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N. Vavilov, V. Migrin

ENHANCED DYNKIN DIAGRAMS DONE RIGHT

Abstract. In the present paper we slightly modify the Dynkin–Minchenko construction of enhanced Dynkin diagrams and construct signed enhanced Dynkin diagrams of exceptional types $\Phi = \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$. We observe that these diagrams contain as subdiagrams all Carter–Stekolshchik diagrams of conjugacy classes of the Weyl groups $W(\Phi)$.

Introduction

In the present paper we draw the **enhanced Dynkin diagrams** of Eugene Dynkin and Andrei Minchenko [8] for senior exceptional types $\Phi = \mathrm{E}_6, \mathrm{E}_7$, and E_8 in a right way, *à la* Rafael Stekolshchik [26], indicating not just adjacency, but also the *signs* of inner products. Two vertices with inner product -1 are joined by a *solid line*, whereas two vertices with inner product $+1$ are joined by a *dotted line*.

Provisionally, in the absense of a better name, we call these creatures *signed* enhanced Dynkin diagrams. They are uniquely determined by the root system Φ itself, up to [a sequence of] the following tranformations: changing the sign of any vertex and simultaneously switching the types of all edges adjacent to that vertex.

Theorem 1. *Signed enhanced Dynkin diagrams of types* E_6*,* E_7*, and* E_8 *are depicted in Fig.* 1*,* 2 *and* 3*, respectively.*

In this form such diagrams contain not just the *extended Dynkin diagrams* of all root subsystems of Φ – that they did already by Dynkin and Minchenko [8] – but also all *Carter diagrams* [6] of conjugacy classes of the corresponding Weyl group $W(\Phi)$.

Theorem 2. *The signed enhanced Dynkin diagrams of types* E_6*,* E_7*, and* E_8 *contain all Carter–Stekolshchik diagrams of conjugacy classes of the Weyl groups* $W(\mathrm{E}_6)$*,* $W(\mathrm{E}_7)$*, and* $W(\mathrm{E}_8)$*.*

Key words and phrases: Exceptional root systems, root subsystems, conjugacy classes of the Weyl group, Dynkin diagrams, Carter diagrams.

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In both cases the non-parabolic root subsystems, and the non-Coxeter conjugacy classes occur by *exactly* the same single reason, the exceptional behaviour of D_4, where the fundamental subsystem can be rewritten as $4\,\mathrm{A}_1$, or as a 4-cycle, respectively, which constitutes one of the most manifest cases of the **octonionic mathematics** [2], so plumbly neglected by Vladimir Arnold (MATHEMATICA EST OMNIS DIVISA IN PARTES TRES, see [1]).

As a most immediate benefit, this provides an *extremely* powerful mnemonic tool, to easily reconstruct both the complete lists of root subsystems, *and* the complete lists of conjugacy classes of $W(\mathrm{E}_6)$, $W(\mathrm{E}_7)$, and $W(\mathrm{E}_8)$ within quarter of an hour on a scrap of paper. But as every powerful tool, it may have more than one use. In a subsequent paper, we plan to establish several further combinatorial results concerning these diagrams. However, here we concentrate on a construction of the pictures themselves, in the hyperbolic realisation of the root systems E_6, E_7, and E_8, see [15].

The present paper is organised as follows. In §1, we reproduce some historical background to place our diagrams in context. In §2, we recall some basic notation related to root systems and to the hyperbolic realisations of the exceptional root systems E_6, E_7, and E_8, that are used in subsequent calculations. In §3, we perform the inductive procedure *à la* Dynkin–Minchenko, to construct Figures 1–3 and prove Theorem 1. In §4, we list all non-Coxeter classes of the Weyl groups $W(\mathrm{E}_6)$, $W(\mathrm{E}_7)$, and $W(\mathrm{E}_8)$, depict all irreducible admissible diagrams with cycles occuring in these groups, Figures 5–6, and observe that all of them occur inside Figures 1–3, thus proving Theorem 2. Finally, in §5 we make some further comments regarding these pictures and some of their uses.

However, the main new bid of the present paper are the diagrams themselves, Figures 1–3. They are bound to have many further uses, apart from the ones of which we are aware today.

The present work, together with [19] constitutes a part of the *Bachelor Qualifying Paper* of the second-named author under the supervision of the first-named author.

§1. Some background

While reconsidering the combinatorial structure of the Gosset–Elte polytopes and calculating the corresponding cycle indices [19] we had an occasion to take another look at the subsystems of the exceptional root systems, and the conjugacy classes of their Weyl group.

Classification of both stocks of creatures are very classical and well-known. Up to conjugacy, susbsystems of root systems were determined by Armand Borel and Jacques de Siebenthal, who proposed a general method, and by Eugene Dynkin [7], who came up with explicit lists, in the late 1940-s and early 1950-s.

The [moderately] challenging cases were E_7 and E_8, which accomodate non-conjugate isomorphic subsystems. See also [32] and [20] for alternative approach and our papers [12, 30] for the explicit lists and some further related details.

1.1. Carter diagrams. Approximately simultaneously with the above, conjugacy classes of the Weyl groups $W(\mathrm{E}_6)$ and $W(\mathrm{E}_7)$ were determined by Sutherland Frame [9], the senior case of $W(\mathrm{E}_8)$ came later [10]. However, unlike the description of root subsystems, all of these papers addressed various types individually. The first [somewhat] uniform approach was only developed by Roger Carter [5, 6].

Very roughly, Carter's description looks as follows. Predominantly, conjugacy classes of the Weyl group $W(\Phi)$ are represented by Coxeter elements of subsystems of the root system $\Delta \leqslant \Phi$. Let Γ be a fundamental subsystem of Δ. Recall that a Coxeter element $w_\Delta = w_\Gamma$ of $W(\Delta) \leqslant W(\Phi)$ is the product of fundamental reflections w_α, $a \in \Gamma$, corresponding to the fundamental roots $\alpha \in \Gamma$ of Δ, their order is immarterial, since all such elements are conjugate. Overwhelmingly, non-conjugate root subsystems produce different conjugacy classes. However, not all conjugacy classes arise that way.

The missing conjugacy classes result from what Carter himself calls *admissible diagrams*, that nowadays are usually called **Carter diagrams**. Basically, these are diagrams constructed from *linearly independent subsets* of roots in exactly the same way as Dynkin diagrams. Namely, two roots α and β are joined with a single bond if the product $w_\alpha w_\beta$ of the corresponding reflections has order 3. Similarly, they are joined with a double bond if the order of $w_\alpha w_\beta$ is 4. We may safely forget G_2, where nothing new can possibly occur.

Unlike Dynkin diagrams themselves, Carter diagrams can contain cycles, admissibility amounts to the requirement that all of their cycles are even. Every Carter diagram C produces a conjugacy class in $W(\Phi)$ as follows. Unlike Coxeter elements, we cannot simply designate w_C as the product of all w_α, $a \in C$, since in the presence of cycles the conjugacy class of such a product can – and does! – depend on the order of factors.

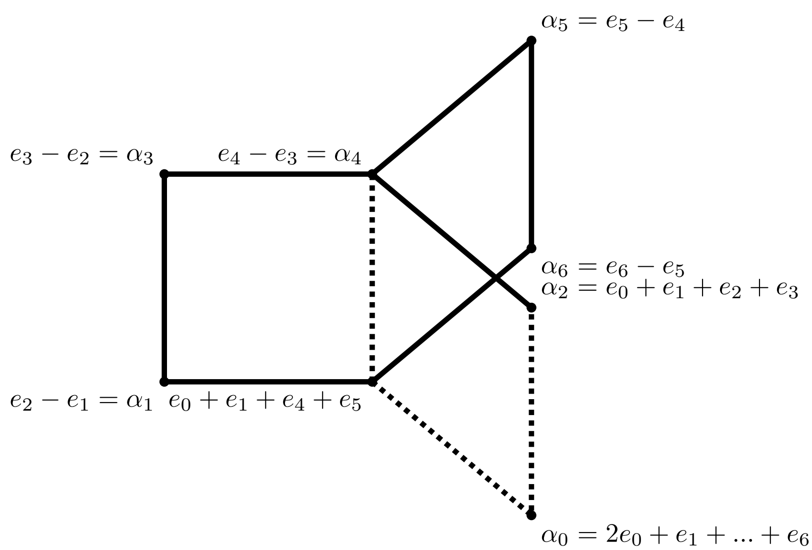

Figure 1. Enhanced Dynkin diagram of type E_6.

However, since all cycles of C are even, the graph C is bipartite, its vertices can be partitioned into two disjoint subsets X and Y consisting of pairwise orthogonal roots. Let w_X and w_Y be the products of reflections w_α, where $\alpha \in X$ or $\alpha \in Y$, respectively. Obviously, w_X and w_Y are involutions, and the conjugacy class of the **semi-Coxeter element** represented by C is $w_C = w_X w_Y$ does not depend on the choice of such X and Y.

There are further complications, but eventually, as a result of lengthy arguments reminiscent of the classification of Dynkin diagrams themselves, and arduous computations, Carter succeeds in mustering a collection of diagrams that produce all conjugacy classes of $W(\Phi)$ in this fashion.

In fact, almost immediately Pawan Bala and Roger Carter discovered the close connection between conjugacy classes of the Weyl groups and the **unipotent conjugacy classes** of the corresponding Chevalley groups [3]. This connection was then made more precise and explicit, and then extensively studied by Tonny Springer, Nicolas Spaltenstein, David Kazhdan and especially by George Lusztig, see, in particular, [14].

1.2. Stekolshchik diagrams. Subsequently, several alternative approaches to the classification of conjugacy classes were proposed, including the extremely illuminating work of Tonny Springer [22] and of Meinolf Geck and Götz Pfeiffer, see [11] and references there. Nonetheless, Carter's list itself remained somewhat misterious. Part of that mistery was lifted by Rafael Stekolshchik in [23, 24, 25]; the final version was published in 2017 in the journal of Lugansk University [26].

In these texts, Stekolshchik made several extremely pertinent observations.

• Diagrams with cycles of arbitrary even length can be reduced to diagrams with cycles of length 4 alone. In particular, this explains why the admissible diagrams with cycles of lengths 6 and 8 that appear for the types E_7 and E_8 do not make their way to the lists of conjugacy classes.

Of course, this is what eventually transpires in Carter's proof as well, but there it only happens at the very last step, when the admissible diagrams with long cycles are eliminated as a result of summing up the orders of the conjugacy classes obtained so far. Stekolshchik provides a direct case by case verification of the fact that the semi-Coxeter element constructed from an admissible diagram with long cycles is conjugate to the semi-Coxeter element constructed from another admissible diagram with cycles of length 4 alone.

• One should explicitly mark the sign of the inner product in Carter diagrams. Stekolshchik himself denotes the negative inner product of two roots by *solid bonds*, and the positive inner product – by *dotted bonds*. We follow this convention in the present paper[1].

The vertices of an extended Dynkin diagram are linearly dependent. It follows that they cannot form a part of an admissible diagram. In particular, any diagram containing a cycle consisting of solid edges alone is not admissible – any such cycle should contain an odd number of dotted edges.

• The cyclic order of reflections with respect to the roots forming a 4-cycle in D_4 leads to the Coxeter class D_4 of $W(\mathrm{D}_4)$ of order 32, whereas

[1]It should be noted that in [8] dotted lines are charged with *three* completely different meanings. In Fig. 2 the *bold* dotted bond denotes the [unique] bond that completes the initial Dynkin diagram of Φ to the extended Dynkin diagram. In the Figs. 2, 3, and 4 the *thin* dotted bonds denote the bonds *hidden* in the usual Euclidean picture. In Fir. 3 the *bold* dotted bonds denote the *emerging* bonds at a certain step of the inductive procedure.

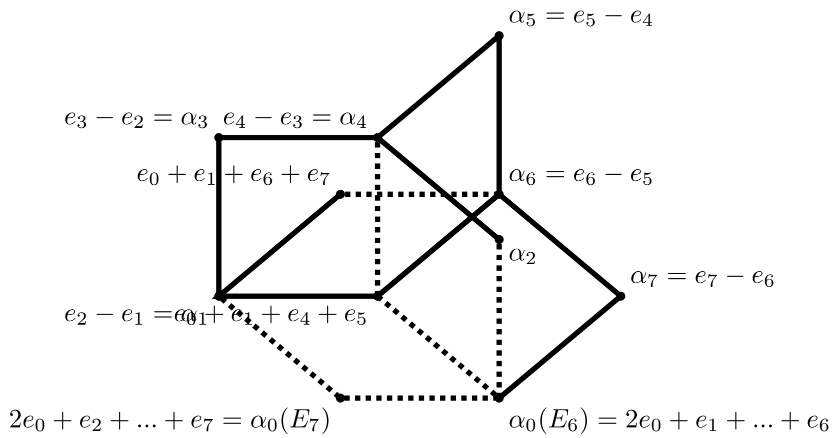

Figure 2. Enhanced Dynkin diagram of type E_7.

the bipartite order, as described above, leads to the semi-Coxeter class $\mathrm{D}_4(a_1)$ of order 12.

Essentially, Stekolshchik proves that all non-Coxeter classes are explained by this single phenomenon. Some elements in the Coxeter class of D_4 can be rewritten as products of the reflections corresponding to 4-cycles, in cyclic order. By iterating this procedure for various copies of D_4, one can eventually obtain all diagrams in Carter's lists.

1.3. Dynkin–Minchenko diagrams. However, some time before that Dynkin and Minchenko [8] made another extremely important observation. As we know, by 2010 both the algorithm to construct root subsystems, and the lists of those were known for some 60 years. However, the *genuine* explanation of these lists was missing. Here are the key new observations of [8].

• All root subsystems of Φ are uniformly constructed using their maximal subsets of pairwise orthogonal roots. Again, occurence of all such non-parabolic subsystems is explained by a single exceptional phenomenon, the presence of 4 pairwise orthogonal roots in D_4.

• All instances, where two isomorphic subsystems of E_7 and E_8 are non-conjugate, are uniformly explained by the presence of 4 pairwise orthogonal roots in D_4 – they either have **charge** 4 themselves, or [in the case of E_7]

are orthogonal completions to one root in systems of charge 4 (and thus have charge 3). This led the authors of [8] to the discovery of what they call **enhanced Dynkin diagrams**. These diagrams are built up inductively as follows.

• We start with the usual Dynkin diagram of type Φ.

• For every node of degree three we look at the corresponding copy of D_4 spanned by this node and the three adjacent nodes. We add the maximal root of this copy of D_4 – or, actually, its opposite.

• However, the roots that emerged at the previous step can be themselves joined to some other roots, which would then produce new nodes of degree three.

• This procedure should be repeated *quantum sufficit* = until complete satisfaction, the step, where no new vertices of degree three occur.

The resulting diagram contains Dynkin diagrams of all root subsystems *up to conjugacy*. By picking up its subsets – obviously, it suffices to take *linearly independent* ones – we now get representatives of all fundamental subsystems of all possible root subsystems.

As another interesting feature, observe that the *exceptional* enhanced Dynkin diagrams contain the **extended Dynkin diagrams** = the usual Dynkin diagrams augmented by the [negative] maximal root of the initial subsystem. For classical types, this is not necessarily the case. For $\Phi = \mathrm{A}_l$, all root subsystems and all conjugacy classes of $W(\mathrm{A}_l) = S_{l+1}$ are parabolic; they correspond to partitions of $l+1$ and thus the enhanced Dynkin diagram is the *usual* Dynkin diagram. For $\Phi = \mathrm{D}_l$, the answer depends on the parity of l. The enhanced Dynkin diagram of type D_4 is the *extended* Dynkin diagram and similarly for all *even* $l = 2m$ the enhanced Dynkin diagram contains the maximal root of D_l and thus the extended Dynkin diagram. But for *odd* $l = 2m+1$ it does not!

Since the authors of [8] were only interested in root subsystems, they were effectively looking not at the roots of Φ, but at its subsystems of type A_1 – what they call **projective roots**.

In the present paper, we effectively merge both approaches, which leads to the diagrams that contain representatives of both Dynkin diagrams of all root subsystems of Φ, *and* Stekolshchik diagrams of all conjugacy classes of $W(\Phi)$.

§2. Notation

2.1. Root systems. In all that regards root systems, including the numbering of their fundamentral roots, we follow Bourbaki [4]. In particular, Φ is a reduced irreducible root system of rank l, $W = W(\Phi)$ is its Weyl group.

Fix an order on Φ, and let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be the corresponding set of fundamental roots, Φ^+ and Φ^- be the corresponding sets of positive and negative roots, respectively. Let $\bar{\Pi}$ be the **extended fundamental system** of Φ. It is obtained by appending to Π the root $\alpha_0 = -\delta$, where δ is the highest root of Φ with respect to the fundamental system Π.

Recall that in the Dynkin form, the highest roots of E_6, E_7, and E_8 are depicted as

$$\begin{smallmatrix}12321\\2\end{smallmatrix}, \qquad \begin{smallmatrix}234321\\2\end{smallmatrix}, \qquad \begin{smallmatrix}2465432\\3\end{smallmatrix}.$$

For a root $\alpha \in \Phi$, we denote by $w_\alpha \in W$ the corresponding root reflection. It is clear that $w w_\alpha w^{-1} = w_{w_\alpha}$ for all $w \in W$. Usually we denote the fundamental root reflection w_{α_i} simply by w_i. Observe that, in many books on Lie algebras and Coxeter groups, it is denoted by s_i. It is well known that the fundamental reflections generate the Weyl group, $W = \langle w_1, \dots, w_l \rangle$.

For two root systems Δ and Σ, we denote by $\Delta+\Sigma$ their orthogonal sum. In particular, $k\Delta = \Delta_1 + \dots + \Delta_k$ is the orthogonal sum of k isomorphic summands. It is sometimes convenient to consider also the empty root system $A_0 = \emptyset$ of rank 0. Recall that $D_1 = D_0 = \emptyset$.

2.2. Hyperbolic realization of E_l. In the present paper, we are predominantly interested in the cases $\Phi = \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$. As in [12, 27, 30], we use the hyperbolic realization of these systems in the $(l+1)$-dimensional Minkowsky space [15]. This realization is considerably more suitable for large-scale calculations than the usual realizations in Euclidean space.

Consider a real vector space $V = \mathbb{R}^{l,1}$ of dimension $l+1$ endowed with a nondegenerate symmetric inner product $(\ ,\ ) \colon V \times V \longrightarrow \mathbb{R}$ of signature $(l, 1)$. Fix an orthonormal base $e_0, e_1, \dots, e_l$ such that $(e_0, e_0) = -1$ and $(e_i, e_i) = 1$ for all $1 \leqslant i \leqslant l$. We are primarily interested in the case $l = 8$.

Fix the following fundamental system $\Pi = \{\alpha_1, \dots, \alpha_8\}$ in $\Phi = \mathrm{E}_8$:

$$\begin{array}{llll} \alpha_1 = e_2 - e_1, & \alpha_2 = e_0 + e_1 + e_2 + e_3, & \alpha_3 = e_3 - e_2, & \alpha_4 = e_4 - e_3, \\ \alpha_5 = e_5 - e_4, & \alpha_6 = e_6 - e_5, & \alpha_7 = e_7 - e_6, & \alpha_8 = e_8 - e_7. \end{array}$$

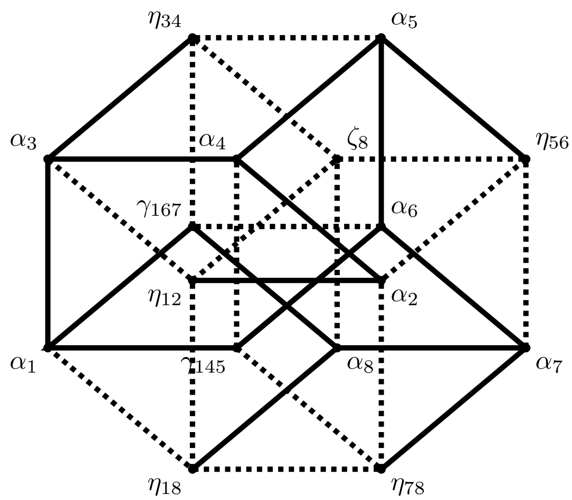

Figure 3. Enhanced Dynkin diagram of type E_8.

To obtain the root system E_7, it suffices to take roots in E_8 such that α_8 does not occur in their linear expansion with respect to the fundamental roots. By the same token, to get a root system of type E_6 it suffices to take roots in E_8 such that both α_7 and α_8 do not occur in their linear expansion with respect to the fundamental roots.

In particular, every element of Φ^+ has one of the following forms:

$$\begin{aligned}
&\beta_{ij} = e_i - e_j, \qquad i > j,\\
&\gamma_{ijh} = e_0 + e_i + e_j + e_h,\\
&\eta_{ij} = 2e_0 + e_1 + \ldots + \widehat{e_i} + \ldots. + \widehat{e_j} + \ldots + e_8,\\
&\zeta_i = 3e_0 + e_1 + \ldots + 2e_i + \ldots + e_8,
\end{aligned}$$

where the indices $i, j, h = 1, \ldots, 8$ are pairwise distinct, while the hat $\widehat{\ }$ over a summand signifies that this summand should be omitted.

§3. Proof of Theorem 1

Now we are all set to provide detailed constructions of the enhanced Dynkin diagrams and thus finish the proof of Theorem 1 stated in the introduction.

We build up the diagrams inductively by the same procedure as Dynkin and Minchenko, but controlling the signs of the resulting roots. Moreover, for senior cases we do not start the construction from scratch, but rather explicitly use the embeddings $\mathrm{E}_6 \leqslant \mathrm{E}_7 \leqslant \mathrm{E}_8$. Thus, we start with 6 fundamental roots of E_6 and consecutively adjoin $2+3+5$ further roots. However, two of these new nodes, namely α_7 for E_7 and α_8 for E_8 come gratis, so that we only have to repeat the inductive step 8 times.

3.1. Type E_6. We start with the root system of type E_6 generated by the above fundamental roots $\alpha_1, \ldots, \alpha_6$.

• This Dynkin diagram has a single node of degree 3, represented by the root α_4. As a first step of the construction, we adjoin the maximal root of the subsystem of type D_4 spanned by $\alpha_2, \alpha_3, \alpha_4, \alpha_5$. Clearly, this is

$$\gamma_{145} = e_0 + e_1 + e_4 + e_5 = \begin{smallmatrix} 01210 \\ 1 \end{smallmatrix},$$

which has negative inner products both with α_1 and α_6.

• The previous step engenders a new node of degree 3 and the procedure should be repeated for the resulting copy of D_4 spanned by $\alpha_1, \alpha_4, \alpha_6$ as terminal nodes and γ_{145} as the central node. Clearly, the maximal root of this system is

$$2\gamma_{145} + \alpha_1 + \alpha_6 - \alpha_4,$$

the minus sign is explained by the fact that the inner product of γ_{145} and α_4 is positive. Thus the new node to be adjoined is the maximal root of the initial root system $\Phi = \mathrm{E}_6$,

$$\eta_{78} = \begin{smallmatrix} 12321 \\ 2 \end{smallmatrix},$$

which has negative inner products with γ_{145} and with α_2.

All nodes in the resulting diagram have degrees 2 or 4, so that the inductive procedure for E_6 is complete.

3.2. Type E_7. Now we take as input the enhanced Dynkin diagram of type E_6 constructed in the previous subsection, and adjoin the new fundamental root α_7. This engenders *two* new degree 3 nodes, namely α_6 is now joined to α_5, α_7, and γ_{145}, whereas η_{78} is now joined to α_2, α_7, and γ_{145} and we should repeat the inductive step for each of these two central nodes.

• First, consider the copy of D_4 generated by α_5, α_7, and γ_{145} as terminal nodes and α_6 as the central node. Clearly, the maximal root of this system is

$$\gamma_{145} + \alpha_5 + 2\alpha_6 + \alpha_7 = \gamma_{167} = e_0 + e_1 + e_6 + e_7 = \begin{smallmatrix} 012221 \\ 1 \end{smallmatrix}.$$

By construction, it has positive inner product with α_6. But since the only fundamental root occuring in γ_{145} that has nonzero inner product with α_1 is α_3, it follows that γ_{145} has negative inner product with α_1.

This makes α_1 a *new* node of degree 3. But hold on, we are not yet finished with the nodes of degree 3 that cropped up at the previous step.

• Next, consider the copy of D_4 generated by α_2, α_7, and γ_{145} as terminal nodes and η_{78} as the central node. Clearly, the maximal root of this system is

$$2\eta_{78} - \alpha_2 + \alpha_7 - \gamma_{145} = \eta_{18} = 2e_0 + e_2 + \ldots + e_7 = \begin{smallmatrix} 234321 \\ 2 \end{smallmatrix},$$

which is the maximal root of the root system $\Phi = \mathrm{E}_7$. By the very construction, it has positive inner product with η_{78}. But it also happens to have positive inner product with α_1.

All nodes in the resulting diagram have degrees 2 or 4, so that the inductive procedure for E_7 is complete.

3.3. Type E_8**.** Again, we take as input the enhanced Dynkin diagram of type E_7 constructed in the previous subsection, and adjoin the new fundamental root α_8. Obviously, α_8 is joined to all degree 2 nodes α_7, γ_{167}, and η_{18}. Thus, we have to perform the usual induction step for α_8 itself. Moreover, all *three* nodes of degree 2 in the lower plane the enhanced Dynkin diagram of type E_7 become nodes of degree 3. Specifically, α_7 is now joined to α_6, α_8, and η_{78}; whereas γ_{167} is now joined to α_1, α_6, and α_8; and finally η_{17} which is now joined to α_1, α_8, and η_{78}. Thus, we should repeat the inductive step for each of these *four* central nodes. In the meantime, new nodes of degree 3 could occur, but as we know from the previously subsection, we should not be concerned, since further bonds may arise while we are completing the inductive steps for those nodes that have cropped up already.

• First, consider the copy of D_4 generated by α_7, γ_{167}, and η_{18} as terminal nodes and α_8 as the central node. Clearly, the maximal root of this system is

$$\eta_{18} + \gamma_{167} + \alpha_7 + 2\alpha_8 = \zeta_8 = 3e_0 + e_1 + \ldots + e_7 + 2e_8 = \begin{smallmatrix} 2465432 \\ 3 \end{smallmatrix},$$

which is the maximal root of the root system $\Phi = \mathrm{E}_8$.

D_4	$D_4(a_1)$
D_5	$D_5(a_1)$
D_6	$D_6(a_1)$
	$D_6(a_2)$
D_7	$D_7(a_1)$
	$D_7(a_2)$
D_8	$D_8(a_1)$
	$D_8(a_2)$
	$D_8(a_3)$

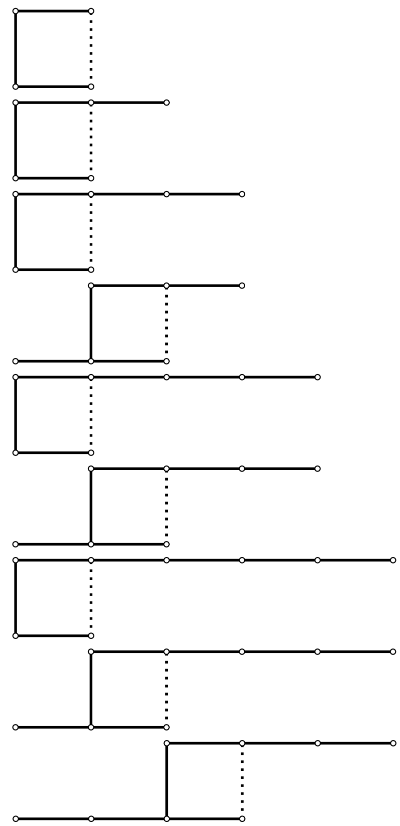

Figure 4. Irreducible admissible diagrams of types D_l, $4 \leqslant l \leqslant 8$.

So far it is not connected to any further root apart from α_8 itself, but hold on, hold on, we are not finished yet. Eventually, it will be connected with all three emerging nodes, and will have degree 4, as any other root.

• Next, consider the copy of D_4 generated by α_6, α_8, and η_{78} as terminal nodes and α_7 as the central node. Clearly, the maximal root of this system is

$$\eta_{78} + \alpha_6 + 2\alpha_7 + \alpha_8 = \eta_{56} = 2e_0 + e_1 + \ldots + e_4 + e_7 + e_8 = \begin{smallmatrix}1232221\\ 2\end{smallmatrix}.$$

Obviously, it has negative inner product with α_5 and positive inner products with α_2 and with ζ_8.

• Further, consider the copy of D_4 generated by α_1, α_6, and α_8 as terminal nodes and γ_{167} as the central node. Clearly, the maximal root of this system is

$$2\gamma_{167} + \alpha_1 - \alpha_6 + \alpha_8 = \eta_{34} = 2e_0 + e_1 + e_2 + e_5 + \ldots + e_8 = \begin{smallmatrix}1244321\\2\end{smallmatrix}.$$

Obviously, it has negative inner product with α_3 and positive inner products with α_5 and with ζ_8.

• Finally, consider the copy of D_4 generated by α_1, α_8, and η_{78} as terminal nodes and η_{17} as the central node. Clearly, the maximal root of this system is

$$2\eta_{17} - \alpha_2 - \eta_{78} + \alpha_8 = \eta_{12} = 2e_0 + e_3 + \ldots + e_8 = \begin{smallmatrix}2454321\\2\end{smallmatrix}.$$

Obviously, it has negative inner product with α_2 and positive inner products with α_3 and with ζ_8.

All nodes in the resulting diagram have degree 4, so that the inductive procedure for E_8 and the proof of theorem stated in the introduction are now complete.

§4. Proof of Theorem 2

Here we do not give an *a priori* proof of the fact that all admissible diagrams are in fact contained in the signed enhanced Dynkin diagrams. Such a proof, combining the ideas of [6, 11, 8, 26] could be given, and would provide an alternative description of conjugacy classes of the Weyl groups $W(\mathrm{E}_l)$, $l = 6, 7, 8$. But it would requre a detailed combinatorial analysis of the diagrams themselves, and we plan to return to it in a subsequent work, see the last section.

Instead, here we provide an *a posteriori* observation that all Carter–Stekolshchik diagrams are, in fact, subdiagrams of the [signed] ehnanced Dynkin diagrams – in the style of ancient "look" or the fashionable present-day computer "experimental" mathematics.

Such diagrams without cycles are in fact Dynkin diagrams of root subsystems in Φ. They are trees and, thus, the distinction between solid and dotted lines does not play a role. That all of them are subdiagrams of the corresponding enhanced Dynkin diagram is the main result of [8, Theorem 1.1.]. Initially, it was the main mission of the enhanced Dynkin diagrams.

E_6 $E_6(a_1)$

$E_6(a_2)$

E_7 $E_7(a_1)$

$E_7(a_2)$

$E_7(a_3)$

$E_7(a_4)$

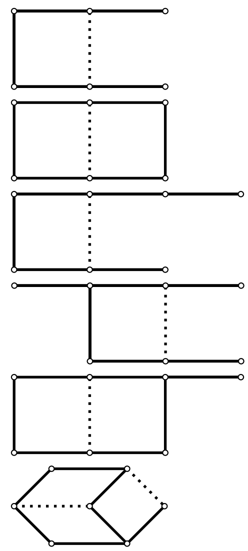

Figure 5. Irreducible admissible diagrams of types E_6 and E_7.

Thus, we only have to look at the admissible diagrams with cycles. The number of such diagrams, including the reducible ones, and those that come from smaller ranks, are 4 for $W(\mathrm{E}_6)$, 13 for $W(\mathrm{E}_7)$ and, finally, 36 for $\mathrm{W}(\mathrm{E}_8)$. Let us list them all in the order they occur in Carter's [6], Tables 8–10, – of course, the first two of these lists are sublists of the next ones, sometimes more than once.

- For type E_6:

$$\mathrm{D}_4(a_1), \qquad \mathrm{D}_5(a_1), \qquad \mathrm{E}_6(a_1), \qquad \mathrm{E}_6(a_2).$$

- For type E_7:

$$\mathrm{D}_4(a_1), \quad \mathrm{D}_4(a_1)+\mathrm{A}_1, \quad \mathrm{D}_5(a_1), \quad \mathrm{D}_5(a_1)+\mathrm{A}_1, \quad \mathrm{D}_6(a_1), \quad \mathrm{D}_6(a_2),$$
$$\mathrm{E}_6(a_1), \quad \mathrm{E}_6(a_2), \quad \mathrm{D}_6(a_2)+\mathrm{A}_1, \quad \mathrm{E}_7(a_1), \quad \mathrm{E}_7(a_2), \quad \mathrm{E}_7(a_3), \quad \mathrm{E}_7(a_4).$$

• For type E_8:

$$\mathrm{D}_4(a_1),\quad \mathrm{D}_4(a_1)+\mathrm{A}_1,\quad \mathrm{D}_5(a_1),\quad \mathrm{D}_4(a_1)+\mathrm{A}_2,\quad \mathrm{D}_5(a_1)+\mathrm{A}_1,$$
$$\mathrm{D}_6(a_1),\quad \mathrm{D}_6(a_2),\quad \mathrm{E}_6(a_1),\quad \mathrm{E}_6(a_2),\quad \mathrm{D}_4(a_1)+\mathrm{A}_3,\quad \mathrm{D}_5(a_1)+\mathrm{A}_2,$$
$$\mathrm{D}_6(a_2)+\mathrm{A}_1,\quad \mathrm{E}_6(a_1)+\mathrm{A}_1,\quad \mathrm{E}_6(a_2)+\mathrm{A}_1,\quad \mathrm{D}_7(a_1),\quad \mathrm{D}_7(a_2),$$
$$\mathrm{E}_7(a_1),\quad \mathrm{E}_7(a_2),\quad \mathrm{E}_7(a_3),\quad \mathrm{E}_7(a_4),\quad 2\,\mathrm{D}_4(a_1),\quad \mathrm{D}_5(a_1)+\mathrm{A}_3,$$
$$\mathrm{D}_8(a_1),\quad \mathrm{D}_8(a_2),\quad \mathrm{D}_8(a_3),\quad \mathrm{E}_6(a_2)+\mathrm{A}_2,\quad \mathrm{E}_7(a_2)+\mathrm{A}_1,\quad \mathrm{E}_7(a_4)+\mathrm{A}_1,$$
$$\mathrm{E}_8(a_1),\quad \mathrm{E}_8(a_2),\quad \mathrm{E}_8(a_3),\quad \mathrm{E}_8(a_4),\quad \mathrm{E}_8(a_5),\quad \mathrm{E}_8(a_6),\quad \mathrm{E}_8(a_7),\quad \mathrm{E}_8(a_8).$$

The *irreducible* admissible diagrams of types D_l, $4 \leqslant l \leqslant 8$, that occur in these lists are all listed in Fig. 4, whereas all those of types E_6 and E_7 are reproduced in Fig. 5, and those of type E_8 – in Fig. 6.

We leave it to the reader as an exercise to find all admissible diagrams of the corresponding types in Figs. 1–3. That's exactly an observation with which the present work started. After you succeed in doing that for $\mathrm{E}_6(a_1)$, the rest becomes obvious.

§5. Final remarks

Let us make some further scattered observations concerning the symmetry of the above diagrams, their further uses, and some of our immediate plans.

• The enhanced Dynkin diagrams of types E_6 and E_8 are *extremely* symmetric. Both are bipartite graphs consisting of two maximal subsets of pairwise orthogonal roots = **mosets**, in the terminology of [8]. Like $4\,\mathrm{A}_1 \sqcup 4\,\mathrm{A}_1$ in the case of E_6 and $8\,\mathrm{A}_1 \sqcup 8\,\mathrm{A}_1$ in the case of E_8.

At the same time, the enhanced Dynkin diagrams of types E_7 looks weird. It is again a bipartite graph, but now of the form $7\,\mathrm{A}_1 \sqcup 4\,\mathrm{A}_1$, where $7\,\mathrm{A}_1$ is, as above, a moset of E_7, but $4\,\mathrm{A}_1$ is clearly a moset of E_6. Worse than that, the roots of $7\,\mathrm{A}_1$ loose their symmetry with respect to a specific copy of $4\,\mathrm{A}_1$. Namely, whereas 6 of the roots forming $7\,\mathrm{A}_1$ are nodes of degree 2 in the exhanced Dynkin diagram, one of them has degree 4. However, modulo sign changes the normaliser of $7\,\mathrm{A}_1$ in $W(\mathrm{E}_7)$ acts as $\mathrm{SL}(3,2)$ and thus is transitive on the seven copies of A_1. This means that the symmetry breaking depends on a specific choice of $4\,\mathrm{A}_1$.

The first-named author has already encountered such similar phenomenon on several instances. It seems that E_7 invariably exhibits much less symmetry than do E_6 and E_8, see, in particular, [28, 29]. Actually,

E_8

$E_8(a_1)$ $E_8(a_2)$ $E_8(a_3)$ $E_8(a_4)$ $E_8(a_5)$ $E_8(a_6)$ $E_8(a_7)$ $E_8(a_8)$

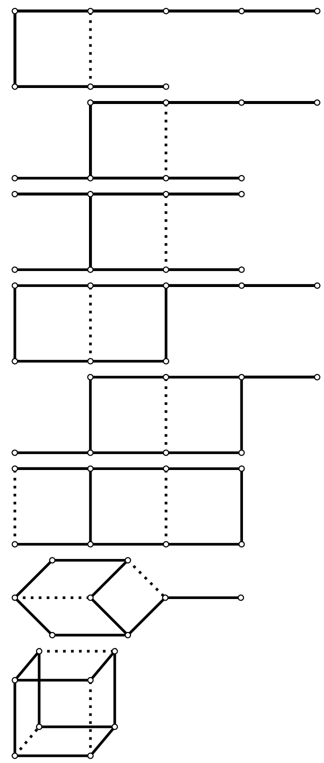

Figure 6. Irreducible admissible diagrams of type E_8.

in 1935 Daniil Kharms already commented this situation[2]: "We went to the Summer Garden and started to count trees there. But when the count reached 6, we stopped and began to dispute: some speculated that 7 would follow, and some that 8."

Similarly, of the cases E_6 and E_8 the case of E_8 seems to be much more symmetric. The case of E_6 exhibits obvious triality, in each moset occuring in the enhanced Dynkin diagram one node has degree 4, whereas the other 3 have degree 2.

For E_8, there are no preferred nodes, all of them have degree 4. Visualising the enhanced Dynkin diagram of type E_8 as a 4-cube, as we do, the two

[2]Translated from Russian by Sergei Kisliakov.

copies of $8\,\mathrm{A}_1$ become the vertices of the positive and negative *demicubes*, respectively.

• Dynkin and Minchenko visualise the enhanced Dynkin diagrams of types E_7 and E_8 differently. In their realisation the diagram for E_7 consists of 4 vertices of a tetrahedron + 6 midpoint of its edges + the center joined to the vertices, but not to the midpoints. Observe that the midpoints come as 3 pairs, corresponding to the pairs of opposite edges of the tetrahedron. If you wish, you can visualise the copies of A_1 in $7\,\mathrm{A}_1$ as the points of the **Fano plane**, with the central node = central point, and other points coming in pairs collinear with the central point.

Their diagram for E_6 is the 4×4 net on a 2-dimensional torus. The exceptional behaviour of this net, in particular, that it is much more symmetric than the nets of different sizes, was simultaneously observed by other authors, notably by Vladimir Kornyak [13]. Of course, again this is related to the exceptional behavious of D_4 and the additional symmetries that come from $W(\mathrm{F}_4)$.

Observe that the same graphs also occur in a completely different context, as graphs with certain extremal properties for their eigenvalues, see [17, 18]. Probably, there is much more to it, than what we see today.

• Concerning the terminology itself, we do not think *enhanced* Dynkin diagram a good name for this object, and it does not naturally translate to Russian. Even less so for the *signed* enhanced diagram. Boris Kuniavsky suggested the name **enriched Dynkin diagrams**, which already sounds much better. However, a posteriori, the best solution would be to completely renounce the use of the term *extended* Dynkin diagrams in the sense of **affine Dynkin diagrams** [4] and reserve the term **extended Dynkin diagrams** to some form of Figs. 1–3.

• The relation of these pictures to the arithmetic of quaternions and octonions seems to be preeminent at all levels. In particular, there are manifest connections to the construction of forms of simple Lie algebras and simple algebraic groups of types E_7 and E_8 in terms of $7\,\mathrm{A}_1$ and $8\,\mathrm{A}_1$ in the works of Laurent Manivel [16] and Victor Petrov [21].

We are positive that there are scores of similar covert beauties around, waiting their time to be discovered and explained. The very special role of D_4 comes over and over again in a vast variety of situations. Thus, with respect to a given base the study of semisimple root elements, triples of unipotent root elements, and many other important structural elements of simple Lie algebras and simple algebraic groups are all reduced to the case of D_4. See [31] for one such instance, and many further related references.

• In a subsequent publication, we plan to return to the specific combinatorial study of these pictures. In particular, we plan to explicitly enumerate subsets of roots in $\Phi = E_6, E_7, E_8$ having the symmetry of an enhanced Dynkin diagram. Also, we intend to clarify the connection with the classification of conjugacy classes of the corresponding Weyl group $W(\Phi)$, and give an a priori explanation thereof.

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Dept. of Mathematics
and Computer Science
St Petersburg State University
St Petersburg, Russia
E-mail: `nikolai-vavilov@yandex.ru`
E-mail: `v-v-m12@mail-ru`

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