have shown that there is no $m\in\{1,2,3\}$ and $i_1,\dots,i_m\in\{\pm1\}$ such that $S_{\mathrm{short}}\subseteq C^{i_1}\dots C^{i_m}$. It follows that $\mathrm{scn}_C(S_{\mathrm{short}})\geqslant 4$. Thus $\mathrm{scn}_{\mathcal{C}}(S_{\mathrm{short}})=4$ by Theorem 33. □

4.2. Elementary covering numbers with respect to conjugacy classes of level $(0, K \times 0)$**.** In this subsection we assume that $\mu = 0$ and $K \times 0 \subseteq \Delta$. We denote by $\mathcal{D}$ the set of all conjugacy classes of level $(0, K \times 0)$ and by T the set of all nontrivial $(0, K \times 0)$-elementary extra short root transvections. We will determine $\text{scn}_{\mathcal{D}}(T)$.

Lemma 37. *Any two elements of* T *are conjugated.*

Proof. The lemma follows from Lemmas 20 and 21. □

Lemma 38. *Let* $\mathcal{D}$ *denote the set of all conjugacy classes of level* $(0, K \times 0)$*, and* S *the set of all nontrivial* $(0, K \times 0)$*-elementary extra short root transvections. Then* $\text{scn}_{\mathcal{D}}(T) \leqslant 2$.

Proof. Let $D \in \mathcal{D}$ and $\sigma \in D$. In order to prove the theorem it suffices to show that $\text{scn}_D(T) \leqslant 2$. Since the level of D equals $(0, K \times 0)$, there is an $x \in K$ and $u, v \in \text{M}_{1\times n}(K)$ such that

$$\sigma = \begin{pmatrix} e_{n\times n} & 0 & 0 \\ u & x & v \\ 0 & 0 & e_{n\times n} \end{pmatrix}.$$

Case 1. Suppose that $\sigma_{0i} \neq 0$ for some $i \in \Theta_{\text{hb}}$. We may assume that $\sigma_{0j} = 0$ for some $j \in \Theta_{\text{hb}} \setminus \{\pm i\}$ (conjugate σ by $T_{ij}(-\sigma_{0i}^{-1}\sigma_{0j})$). One checks easily that $[\sigma, T_{i,-j}(1)] = T_j(\sigma_{0i}, 0)$. It follows that $T_j(\sigma_{0i}, 0) \in DD^{-1}$. Thus, $S \subseteq DD^{-1}$ by Lemma 37.

Suppose that $\sigma_{0i} = 0$ for any $i \in \Theta_{\text{hb}}$. Then $x \neq 1$ since the level of D equals $(0, K \times 0)$. One checks easily that $[\sigma, T_1(1, 0)] = T_1(x - 1, 0)$. It follows that $T_1(x - 1, 0) \in DD^{-1}$. Thus, $S \subseteq DD^{-1}$ by Lemma 37. □

Theorem 39. $\text{scn}_{\mathcal{D}}(T) = 1$ *if* $K = \mathbb{F}_2$ *and* $(0, 1) \in \Delta$*, and* $\text{scn}_{\mathcal{D}}(T) = 2$ *otherwise.*

Proof. Case 1. Suppose that $K = \mathbb{F}_2$ and $(0, 1) \in \Delta$. Let $D \in \mathcal{D}$ and $\sigma \in D$. Since the level of D equals $(0, K \times 0)$, there are $u, v \in \text{M}_{1\times n}(K)$ such that

$$\sigma = \begin{pmatrix} e_{n\times n} & 0 & 0 \\ u & 1 & v \\ 0 & 0 & e_{n\times n} \end{pmatrix}.$$

Moreover, $\sigma_{0i} = 1$ for some $i \in \Theta_{\text{hb}}$. One checks easily that $\sigma^\tau = T_{-i}(1, 0)$, where $\tau = (\prod_{j \neq \pm i} T_{ij}(*))T_i(0, *)$. It follows that $T_{-i}(1, 0) \in D$. Thus, $S \subseteq D$ by Lemma 37.

Suppose that $K=\mathbb{F}_2$ and $(0,1)\notin\Delta$. Set $\sigma:=T_1(1,0)T_{-1}(1,0)\in G$ and let D be the conjugacy class of σ. Then the level of D equals $(0,K\times 0)$. Assume that $\operatorname{scn}_D(T)=1$. Then there is a $\tau\in H$ such that ${}^{\tau}\sigma=T_{-1}(1,0)$. Since $\sigma=e+e_{01}+e_{0,-1}=e+e_0(e_1^t+e_{-1}^t)$, we obtain

$$\begin{aligned}&{}^{\tau}\sigma=T_{-1}(1,0)\\ \Leftrightarrow\ & e+e_0(e_1^t+e_{-1}^t)\tau^{-1}=e+e_0e_1^t\\ \Leftrightarrow\ & e_0(e_1^t+e_{-1}^t)=e_0e_1^t\tau\\ \Leftrightarrow\ & e_0(e_1^t+e_{-1}^t)=e_0\tau_{1*}\\ \Leftrightarrow\ & e_1^t+e_{-1}^t=\tau_{1*}.\end{aligned}$$

It follows from Lemma 12 that $\tau'_{*,-1}=e_1+e_0x+e_{-1}$ for some $x\in\mathbb{F}_2$. Hence, $(x,1)=Q(\tau'_{*,-1})\in\Delta$ (also by Lemma 12). Since by assumption $(0,1)\notin\Delta$, we obtain $(1,1)\in\Delta$. Since $(1,0)\in\Delta$, it follows that $(0,1)=(1,1)\dotplus(1,0)\in\Delta$ which contradicts the assumption that $(0,1)\notin\Delta$. Hence $\operatorname{scn}_D(T)\geqslant 2$. It follows from Lemma 38 that $\operatorname{scn}_{\mathcal{D}}(T)=2$.

Suppose that $K\neq\mathbb{F}_2$. Choose an $x\in K\setminus\{0,1\}$. Let D be the conjugacy class of $\operatorname{diag}(1,\dots,1,x,1,\dots,1)\in G$, where x is in position $(0,0)$. Then the level of D equals $(0,K\times 0)$. Since $\det(\operatorname{diag}(1,\dots,1,x,1,\dots,1))=x\neq 1$, we have $\operatorname{scn}_D(T)\geqslant 2$. It follows from Lemma 38 that $\operatorname{scn}_{\mathcal{D}}(T)=2$. □

4.3. Some open questions. As in Subsec. 4.1 we denote by $\mathcal{C}$ the set of all conjugacy classes of level (K,Δ), by S_{short} the set of all nontrivial short root transvections and by S_{extra} the set of all nontrivial extra short root transvections. By Theorem 33, we have $\operatorname{scn}_{\mathcal{C}}(S_{\text{short}})\leqslant 4$. By Theorem 36, 4 is the optimal uniform bound for $\operatorname{scn}_{\mathcal{C}}(S_{\text{short}})$ (valid for all Hermitian form fields (K,Δ) and $n\geqslant 3$). One can ask Questions 40 and 41 below.

Question 40. Can the bound $\operatorname{scn}_{\mathcal{C}}(S_{\text{short}})\leqslant 4$ be improved if one restricts to Hermitian form fields (K,Δ), where $\mu\neq 0$ (i.e., the Hermitian form B is nondegenerate) or 2 is invertible?

Question 41. What is the optimal bound for $\operatorname{scn}_{\mathcal{C}}(S_{\text{short}})$ for the classical Chevalley groups $\mathrm{Sp}_{2n}(K)$, $\mathrm{O}_{2n}(K)$ and $\mathrm{O}_{2n+1}(K)$, respectively?

By Corollary 34, we have $\operatorname{scn}_{\mathcal{C}}(S_{\text{extra}})\leqslant 12$. But it could be the case that this bound is not optimal.

Question 42. What is the optimal bound for $\operatorname{scn}_{\mathcal{C}}(S_{\text{extra}})$?

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