Записки научных
семинаров ПОМИ
Том 498, 2020 г.

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MULTIBRANCHED SURFACES IN 3-MANIFOLDS

Abstract. This article is a survey of recent works on embeddings of multibranched surfaces into 3-manifolds.

Throughout this article, we will work in the piecewise linear category. All topological spaces are assumed to be second countable and Hausdorff.

Given a pair (X, Y) of topological spaces, we regard the following problems as fundamental ones.

(1) Can X be embedded into Y?
(2) If X can be embedded into Y, then
   (a) In which cases are two embeddings of X into Y equivalent (with respect to the equivalence relation according to the situation)?
   (b) In what ways can X be embedded into Y?

In this article, we consider the case where X is a multibranched surface and Y is a closed orientable 3-manifold.

We say that a 2-dimensional CW complex is a *multibranched surface* if removing all points whose open neighborhoods are homeomorphic to the 2-dimensional Euclidean space yields a 1-dimensional complex homeomorphic to a disjoint union of simple closed curves.

Multibranched surfaces naturally arise in several areas:

- polycontinuous patterns – a mathematical model of microphase-separated structures made by block copolymers ([13, 24, 25]),
- 2-stratifolds – as spines of closed 3-manifolds ([17–19]),
- trisections, multisections – as an analog of Heegaard splittings ([16, 28, 38]),
- essential surfaces – as non-meridional essential surfaces in link exteriors ([9, 10]), essential surfaces in handlebody-knot exteriors ([27]) and in manifolds obtained by Dehn surgeries ([9, 22]).

Key words and phrases: multibranched surface, 3-manifold, 3-sphere, embedding, obstruction, graph, complex, essential surface.

The author is partially supported by Grant-in-Aid for Scientific Research (C) (No. 17K05262) and (B) (No. 16H03928), The Ministry of Education, Culture, Sports, Science, and Technology, Japan.

The article is organized as follows. In Sec. 1, we define several concepts related to multibranched surfaces. In Sec. 2, we describe some of the backgrounds for multibranched surfaces. In Sec. 3, we study embeddings of multibranched surfaces into closed orientable 3-manifolds. In Sec. 4, we consider multibranched surfaces that cannot be embedded into the 3-sphere.

§1. Preliminaries

1.1. Definition. Let $\mathbb{R}^2_+$ be the closed upper half-plane

$$\{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geqslant 0\}.$$

The *multibranched Euclidean plane*, denoted by $\mathbb{R}^2_i$ $(i \geqslant 1)$, is the quotient space obtained from i copies of $\mathbb{R}^2_+$ by identifying their boundaries

$$\partial\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = 0\}$$

via the identity map. See Fig. 1 for the multibranched Euclidean plane $\mathbb{R}^2_5$.

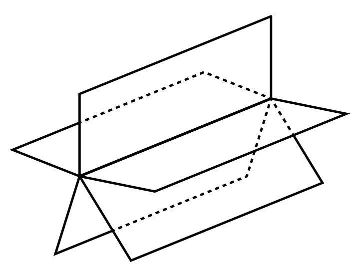

Figure 1. The multibranched Euclidean plane $\mathbb{R}^2_5$.

A second countable Hausdorff space X is called a *multibranched surface* if X contains a disjoint union of simple closed curves $l_1, \dots, l_n$ satisfying the following:

(1) For each point $x \in l_1 \cup \cdots \cup l_n$ there exist an open neighborhood U of x and a positive integer i such that U is homeomorphic to $\mathbb{R}^2_i$.
(2) For each point $x \in X - (l_1 \cup \cdots \cup l_n)$ there exists an open neighborhood U of x such that U is homeomorphic to $\mathbb{R}^2$.

1.2. Construction. To construct a compact multibranched surface, we prepare a closed 1-dimensional manifold B (corresponding to $l_1, \dots, l_n$), a compact 2-dimensional manifold S (corresponding to the union of the closures of the components of $X-(l_1 \cup \cdots \cup l_n)$), and a map $\phi\colon \partial S \to B$ such that for every connected component c of ∂S, the restriction $\phi|_c\colon c \to \phi(c)$ is a covering map. Then a multibranched surface X can be constructed from the triple $(B, S; \phi)$ as the quotient space $X = B \cup_\phi S$.

A connected component of B, S, or ∂S is said to be a *branch*, *sector*, or *prebranch*, respectively. The set consisting of all branches or sectors is denoted by $\mathcal{B}(X)$ or $\mathcal{S}(X)$, respectively.

1.3. Degrees, oriented degrees, and regularity. For a prebranch c of a multibranched surface X, the covering degree of $\phi|_c\colon c \to \phi(c)$ is called the *degree* of c and denoted by $d(c)$. We give an orientation for each branch and each prebranch c of X. (In the case where a sector s is orientable and oriented, the orientations of the prebranches in ∂s are induced by that of s.) The *oriented degree* of a prebranch c of X is defined as follows: if the covering map $\phi|_c\colon c \to \phi(c)$ is orientation-preserving, then the *oriented degree* $\mathrm{od}(c)$ of c is defined by $\mathrm{od}(c) = d(c)$, and if it is orientation-reversing, then the oriented degree is defined by $\mathrm{od}(c) = -d(c)$.

A prebranch c of X is said to be *attached* to a branch l if $\phi(c) = l$. We denote by $\mathcal{A}(l)$ the set consisting of all prebranches attached to l; the number of elements of $\mathcal{A}(l)$ is called the *index* of l and denoted by $i(l)$.

A multibranched surface X is *regular* if for each branch l and each pair of prebranches $c, c' \in \mathcal{A}(l)$, the condition $d(c) = d(c')$ holds. Let X be a regular multibranched surface, and let l be a branch of X. Since each pair of prebranches $c, c' \in \mathcal{A}(l)$ has the same degree, the *degree* of l is well defined as $d(l) = d(c) = d(c')$.

1.4. Graph representations. Let X be a compact multibranched surface obtained from $(B, S; \phi)$ such that all components of S are orientable and oriented and have nonempty boundary. (Hereafter, we assume that the multibranched surfaces under consideration satisfy these conditions unless otherwise stated.) The multibranched surface $X = B \cup_\phi S$ has a graph representation ([10]) defined as follows. Let $G = (V_S \cup V_B, E)$ be a bipartite graph such that $|V_S| = |\mathcal{S}(X)|$ and $|V_B| = |\mathcal{B}(X)|$. To each sector $s \in \mathcal{S}(X)$, we assign a vertex $v(s) \in V_S$ labeled by $g(s)$, where $g(s)$ denotes the genus of s. To each branch $l \in \mathcal{B}(X)$, we assign a vertex $v(l) \in V_B$. To a prebranch $c \subset \partial s$, we assign an edge $e \in E$ connecting $v(s)$ and

$v(l)$ and labeled by $\mathrm{od}(c)$, where $c \in \mathcal{A}(l)$. A concept similar to this graph representation was defined in [17].

Example 1.1. A closed nonorientable surface of crosscap number h can be regarded as a multibranched surface X with h branches $B = l_1 \cup \cdots \cup l_h$ and a planar surface S with h boundary components such that $d(c) = 2$ for any prebranch $c \subset \partial S$. Then X has a graph representation G as shown in Fig. 2.

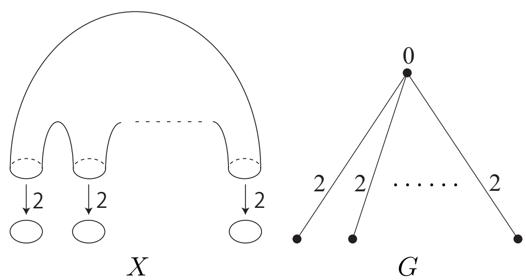

Figure 2. A multibranched surface X and its graph representation G.

1.5. Incidence matrices. For a sector $s \in \mathcal{S}(X)$ and a branch $l \in \mathcal{B}(X)$ of a multibranched surface X, we define the *algebraic degree* $d(l; s)$ as follows:

$$d(l; s) = \sum_{c \in \mathcal{A}(l) \cap \partial s} \mathrm{od}(c).$$

Then, we define the *incidence matrix* $M_X = (a_{ij})$ $(i = 1, \ldots, n; j = 1, \ldots, m)$ by

$$a_{ij} = d(l_i; s_j),$$

where $\mathcal{B}(X) = \{l_1, \ldots, l_n\}$ and $\mathcal{S}(X) = \{s_1, \ldots, s_m\}$.

1.6. The first homology group. The multibranched surface obtained by removing an open disk from each sector except its collar is denoted by $\dot{X}$.

Theorem 1.2 ([31, Theorem 4.1])**.** *Let X be a regular multibranched surface with $\mathcal{B}(X) = \{l_1, \ldots, l_n\}$ and $\mathcal{S}(X) = \{s_1, \ldots, s_m\}$. Then*

$$H_1(X) = \left\langle l_1, \ldots, l_n \;\middle|\; \sum_{k=1}^{n} d(l_k; s_1) l_k, \ldots, \sum_{k=1}^{n} d(l_k; s_m) l_k \right\rangle \oplus \mathbb{Z}^{r'(X)},$$

where $r'(X) = \operatorname{rank} H_1(\dot{X}) - n$.

Therefore, the torsion subgroup of $H_1(X)$ can be calculated from the incidence matrix M_X.

Example 1.3. Let X be a multibranched surface with the graph representation shown in Fig. 6, where we consider the case of $n = 4$, $g_i = 0$ ($i = 1, 2, 3, 4$), and all degrees equal to 1. In [31, Example 4.2], the first homology group is calculated by using Theorem 1.2 to be $H_1(X) = (\mathbb{Z}/3\mathbb{Z}) \oplus \mathbb{Z}^4$.

As we shall see later, the incidence matrix of X is

$$M_X = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

This matrix is equivalent to the matrix (3) as follows:

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 3 & 3 & 3 & 3 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 3 \end{pmatrix}.$$

This shows that the torsion subgroup of $H_1(X)$ is $\mathbb{Z}/3\mathbb{Z}$.

On the other hand, a natural presentation for the fundamental group of a 2-stratifold was given in [18]. Thus, we can also obtain the first homology group via abelianization.

1.7. Circular permutation systems and slope systems. A *permutation* of a set A is a bijection from the additive group $\mathbb{Z}/n\mathbb{Z}$ into A. Two permutations σ and σ' of A are *equivalent* if there is an element $k \in \mathbb{Z}/n\mathbb{Z}$ such that $\sigma'(x) = \sigma(x + k)$ ($x \in \mathbb{Z}/n\mathbb{Z}$). The equivalence class of a permutation of A is a *circular permutation*.

For a regular multibranched surface X, we define the "circular permutation system" and "slope system" of X as follows. A circular permutation of $\mathcal{A}(l)$ is called a *circular permutation on the branch* l. A collection $\mathcal{P} = \{\mathcal{P}_l\}_{l \in \mathcal{L}(X)}$ is called a *circular permutation system* of X if $\mathcal{P}_l$ is a circular permutation on l. For a branch l, a rational number p/q with $q = d(l)$

is called a *slope* of l. A collection $\{\mathcal{S}_l\}_{l\in\mathcal{L}(X)}$ is called a *slope system* of X if $\mathcal{S}_l$ is a slope of l.

1.8. Neighborhoods. Let $X = B \cup_\phi S$ be a regular multibranched surface, and let $\mathcal{P} = \{\mathcal{P}_l\}_{l\in\mathcal{L}(X)}$ and $\mathcal{S} = \{\mathcal{S}_l\}_{l\in\mathcal{L}(X)}$ be a permutation system and a slope system of X, respectively. We will construct a compact orientable 3-manifold that is uniquely determined up to a homeomorphism by the pair of $\mathcal{P}$ and $\mathcal{S}$, by the following procedure.

First, for each branch $l \in \mathcal{B}(X)$ and each sector $s \in \mathcal{S}(X)$, we take a solid torus $l \times D^2$, where D^2 is a 2-disk, and take the product $s \times [-1,1]$. If s is nonorientable, then we take a twisted I-bundle $s\tilde{\times}[-1,1]$ over s. We endow these 3-manifolds with orientations.

Next, we glue them together according to the permutation system $\mathcal{P}$ and the slope system $\mathcal{S}$, where we assign the slope $\mathcal{S}_l$ of l to the isotopy class of a loop k_l in $\partial(l \times D^2)$, by an orientation-reversing map

$$\Phi\colon \partial S \times [-1,1] \to \partial(B \times D^2)$$

satisfying the condition that for each branch l and each prebranch c with $\phi(c) = l$, the restriction $\Phi|_{c\times[-1,1]}\colon c \times [-1,1] \to N\left(k_l; \partial\left(l \times D^2\right)\right)$ is a homeomorphism.

Then, we uniquely obtain a compact orientable 3-manifold with boundary, denoted by $N(X;\mathcal{P},\mathcal{S})$. The 3-manifold $N(X;\mathcal{P},\mathcal{S})$ is called the *neighborhood* of X with respect to $\mathcal{P}$ and $\mathcal{S}$. The set consisting of all neighborhoods of X is denoted by $\mathcal{N}(X)$.

§2. Background

2.1. Graphs. A graph G can be regarded as a 1-dimensional CW complex, where a vertex and an edge correspond to a 0-cell and 1-cell, respectively, and the vertices of an edge specify the attaching map for the 1-cell to 0-cells. This structure can be extended to 2-dimensional objects as in Sec. 1.2, that is, we extend vertices, edges, and the attaching map to a closed 1-dimensional manifold B (branch), a compact 2-dimensional manifold without closed components S (sector), and a covering map $\phi\colon \partial S \to B$, respectively. Then a multibranched surface X can be obtained as the quotient space $X = B \cup_\phi S$.

Kuratowski ([29]) proved that a graph G as a 1-dimensional CW complex cannot be embedded into $\mathbb{R}^2$ if and only if G contains the complete graph K_5 or the complete bipartite graph $K_{3,3}$ as a subspace. At the present time, this result is stated in the following form: G cannot be embedded into $\mathbb{R}^2$ if and only if G has K_5 or $K_{3,3}$ as a minor. Robertson and Seymour ([37]) showed that for any minor-closed property P, the set of minor-minimal graphs that do not have P is finite. This motivates us to consider the following problem: Characterize all "minor-minimal" multibranched surfaces that cannot be embedded in $\mathbb{R}^3$ (Problem 4.1). Since all closed nonorientable surfaces are minor-minimal multibranched surfaces, the set of "minor-minimal" multibranched surfaces that cannot be embedded in $\mathbb{R}^3$ is infinite. We will give the details in Sec. 4.

2.2. 2-Dimensional complexes. A 2-dimensional CW complex is a multibranched surface if removing all points whose open neighborhoods are homeomorphic to $\mathbb{R}^2$ yields a 1-dimensional complex homeomorphic to a disjoint union of simple closed curves. Thus, the set of multibranched surfaces is a subset of the set of 2-dimensional CW complexes.

Embeddings of 2-complexes into manifolds are widely studied in [23].

Matoušek, Sedgwick, Tancer, and Wagner ([30]) showed that there is an algorithm that, given a 2-dimensional simplicial complex K, decides whether K can be embedded (piecewise linearly or, equivalently, topologically) in $\mathbb{R}^3$.

Carmesin ([1–5]) proved that a locally 3-connected simply connected 2-dimensional simplicial complex has a topological embedding into the 3-space if and only if it has no space minor from a finite explicit list $\mathcal{Z}$ of obstructions.

2.3. Essential surfaces. The embedding of multibranched surfaces in the 3-sphere S^3 is closely related to the existence of essential surfaces in link exteriors. Let L be a link in S^3, and let F be an essential surface properly embedded in the exterior $E(L)$ of L whose boundary ∂F is non-meridional. By shrinking the regular neighborhood $N(L)$ into L and extending F along it, we obtain an essential multibranched surface X embedded in S^3, where we say that a multibranched surface X with branches B and sectors S embedded in S^3 is *essential* if $S \cap E(B)$ is essential, namely, incompressible, boundary-incompressible, and not boundary-parallel in $E(B)$. Conversely, let X be an essential multibranched surface with branches B and sectors S embedded in S^3. Then B is a link in S^3 and $S \cap E(B)$ is an essential surface properly embedded in $E(B)$ whose boundary is non-meridional. Therefore, the set of all pairs (L, F) of a link L in S^3 and an essential surface F properly embedded in the exterior of L whose boundary ∂F is non-meridional coincides with the set of all essential multibranched surfaces embedded in S^3.

2.4. The fundamental problem. The Menger–Nöbeling theorem ([7, Theorem 1.11.4]) shows that any finite 2-dimensional CW complex can be embedded in $\mathbb{R}^5$. Furthermore, any multibranched surface can be embedded in $\mathbb{R}^4$ ([31, Proposition 2.3]). More generally, any finite 2-dimensional simplicial complex whose intrinsic 1-skeleton is a proper subset of K_7 embeds in $\mathbb{R}^4$ ([14]).

If for a branch l there exist prebranches $c, c' \in \mathcal{A}(l)$ such that $d(c) \neq d(c')$, then the multibranched surface embeds in no 3-manifold. The converse also holds; namely, we have shown that a multibranched surface can be embedded in some closed orientable 3-manifold if and only if the multibranched surface is regular ([36, Corollary 2.4], [31, Proposition 2.7]).

We remark that any 3-manifold can be embedded in $\mathbb{R}^5$ ([43]). Thus, we obtain the following diagram showing the embeddability of multibranched surfaces (Fig. 3).

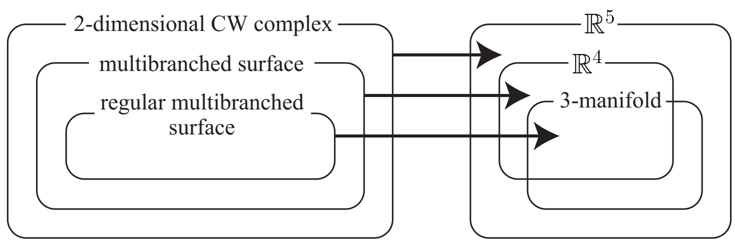

Figure 3. The embeddability of multibranched surfaces.

The following problems are fundamental for embeddings of multibranched surfaces.

Problem 2.1. For a regular multibranched surface X, find a simplest closed orientable 3-manifold M in which X can be embedded. Moreover, determine the minimal Heegaard genus of such a 3-manifold M.

Problem 2.2. For a regular multibranched surface X, determine whether or not X can be embedded in the 3-sphere S^3.

We consider Problem 2.1 in Sec. 3 and Problem 2.2 in Sec. 4.

§3. Embeddings into 3-manifolds

3.1. The Heegaard genus. For a closed orientable 3-manifold M, the Heegaard genus is a fundamental index. The *Heegaard genus* $g(M)$ of M is defined as the minimal genus of a closed orientable surface F embedded in M such that F separates M into two orientable handlebodies.

For an orientable compact 3-manifold N with boundary, the minimal Heegaard genus of closed orientable 3-manifolds in which N can be embedded is denoted by $\mathrm{eg}(N)$ and called the *embeddable genus* of N. We remark that $\mathrm{eg}(N) \leqslant g(N)$ ([31, Proposition 3.1]), where $g(N)$ denotes the minimal genus of Heegaard splittings of N in the sense of Casson and Gordon ([6]).

For a regular multibranched surface X, we define the *minimum genus* $\min g(X)$ and *maximum genus* $\max g(X)$, respectively, as follows:

$$\min g(X) = \min\{\mathrm{eg}(N) \mid N \in \mathcal{N}(X)\},$$
$$\max g(X) = \max\{\mathrm{eg}(N) \mid N \in \mathcal{N}(X)\}.$$

3.2. Upper bounds. The inequalities in the following theorem give upper bounds for the minimum and maximum genera. In fact, Theorem 3.5 of [31] states only that $\min g(X) \leqslant |\mathcal{B}(X)| + |\mathcal{S}(X)|$, but its proof is still effective for $\max g(X)$ and implies the latter half.

Theorem 3.1 ([31, Theorem 3.5]). *If X is a regular multibranched surface, then*

$$\max g(X) \leqslant |\mathcal{B}(X)| + |\mathcal{S}(X)|.$$

Moreover, if the degree of each branch of X is 1*, then*

$$\max g(X) \leqslant |\mathcal{S}(X)|.$$

Remark 3.2. In the proof of [31, Theorem 3.5], it is shown that X can be embedded in a connected sum of $|\mathcal{B}(X)|$ lens spaces and $|\mathcal{S}(X)|$ copies of $S^2 \times S^1$. Yuya Koda asked me whether any closed orientable 3-manifold contains a minimal genus embedding of some multibranched surfaces.

The next theorem follows from the two cited results and gives an estimate for the embeddable genus of a neighborhood of a regular multibranched surface.

Theorem 3.3 ([31, Theorem 3.6], [11, Lemma 2.2]). *If X is a regular multibranched surface and $N \in \mathcal{N}(X)$ is a neighborhood of X, then*

$$\mathrm{rank}\, H_1(X) - g(\partial N) \leqslant \mathrm{eg}(N) \leqslant \mathrm{rank}\, H_1(G_N) + g(\partial N),$$

where G_N denotes the abstract dual graph of N and $g(\partial N)$ is the sum of the genera of all components of ∂N.

3.3. Lower bounds. The following lower bounds for the minimum and maximum genera are known.

Theorem 3.4 ([11], cf. [40, Theorem 1.3])**.** *If X is a regular multibranched surface, then*

$$\min g(X) \geqslant \operatorname{rank} H_1(X) - \max_{N \in \mathcal{N}(X)} g(\partial N), \tag{3.1}$$

$$\max g(X) \geqslant \operatorname{rank} H_1(X) - \min_{N \in \mathcal{N}(X)} g(\partial N). \tag{3.2}$$

3.4. The graph product $G \times S^1$. For a graph G, we obtain a regular multibranched surface by taking the product with S^1. We consider the genus of a regular multibranched surface that forms $G \times S^1$ and, using Theorem 3.4, obtain the following theorem, which shows an interplay between the genus of a graph G and the genus of the multibranched surface $G \times S^1$.

The *minimum genus* $\min g(G)$ of a graph G is defined as the minimal genus of closed orientable surfaces in which G can be embedded. The *maximum genus* $\max g(G)$ of a graph G is defined as the maximal genus of closed orientable surfaces in which G can be embedded so that the complement of G consists of open disks. It is remarkable that Xuong and Nebeský determined the maximum genus of a graph by a completely combinatorial formula ([44, Theorem 3], [33, Theorem 2]).

Theorem 3.5 ([40, Corollary 1.2], [11])**.** *If G is a graph, then*

$$\min g(G \times S^1) = 2 \min g(G), \tag{3.3}$$

$$\max g(G \times S^1) = 2 \max g(G). \tag{3.4}$$

In [40, Corollary 1.2], it was shown that the minimum of $\dim H_1(M; \mathbb{F})$, where $\mathbb{F} = \mathbb{Z}_p$ or $\mathbb{Q}$, for a closed orientable 3-manifold M containing $G \times S^1$ is equal to $2 \min g(G)$. It is well known that $g(M) \geqslant \dim H_1(M; F)$. Hence, the inequality $\min g(G \times S^1) \geqslant 2 \min g(G)$ in Theorem 3.5 holds.

3.5. Spines of closed 3-manifolds. A multibranched surface X is called a 2-*stratifold* if each prebranch c of X satisfies $d(c) > 2$. Gómez-Larrañaga, González-Acuña, and Heil studied 2-stratifolds from the viewpoint of 3-manifold groups. They asked the following questions.

Question 3.6. Which 3-manifolds have fundamental groups isomorphic to the fundamental group of a 2-stratifold?

Question 3.7. Which closed 3-manifolds have spines that are 2-stratifolds?

Recall that a subpolyhedron P of a 3-manifold M is a *spine* of M if $M - \mathrm{int}(B^3)$ collapses to P, where B^3 is a 3-ball in M. An equivalent definition is that $M - P$ is homeomorphic to an open 3-ball.

Gómez-Larrañaga, González-Acuña, and Heil completely answered these questions.

Theorem 3.8 ([19, Theorem 1])**.** *Let M be a closed 3-manifold and X_G be a 2-stratifold. If $\pi_1(M) \cong \pi_1(X_G)$, then $\pi_1(M)$ is a free product of groups where each factor is cyclic or $\mathbb{Z} \times \mathbb{Z}_2$.*

Theorem 3.9 ([19, Theorem 2])**.** *A closed 3-manifold M has a 2-stratifold as a spine if and only if M is a connected sum of lens spaces, S^2-bundles over S^1, and copies of $P^2 \times S^1$.*

3.6. The neighborhood equivalence. In this subsection, we assume that a multibranched surface is regular, has no disk sectors, and the degree is greater than 2 for each branch. Let A be either an annulus sector of X whose boundary consists of two branches with at least one branch of degree 1 or a Möbius-band sector of X whose boundary has degree 1. An *IX-move* along A is a transformation shrinking A into the core circle, and an *XI-move* is a transformation reverse to an IX-move.

If two multibranched surfaces X and X' embedded in a 3-manifold M are related by IX-moves and XI-moves, then the regular neighborhoods $N(X)$ and $N(X')$ are isotopic in M. The following theorem states that the converse holds.

Theorem 3.10 ([26])**.** *Let X and X' be two multibranched surfaces embedded in an orientable 3-manifold M. If $N(X)$ is isotopic to $N(X')$ in M, then X can be transformed into X' by a finite sequence of IX-moves, XI-moves, and isotopies.*

For a larger class, the Matveev–Piergallini theorem is known: two simple polyhedra embedded in a 3-manifold have isotopic neighborhoods if and only if they are connected by a sequence of $2 \leftrightarrow 3$ *moves*, $0 \leftrightarrow 2$ *moves*, and isotopies ([32, 35]).

3.7. Neighborhood partial orders. Let X be an essential multibranched surface embedded in a closed orientable 3-manifold M. We say that a sector s of X is *excessive* if s is boundary-parallel in $M - \operatorname{int} N(X - s)$. A multibranched surface X is said to be *efficient* if every sector is not excessive.

In this subsection, we restrict multibranched surfaces to the set $\mathcal{X}$ of all connected compact multibranched surfaces X embedded in a closed orientable 3-manifold M satisfying the following conditions: X is maximally spread (that is, no XI-move is applicable to X), essential, and efficient in M, and has neither open disk sectors nor branches of degree less than 3.

Under the influence of Theorem 3.10, we define an equivalence relation on $\mathcal{X}$ as follows. Two multibranched surfaces X and X' in $\mathcal{X}$ are *neighborhood equivalent*, denoted by $X \overset{\mathrm{N}}{\sim} X'$, if X can be transformed into X' by a finite sequence of IX-moves and XI-moves. Moreover, we define a binary relation $\leqslant$ over $\mathcal{X}$ as follows. (As in Sec. 1.2, put $X = B_X \cup_{\phi_X} S_X$ and $Y = B_Y \cup_{\phi_Y} S_Y$; where by using the same symbols we assume that B_X, S_X, B_Y, S_Y are embedded in M.)

Definition 3.11. For $X = B_X \cup_{\phi_X} S_X$ and $Y = B_Y \cup_{\phi_Y} S_Y$ in $\mathcal{X}$, we set $X \leqslant Y$ if

(1) there exists an isotopy of Y in M such that $Y \subset N(X)$ and $B_Y \subset N(B_X)$, and
(2) there exists no essential annulus in $N(X) - Y$.

We define the *neighborhood partial order* $\preceq$ over the set $\mathcal{X}/\overset{\mathrm{N}}{\sim}$ by setting $[X] \preceq [Y]$ if $X \leqslant Y$ for equivalence classes $[X]$ and $[Y]$ in $\mathcal{X}/\overset{\mathrm{N}}{\sim}$.

Theorem 3.12 ([34])**.** *The relation $\preceq$ on the set $\mathcal{X}/\overset{\mathrm{N}}{\sim}$ is well defined, and $(\mathcal{X}/\overset{\mathrm{N}}{\sim}; \preceq)$ is a partially ordered set.*

We say that B_X is *toroidal* if there exists an essential torus T in the exterior $E(B_X)$ of B_X in M, that is, T is incompressible in $E(B_X)$ and T is not parallel to a torus in $\partial E(B_X)$. We say that E_X is *cylindrical*, where E_X stands for $E(B_X) \cap X$, if there exists an essential annulus A in $E(B_X)$ with $A \cap X = A \cap E_X = \partial A$, that is, A is incompressible and A is parallel to neither an annulus in E_X nor an annulus in $\partial E(B_X)$.

Theorem 3.13 ([34])**.** *Let $[X]$ and $[Y]$ be equivalence classes in $\mathcal{X}/\overset{\mathrm{N}}{\sim}$. If $[X] \preceq [Y]$ and $[X] \neq [Y]$, then either B_Y is toroidal or E_Y is cylindrical.*

Theorem 3.13 provides a sufficient condition for an equivalence class $[X] \in \mathcal{X}/\overset{\mathrm{N}}{\sim}$ to be minimal with respect to the partial order of $(\mathcal{X}/\overset{\mathrm{N}}{\sim}; \preceq)$, that is, if B_X is atoroidal and E_X is acylindrical, then $[X]$ is minimal.

3.8. Essential decompositions and Eudave-Muñoz knot types. Let X be a multibranched surface embedded into the 3-sphere S^3, and let $V_1, \dots, V_n$ be the regions into which X decomposes S^3. If X is essential, then we call this decomposition $S^3 = V_1 \cup \dots \cup V_n$ an *essential decomposition*. As explained in Sec. 2.3, a link with an essential surface of non-meridional boundary slope gives an essential decomposition.

In this subsection, we recall the concept of Eudave-Muñoz knots ([9]) in the language of multibranched surfaces. Let X be a multibranched surface having a two-holed torus as a unique sector s and a single branch l such that one prebranch c has $\mathrm{od}(c) = 2$ and another prebranch c' has $\mathrm{od}(c') = -2$.

Suppose that X is embedded in S^3 so that it is essential and the two regions of $S^3 - X$ are genus two handlebodies, say H and W. Then, by combining [9] with [22], the branch l forms an Eudave-Muñoz knot. From the point of view that any essential embedding restricts the knot type of the branch, this phenomenon is special for low-dimensional geometric topology.

Eudave-Muñoz knots appear in the last piece of the classification of essential annuli in the exterior of genus two handlebody-knots in the 3-sphere S^3 ([27]). We take a regular neighborhood $N(l)$ and denote two handlebodies $S^3 - N(l) - s$ by H and W again. See Fig. 4 for the configuration. Put $A = N(l) \cap W$. Then H is a genus two handlebody-knot with an essential annulus A of type 4 in [27].

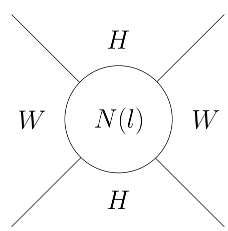

Figure 4. The $(1, 2, 2; 2)$-trisection coming from Eudave-Muñoz knots.

The configuration shown in Fig. 4 also provides a nice example of a trisection. Let X' be a multibranched surface having two branches $b \cup b' = N(l) \cap H \cap W$ and three sectors $s_1 = H \cap W$, $s_2 = N(l) \cap H$, and $s_3 = N(l) \cap W$. Then X' gives an essential decomposition $S^3 = N(l) \cup H \cup W$, where the triple of genera of three handlebodies is $(1, 2, 2)$ and the number of branches is 2. Thus, this gives a $(1,2,2;2)$-trisection of S^3. Moreover, it is shown in [28, Proposition 4.7.1] that this trisection is not a stabilization of any other trisection.

3.9. Efficient embeddings and universal bounds. Recall the relation between essential surfaces in link exteriors and essential multibranched surfaces from Sec. 2.3, and the definition of efficient embedding from Sec. 3.7. Suppose that X is an essential and efficient multibranched surface embedded in a 3-manifold. Then we have a link and essential surfaces in the link exterior, and, moreover, no two essential surfaces are mutually parallel.

Let X be a multibranched surface with a single branch and precisely n sectors each of which is a one-holed torus with oriented degree 1. Suppose that X is embedded in S^3 so that it is essential and efficient and the branch forms a hyperbolic knot. Then we have a hyperbolic knot bounding n pairwise nonparallel Seifert surfaces of genus 1. Tsutsumi first showed that the number n is at most 7 ([41]). After that, Eudave-Muñoz, Ramírez-Losada, and Valdez-Sánchez showed that n is at most 6 and provided an example of such an embedding of X for $n = 5$ ([12]). Finally, Valdez-Sánchez showed that n is at most 5 ([42]) and, therefore, this bound is optimal.

This phenomenon is also special for low-dimensional geometric topology. Typically, contrary to the above, there is no upper bound. Tsutsumi showed that for any positive integer n there is a genus one hyperbolic knot in S^3 that bounds pairwise nonparallel incompressible Seifert surfaces $S, F_1, \dots, F_n$, where S is of genus 1 and F_i is of genus 2 ([41, Theorem 5.5]).

§4. Forbidden minors for S^3

4.1. Minors and obstruction sets. In this subsection, we allow the degree $d(B_i)$ of a branch B_i to be 1 or 2.

We denote by $\mathcal{M}$ the set of all regular multibranched surfaces (modulo homeomorphism). For X and Y in $\mathcal{M}$, we write $X < Y$ if X is obtained from Y either by an IX-move or by removing a sector of Y. We define an equivalence relation $\sim$ on $\mathcal{M}$ as follows: if $X < Y$ and $Y < X$, then $X \sim Y$.

We define a partial order $\prec$ on $\mathcal{M}/\sim$ as follows. Let X, $Y \in \mathcal{M}$. We set $[X] \prec [Y]$ if there exists a finite sequence $X_1, \dots, X_n \in \mathcal{M}$ such that $X_1 \sim X$, $X_n \sim Y$, and $X_1 < \dots < X_n$.

A multibranched surface class $[X]$ is called a *minor* of another multibranched surface class $[Y]$ if $[X] \prec [Y]$. In particular, $[X]$ is called a *proper minor* of $[Y]$ if $[X] \prec [Y]$ and $[Y] \neq [X]$. A subset $\mathcal{P}$ of $\mathcal{M}/\sim$ is said to be *minor-closed* if for any $[X] \in \mathcal{P}$, every minor of $[X]$ belongs to $\mathcal{P}$. For a minor-closed set $\mathcal{P}$, we define the *obstruction set* $\Omega(\mathcal{P})$ as the set of all elements $[X] \in \mathcal{M}/\sim$ such that $[X] \notin \mathcal{P}$ and every proper minor of $[X]$ belongs to $\mathcal{P}$.

The set of all multibranched surfaces embeddable into S^3, denoted by $\mathcal{P}_{S^3}$, is minor-closed. As a 2-dimensional version of Kuratowski's and Wagner's theorems, we consider the following problem.

Problem 4.1. Characterize the obstruction set $\Omega(\mathcal{P}_{S^3})$.

We summarize all known results on $\Omega(\mathcal{P}_{S^3})$ at the present moment. As we shall see later, (2) and (3) in Theorem 4.2 are infinite families of multibranched surfaces; we shall explain the notation X_1, X_2, X_3, and $X_g(p_1, \dots, p_n)$ only after stating Theorem 4.2.

Theorem 4.2. *The following multibranched surfaces belong to* $\Omega(\mathcal{P}_{S^3})$*:*

1. $K_5 \times S^1$ *and* $K_{3,3} \times S^1$ ([39]),
2. *all multibranched surfaces of the forms* X_1*,* X_2*, and* X_3 ([8]),
3. *all multibranched surfaces of the form* $X_g(p_1, \dots, p_n)$ ([31]).

Remark 4.3. (1) Since any proper minor of K_5 and $K_{3,3}$ is planar, any proper minor of $K_5 \times S^1$ and $K_{3,3} \times S^1$ can be embedded in $D^2 \times S^1 \subset S^3$.

(2) We say that a multibranched surface X is *critical* for S^3 if X cannot be embedded in S^3 and $X - x$ can be embedded in S^3 for any $x \in X$. It is shown in [8] that all multibranched surfaces of the forms X_1, X_2, and X_3 are critical for S^3.

(3) Since each multibranched surface of the form $X_g(p_1, \dots, p_n)$ has a single sector, the minimality for $\mathcal{P}_{S^3}$ naturally holds.

Theorem 4.2 (1) was proved in [39, Theorem 1]. It also follows from Theorem 3.5 and Kuratowski's and Wagner's theorems.

The families of multibranched surfaces of the forms X_1, X_2, and X_3 in Theorem 4.2 (2) are defined as follows.

Let X_1 be a multibranched surface having a single branch and obtained from a single sector of genus g with precisely n boundary components by a covering map of degree ϵ_i on each prebranch. See Fig. 5 for a graph representation. We assume that $\epsilon_i = \pm p$ for the regularity of X_1. Then the incidence matrix is $M_{X_1} = \big(\sum_{i=1}^{n} \epsilon_i\big)$. If $\big|\sum_{i=1}^{n} \epsilon_i\big| > 1$, then $H_1(X_1)$ has torsion and X_1 cannot be embedded in S^3. Conversely, if $\left|\sum_{i=1}^{n} \epsilon_i\right| \leqslant 1$, then, by [8, Theorem 3.2], X_1 can be embedded in S^3. Hence, $X_1 \in \Omega(\mathcal{P}_{S^3})$ if and only if $\left|\sum_{i=1}^{n} \epsilon_i\right| > 1$.

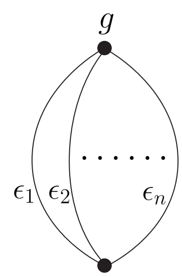

Figure 5. A graph representation of X_1.

Let X_2 be a multibranched surface having a graph representation of the form shown in Fig. 6, where $n \geqslant 3$ and all degrees are 1 (we omit the labels on edges). Then, by [8, Theorem 3.3], $X_2 \in \Omega(\mathcal{P}_{S^3})$.

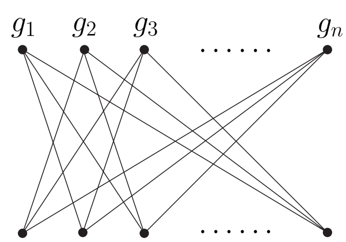

Figure 6. A graph representation of X_2.

The incidence matrix of X_2 is

$$M_{X_2} = \begin{pmatrix} 0 & 1 & \cdots & \cdots & 1 \\ 1 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 0 \end{pmatrix}.$$

Since $\det(M_{X_2}) = (-1)^{n+1}(n-1)$ and $n \geqslant 3$, it follows that $H_1(X_2)$ has torsion.

Let X_3 be a multibranched surface having a graph representation of the form shown in Fig. 7, where $n \geqslant 2$, $k_i \geqslant 1$, $k_1k_2k_3 \cdots k_n \geqslant 3$, and all degrees are 1 unless otherwise specified. Then, by [8, Theorem 3.7], $X_3 \in \Omega(\mathcal{P}_{S^3})$.

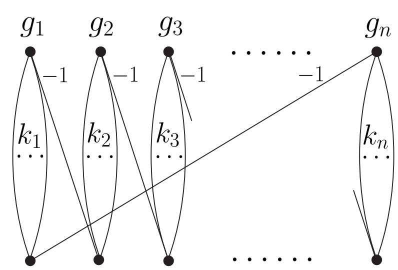

Figure 7. A graph representation of X_3.

The incidence matrix of X_3 is

$$M_{X_3} = \begin{pmatrix} k_1 & 0 & \cdots & \cdots & 0 & -1 \\ -1 & k_2 & 0 & \ddots & \ddots & 0 \\ 0 & -1 & k_3 & \ddots & \ddots & \vdots \\ \vdots & 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & k_{n-1} & 0 \\ 0 & \cdots & \cdots & 0 & -1 & k_n \end{pmatrix}.$$

Since $\det(M_{X_3}) = k_1k_2k_3 \cdots k_n - 1 \geqslant 2$, it follows that $H_1(X_3)$ has torsion.

The multibranched surface $X_g(p_1,\dots,p_n)$ in Theorem 4.2 (3) was first presented in [31, Example 4.3]. Let $X_g(p_1,\dots,p_n)$ be a multibranched surface having a graph representation of the form shown in Fig. 8, where $n \geqslant 1$ and $p = \gcd\{p_1,\dots,p_n\} > 1$. As we have seen in Example 1.1, a closed nonorientable surface of crosscap number n is homeomorphic to $X_0(2,\dots,2)$.

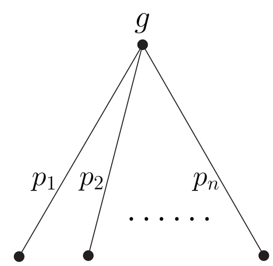

Figure 8. A graph representation of $X_g(p_1,\dots,p_n)$.

As shown in [31, Example 4.3], we have

$$H_1(X_g(p_1,\dots,p_n)) = (\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Z}^{2g+n-1}.$$

Hence, $X_g(p_1,\dots,p_n)$ cannot be embedded in S^3, since $p > 1$.

4.2. Beyond torsion. In the previous subsection, we conclude that some multibranched surfaces cannot be embedded in S^3 because of the torsion part of the first homology group. We recall that $X_g(p_1,\dots,p_n)$ in Theorem 4.2 (3) cannot be embedded in S^3 if $p = \gcd\{p_1,\dots,p_n\} > 1$. Then, the following inverse problem naturally arises.

Problem 4.4 ([31, Problem, p. 631])**.** If $p = 1$, then can $X_g(p_1,\dots,p_n)$ be embedded in S^3?

The following theorem gives a partial answer to Problem 4.4.

Theorem 4.5 ([10, Theorem 1.5])**.** *If $p = 1$, then $X_g(p_1,p_2,p_3)$ can be embedded in S^3 for a sufficiently large g.*

But what can we say about Problem 4.4 when $g = 0$? This is related to a main theme in [10]. In [10], we characterized nonhyperbolic 3-component links, in the 3-sphere, whose exteriors contain essential 3-holed spheres with non-integral boundary slopes. This implies that we can derive a formula for the triple p_1, p_2, p_3 ([10, Proposition 1.4]). For hyperbolic links, we conjectured the following.

Conjecture 4.6 ([10, Conjecture 1.1], cf. [20,21])**.** There does not exist an essential n-punctured sphere with non-meridional, non-integral boundary slope in a hyperbolic link exterior in the 3-sphere.

It can be checked that the triple $(5, 7, 18)$ does not satisfy the formula in [10, Proposition 1.4]. Therefore, assuming Conjecture 4.6, we conclude that $X_0(5, 7, 18)$ cannot be embedded in S^3.

On the other hand, if we allow embeddings in 3-manifolds other than S^3, then Problem 4.4 holds. We use a result of [15] that a compact 3-manifold M with connected boundary can be embedded in a homology 3-sphere if and only if $H_1(M)$ is free and $H_2(M) = 0$. Since for a unique neighborhood $N \in \mathcal{N}(X_g(p_1, \dots, p_n))$, $H_1(N)$ is free and $H_2(N) = 0$ when $p = 1$, we have the following.

Theorem 4.7. *If $p = 1$, then $X_g(p_1, \dots, p_n)$ can be embedded in a homology 3-sphere.*

§5. The prospects

The author would like to conclude this survey article by stating the following prospects.

Firstly, it is important to characterize essential and efficient decompositions of S^3, where we say that a decomposition $S^3 = V_1 \cup \cdots \cup V_n$ by a multibranched surface X is *efficient* if X is efficient. This can be applied to polycontinuous patterns, trisections, essential surfaces as stated in the introduction.

Secondly, it is a fundamental problem to characterize the obstruction set $\Omega(\mathcal{P}_{S^3})$. This problem has a difficulty as stated in Sec. 4.2, but it is also of interest for Conjecture 4.6.

Acknowledgments. The author would like to thank Fico González-Acuña and Arkadiy Skopenkov for information about related results. The author is grateful to the referee for several useful comments on the first submission and to the editors for pointing out many inconsistent things in the second submission.

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Поступило 13 декабря 2020 г.