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PROJECTED AND NEAR-PROJECTED EMBEDDINGS

Abstract. A stable smooth map $f\colon N \to M$ is called *k-realizable* if its composition with the inclusion $M \subset M \times \mathbb{R}^k$ is C^0-approximable by smooth embeddings; and a *k-prem* if the same composition is C^∞-approximable by smooth embeddings, or, equivalently, if f lifts vertically to a smooth embedding $N \hookrightarrow M \times \mathbb{R}^k$.

It is obvious that if f is a k-prem, then it is k-realizable. We refute the so-called "prem conjecture" that the converse holds. Namely, for each $n = 4k + 3 \geqslant 15$ there exists a stable smooth immersion $S^n \looparrowright \mathbb{R}^{2n-7}$ that is 3-realizable but is not a 3-prem.

We also prove the converse in a wide range of cases. A k-realizable stable smooth fold map $N^n \to M^{2n-q}$ is a k-prem if $q \leqslant n$ and $q \leqslant 2k - 3$; or if $q < n/2$ and $k = 1$; or if $q \in \{2k - 1, 2k - 2\}$ and $k \in \{2, 4, 8\}$ and n is sufficiently large.

§1. Introduction

1.1. k-Prems. We call a map $f\colon N \to M$ a (PL/smooth) *k-prem* (*k*-codimensionally **projected embedding)** if it factors into the composition of some (PL/smooth) embedding $N \hookrightarrow M \times \mathbb{R}^k$ and the projection $M \times \mathbb{R}^k \to M$. For example, a constant map f is a k-prem if and only if N embeds in $\mathbb{R}^k$. The abbreviation "prem" was coined by A. Szűcz (see [2,39]). Let us mention some results on the subject (further references can be found in [4,29,30]).

(1) It is well known and easy to see that stable[1] smooth functions $S^1 \to \mathbb{R}^1$ are smooth 1-prems (cf. Example 1.7 below). On the other hand, there exist stable PL functions on trees, "letter H"$\to \mathbb{R}^1$ and "letter X"$\to \mathbb{R}^1$, that are not PL 1-prems (Siekłucki [38]; see also another proof in [7, §3]).

(2) It is not hard to show that stable smooth functions $M^2 \to \mathbb{R}^1$ on an orientable surface are smooth 2-prems ([23]; see also [27, proof of the Yamamoto–Akhmetiev theorem]). Tarasov and Vyalyi

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[1]See [15] concerning stable (i.e., C^∞-stable) smooth maps and [30, Part I, Appendix B] concerning stable PL maps.

constructed a stable PL function $f\colon M^2 \to \mathbb{R}^1$ on an orientable surface of a high genus that is not a PL 2-prem [41]. They also proved that stable PL functions $S^2 \to \mathbb{R}^1$ are 2-prems.

(3) Stable smooth maps $M^2 \to \mathbb{R}^2$ of an orientable surface are smooth 2-prems (Yamamoto [42]). Stable smooth maps $S^2 \to S^2$ are smooth 2-prems (Yamamoto–Akhmetiev; see [27]). Stable PL/smooth maps $S^n \to S^n$, $n \geqslant 3$, are PL/smooth n-prems (Melikhov [30]).

(4) Every stable smooth approximation of the composition $S^2 \to \mathbb{R}P^2 \looparrowright \mathbb{R}^3$ of the 2-covering and the Boy immersion is not a smooth 1-prem (Akhmetiev). This follows from the results of [3] for C^∞-approximations, but the proof there works for C^0-approximations as well. See also [9, 14, 31, 32, 37].

(5) Every finite covering between spaces that are homotopy equivalent to n-polyhedra is an $(n+1)$-prem (Hansen [17]). Every nondegenerate[2] PL map between compact n-polyhedra is a PL $(n+1)$-prem (Melikhov [30]). It is easy to see (cf. Proposition 1.10 and its proof) that every stable PL/smooth map $N^n \to M^m$ is a PL/smooth $(2n+1-m)$-prem.

(6) A stable PL map is a k-prem if and only if it is a PL k-prem (Melikhov [30]).

For a stable smooth map $f\colon N^n \to M^m$, where N is compact and $m \geqslant n$, it is not hard to show that f is a smooth k-prem if and only if the composition $N \xrightarrow{f} M \subset M \times \mathbb{R}^k$ is C^∞-approximable by embeddings (see Proposition 1.10 below).

1.2. k-Realizable maps. We say that a map $f\colon N^n \to M^m$ is (PL/smoothly) *k-realizable* (realization by embeddings) if the composition $N \xrightarrow{f} M \subset M \times \mathbb{R}^k$ is C^0-approximable by (PL/smooth) embeddings. Topological, PL, and smooth realizability are equivalent in the metastable range $2(m+k) \geqslant 3(n+1)$ [16].

Since every compact subset of $\mathbb{R}^k$ can be brought into an arbitrarily small neighborhood of the origin by a diffeomorphism of $\mathbb{R}^k$, every k-prem is k-realizable.

It is quite easy to construct nonstable maps that are k-realizable but not k-prems (see Examples 1.7, 1.8, 1.9 below).

[2]A PL map is called *nondegenerate* if it has no point-inverses of dimension > 0.

The following conjecture appears in the first paragraph of the 2002 paper [7]:

Conjecture 1.3 (prem conjecture)**.** *Smoothly k-realizable general position*[3] *smooth maps $N^n \to \mathbb{R}^m$ are smooth k-prems, at least in the metastable range $2(m+k) \geqslant 3(n+1)$.*

It was proved in [7] that a 1-realizable stable smooth map $N^n \to \mathbb{R}^{2n-1}$, $n \geqslant 3$, is a smooth 1-prem. It was also proved in [7] that a smoothly 1-realizable stable smooth map of an orientable surface $M^2 \to \mathbb{R}^3$ is a smooth 1-prem (the idea of the proof for $M^2 = S^2$ appears already in [3]).

Theorem 1.4. *There exists a stable PL function $f\colon M \to \mathbb{R}^1$ on some closed orientable 2-manifold such that f is PL 2-realizable but is not a PL 2-prem.*

This is an easy consequence of the above-mentioned result of Tarasov and Vyalyi [41].

Proof. Let $f\colon M^2 \to \mathbb{R}^1$ be the stable PL function of [41] that is not a PL 2-prem. Since f is approximable by stable smooth functions, which are 2-prems (see item (2) in §1.1), f is 2-realizable. $\square$

1.5. Main results. We construct a counterexample to the prem conjecture.

Theorem 1. *For each $n = 4k+3 \geqslant 15$ there exists a stable smooth immersion $S^n \looparrowright \mathbb{R}^{2n-7}$ that is 3-realizable but is not a 3-prem.*

We also show that the prem conjecture holds in a wide range of dimensions.

Theorem 2. *Let N^n be a compact smooth manifold (respectively, a compact polyhedron), M^m a smooth (PL) manifold, and $f\colon N \to M$ a stable smooth (PL) map, where $m \geqslant n$ and $2n-m \leqslant 2k-3$. In the smooth case, assume additionally that either f is a fold map or $3n-2m \leqslant k$. Then f is k-realizable if and only if it is a smooth (PL) k-prem.*

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[3]The exact meaning of "general position maps" is not discussed in [7], but this term must have been imported, via [2] and [40], from [39], where "generic maps" are defined as Boardman maps with normal crossings (in the terminology of [15]). Such maps form an open dense subset of $C^\infty(N, M)$ and hence include all stable maps (see [15]).

Theorem 2 implies the first author's 2004 conjecture [4, Conjecture 1.9]: a generic smooth map $S^n \to S^n$ is k-realizable if and only if it is a k-prem, as long as $n \leqslant 2k-3$ (except that the definition of "generic" in [4] has to be improved). On the other hand, Theorem 2 for stable smooth maps falls short of the second author's 2004 announcement in [27], where it was expected to hold under the weaker restriction $4n-3m \leqslant k$.

The proof of these results is based on the following two criteria. Let $f : N \to M$ be a stable PL or smooth map.

- By recent work of the second author [30], f is a k-prem if and only if $\Delta_f := \{(x,y) \in N \times N \setminus \Delta_N \mid f(x) = f(y)\}$ admits a $\mathbb{Z}/2$-equivariant map to S^{k-1}, assuming that either f is a fold map or $3n-2m \leqslant k$ in the smooth case.
- Building on some previous work [4, 26, 34], we show that f is k-realizable if and only if Δ_f admits a *stable* $\mathbb{Z}/2$-map to S^{k-1}, i.e., for some x there exists a $\mathbb{Z}/2$-map $\Delta_f * S^x \to S^{k-1} * S^x = S^{k+x}$.

Due to these two criteria, in the stable range we prove the equivalence of the two notions (k-realizable and k-prem), whereas in the unstable range we have a chance to realize the difference known from homotopy theory by a geometric example. The latter task is nontrivial, which is why there is a huge gap in dimensions between Theorems 1 and 2.

The remainder of the paper is largely devoted to the study of this gap.

It turns out pretty quickly that we could not have had $k = 1$ in Theorem 1.

Theorem 3. *Let N^n be a compact smooth manifold (respectively, a compact polyhedron), M^m a smooth (PL) manifold, and suppose that $2m > 3n$. A stable smooth (PL) map $f\colon N \to M$ is 1-realizable if and only if it is a smooth (PL) 1-prem.*

In fact, "stable equivariant maps" as defined above are not the same as "stable equivariant maps" in the usual sense of homotopy theory. For example, our actions of $\mathbb{Z}/2$ do not have any fixed points, but they must always be present in the traditional setup of equivariant homotopy theory. In §3 we reformulate our problem in the traditional algebraic setup, which already brings some progress.

In a relative situation, we are able to produce a simpler example, with $k = 2$ instead of $k = 3$.

Theorem 4. *For each $n \geqslant 7$ there exists a stable smooth immersion $f\colon S^n \looparrowright \mathbb{R}^{2n-2}$ and its 2-realization $g\colon S^n \hookrightarrow \mathbb{R}^{2n}$ not isotopic through 2-realizations of f to any vertical lift of f.*

Also, due to a little gap between the "stable range" in the traditional algebraic setup and the "stable range" in our initial approach above, with some numeric luck regarding the value of k we are able to go one dimension deeper.

Theorem 5. *Let N^n be a compact smooth manifold (respectively, a compact polyhedron), $M^{2n-2k+2}$ a smooth (PL) manifold, and $f\colon N \to M$ a stable smooth (PL) map, where $n \geqslant 2k-1$. In the smooth case, assume additionally that either f is a fold map or $n \geqslant 3k-4$. If $f\colon N \to M$ is k-realizable and $k \in \{2,4,8\}$, then f is a smooth (PL) k-prem.*

In §4 we further reformulate a part of the problem, which has by now become fully algebraic, in geometric terms (using the Pontryagin–Thom construction). This enables us to construct a secondary obstruction in stable cohomotopy and prove its vanishing (Theorem 4.13), which yields that the stable and the unstable cohomotopy Euler classes of the sum of k copies of a line bundle over an n-polyhedron X are equally strong in the first unstable dimension $n = 2k-1$ when k is even.

As a consequence, we can sometimes go down one more dimension.

Theorem 6. *Let N^n be a compact smooth manifold (respectively, a compact polyhedron), $M^{2n-2k+1}$ a smooth (PL) manifold, and $f\colon N \to M$ a stable smooth (PL) map, where $n \geqslant 2k+1$. In the smooth case, assume additionally that either f is a fold map or $n \geqslant 3k-2$. If $f\colon N \to M$ is k-realizable and $k \in \{2,4,8\}$, then f is a smooth (PL) k-prem.*

This theorem disproves the first author's 2004 conjecture [4, Conjecture 1.10]: "There exists a generic immersion $f\colon S^7 \to \mathbb{R}^{11}$ such that the composition of f with the inclusion $\mathbb{R}^{11} \subset \mathbb{R}^{13}$ is discretely realizable but not C^1-approximable by embedding."

1.6. Examples and remarks. The following simple examples explain why one restricts attention to generic/stable maps in the prem conjecture.

Example 1.7 (a 1-realizable Morse function on S^1 that is not a 1-prem)**.** Consider the composition $S^1 \xrightarrow{p} S^1 \xrightarrow{\varphi} \mathbb{R}$ of the 2-covering p and the simplest Morse function φ with two critical points x, y. If $q\colon S^1 \to \mathbb{R}$ is a map such that $q \times (\varphi p)\colon S^1 \to \mathbb{R} \times \mathbb{R}$ is an embedding, then $q|_{p^{-1}(x)}\colon S^0 \hookrightarrow$

$\mathbb{R}$ is isotopic to $q|_{p^{-1}(y)}\colon S^0 \hookrightarrow \mathbb{R}$ for each choice of a homeomorphism between $p^{-1}(x)$ and $p^{-1}(y)$. Hence S^0 can be everted by an isotopy within $\mathbb{R}^1$, which is absurd.

On the other hand, φp is 1-realizable, since every stable smooth function $f\colon S^1 \to \mathbb{R}$ is a 1-prem. Indeed, let $y \in \mathbb{R}$ be the absolute maximum of f. If $\varepsilon > 0$ is sufficiently small, $f^{-1}([-\infty, y-\varepsilon))$ is an arc I. The graph $\Gamma(f|_I)\colon I \hookrightarrow I \times [-\infty, y-\varepsilon]$ combines with the graph $\Gamma(f|_{\overline{S^1\setminus I}})\colon \overline{S^1 \setminus I} \hookrightarrow I \times [y-\varepsilon, +\infty)$ into an embedding $S^1 \to I \times \mathbb{R}$ that projects onto f.

Example 1.8 (a 2-realizable Morse function on S^2 that is not a 2-prem)**.** The complete graph on 5 vertices K_5 contains a Hamiltonian (i.e., passing once through each vertex) cycle Z_1 of length 5, and the remaining 5 edges of K_5 form another Hamiltonian cycle Z_2. By switching the thus obtained smoothing of each vertex to the opposite one (without changing the orientations of the edges), we get an Eulerian (i.e., passing once through each edge) cycle Z. Attaching three 2-disks to K_5 along Z_1, Z_2, and Z, we obtain a genus two closed surface M and a nonstable Morse function $f\colon M \to \mathbb{R}^1$ whose only degenerate critical levels are: a two-point set $f^{-1}(1)$ (both points are maxima); $K_5 = f^{-1}(0)$ (with a saddle point at each vertex of K_5); a singleton $f^{-1}(-1)$ (which is a minimum). Since every stable Morse function on an orientable surface is a 2-prem (see item 2 in §1.1), f is 2-realizable. However, f is not a 2-prem, since K_5 does not embed in $\mathbb{R}^2$.

Example 1.9 (a 3-realizable branched covering $S^3 \to S^3$ that is not a 3-prem)**.** The join of two copies of the 2-covering $S^1 \to S^1$ is a nonstable map $S^3 \to S^3$. It factors into the composition of the 2-covering $p\colon S^3 \to \mathbb{R}P^3$ and the 2-fold covering $q\colon \mathbb{R}P^3 \to S^3$ branched along the Hopf link. Since p is not a 3-prem by the Borsuk–Ulam theorem (see [27]), nor is qp. However, by [27], qp is 3-realizable.

Proposition 1.10. *Let $f\colon N^n \to M^m$ be a stable smooth map between smooth manifolds, where $m \geqslant n$ and N is compact. Then f is a smooth k-prem if and only if the composition of f with the inclusion $j\colon M \times \{0\} \hookrightarrow M \times \mathbb{R}^k$ is C^∞-approximable by embeddings.*

Proof. Given a smooth map $g\colon N \to \mathbb{R}^k$ such that $f \times g\colon N \to M \times \mathbb{R}^k$ is a smooth embedding, since N is compact, for each $\varepsilon > 0$ one can find $\delta > 0$ such that $f \times \delta g$ is C^∞-ε-close to jf.

Conversely, if $f' \times g\colon N \to M \times \mathbb{R}^k$ is a smooth embedding C^∞-ε-close to jf, then f' is C^∞-ε-close to f. If $\varepsilon > 0$ is sufficiently small, since f is stable, there exist diffeomorphisms $h\colon N \to N$ and $H\colon M \to M$ such that $fh = Hf'$. Then $H \times \mathrm{id}_{\mathbb{R}^k}$ takes $f' \times g$ onto $(fh) \times g$, which is therefore a smooth embedding. Hence $f \times (gh^{-1})$ is also a smooth embedding. So f is a k-prem. $\square$

Remark 1.11. In this remark we use "generic" in the sense of [30]. Let N^n and M^m be smooth manifolds, where $m \geqslant n$ and N is compact.

(a) By Mather's theorem [24], generic smooth maps $f\colon N \to M$ are stable if either $6m \geqslant 7n-7$, or $6m \geqslant 7n-8$ and $m \leqslant n+3$. By the multijet transversality, generic smooth immersions $N \to M$ are stable [15, III.3.3, III.3.11]. More generally, by Morin's canonical form and well-known results of Mather and Boardman, generic corank one maps $N \to M$ are stable (see [30, Part I, Theorem A.2]). In particular, generic fold maps $N \to M$ are stable.

(b) The "only if" part of Proposition 1.10 holds without assuming that f is stable. The "if" part has the following version with a weaker hypothesis and a weaker conclusion. If $f\colon N \to M$ is a generic smooth map such that jf is C^∞-approximable by embeddings, then f is a topological k-prem. This is proved similarly to Proposition 1.10, using that f is C^0-stable [13].

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§2. The prem conjecture

Let $S^k_\circ$ denote the k-sphere endowed with the antipodal involution. Let $\tilde{X} = X \times X \setminus \Delta$ with the factor exchanging involution t. Given a map $f\colon X \to M$, set $\Delta_f = \{(x,y) \in \tilde{X} \mid f(x) = f(y)\}$. Then $\tilde{f}\colon \tilde{X} \setminus \Delta_f \to S^{m-1}_\circ$ is defined by $(x,y) \mapsto \frac{f(x)-f(y)}{||f(x)-f(y)||}$.

Theorem 2.1 (Melikhov [30])**.** *Let N^n be a compact smooth manifold (respectively, a compact polyhedron), M^m a smooth (PL) manifold, and $f\colon N \to M$ a stable smooth (PL) map, where $m \geqslant n$ and $2(m+k) \geqslant 3(n+1)$. In the smooth case, assume additionally that either f is a fold map or $3n-2m \leqslant k$. Then f is a smooth (PL) k-prem if and only if there exists an equivariant map $\Delta_f \xrightarrow{\varphi} S^{k-1}_\circ$.*

Moreover, for any such φ there exists a smooth (PL) embedding $g\colon N \hookrightarrow M \times \mathbb{R}^k$ projecting to f and such that $\tilde{g}|_{\Delta_f}$ is equivariantly homotopic to φ.

Remark 2.2. In the case where f is a triple-point-free immersion, Theorem 2.1 is obvious. Indeed, in this case $\Delta_f \subset N \times N$ projects homeomorphically onto $S_f \subset N$, so any extension of the equivariant map $S_f \to S^{k-1}$ to a smooth (PL) map $\psi\colon N \to \mathbb{R}^k$ yields a smooth (PL) embedding $f \times \psi\colon N \hookrightarrow M \times \mathbb{R}^k$.

Remark 2.3. As discussed in [30], the hypothesis of stability can be somewhat relaxed in the PL case and in the nonfold smooth case of Theorem 2.1. Consequently, this hypothesis can also be similarly relaxed in all our main results of positive type.

Theorem 2.4. *Let X^n be a compact polyhedron, Q^q a PL manifold, and $f\colon X \to Q$ a PL map, where $2q \geqslant 3(n+1)$.*

(a) (A. Skopenkov [34]). *When $Q = \mathbb{R}^q$, f is C^0-approximable by embeddings if and only if $\tilde{f}\colon \tilde{X} \setminus \Delta_f \to S^{q-1}$ extends, after an equivariant homotopy, to an equivariant map $\tilde{X} \xrightarrow{\varphi} S^{q-1}_\circ$.*

Moreover, for any such φ and any $\varepsilon > 0$ there exists a PL embedding g that is C^0-ε-close to f and such that $\tilde{g}$ and φ are equivariantly homotopic with support in an equivariant regular neighborhood of Δ_f.

(b) (Melikhov [26, 1.7(a,a+)]). *In general, f is C^0-approximable by embeddings if and only if $f \times f\colon X \times X \to Q \times Q$ is C^0-approximable by isovariant maps.*

Moreover, there exists $\delta > 0$ such that if $\Phi\colon X \times X \to Q \times Q$ is an isovariant map C^0-δ-close to $f \times f$, then for each $\varepsilon > 0$ there exists a PL embedding g that is C^0-ε-close to f and such that $g \times g$ and Φ are isovariantly homotopic.

2.5. Proof of Theorem 2. Using Theorem 2.4, we will now prove the following.

Theorem 2.6. *Let X^n be a compact polyhedron, M^m a compact PL manifold, and $f\colon X \to M$ a stable PL map, where $2(m+k) \geqslant 3(n+1)$. Then f is k-realizable if and only if $S^{m-1}_\circ * \Delta_f$ admits an equivariant map to $S^{m+k-1}_\circ$.*

Proof. Let $\check{X}$ denote $\overline{X \times X \setminus N}$, where N is the second derived neighborhood of the diagonal in some equivariant triangulation of $X \times X$ in which $f \times f\colon X \times X \to M \times M$ is simplicial. Then $(\tilde{X}, \Delta_f)$ equivariantly deformation retracts onto the pair $(\check{X}, \check{\Delta}_f)$ of compact polyhedra, where $\check{\Delta}_f = \overline{\Delta_f \setminus N}$. Let R be a $\mathbb{Z}/2$-invariant regular neighborhood of $\check{\Delta}_f$ in $\check{X}$, so that $R \cup N$ is a $\mathbb{Z}/2$-invariant regular neighborhood of $\Delta_f \cup \Delta_X$ in X. Since f is stable, $f \times f$ restricted to $\check{X}$ is PL transverse to Δ_M.

Let us focus on the case where M is smoothable. Let τ be the equivariant normal PL disc bundle of Δ_M in $M \times M$, whose total space $D(\tau)$ is a $\mathbb{Z}/2$-invariant regular neighborhood of Δ_M. If φ denotes $f \times f|_{\check{\Delta}_f} \colon \check{\Delta}_f \to \Delta_M$, then $\varphi^*\tau$ is the equivariant normal PL disc bundle of $\check{\Delta}_f$ in $\check{X}$, with total space $D(\varphi^*\tau_M) = R$ [35], [8, §II.4].

In the case $M = \mathbb{R}^m$, τ is equivariantly trivial, so $\varphi^*\tau$ is equivariantly isomorphic to $\check{\Delta}_f \times D^m_\circ \to \check{\Delta}_f$ with the diagonal action of $\mathbb{Z}/2$, where $D^m_\circ$ denotes the m-ball with the antipodal action of $\mathbb{Z}/2$ (with one fixed point). With respect to this trivialization, $\tilde{f}$ restricted to $\check{\Delta}_f \times \partial D^m_\circ$ is precisely the projection onto $\partial D^m_\circ = S^{m-1}_\circ$. Then the restriction $\check{f} \colon \check{X} \setminus \check{\Delta}_f \to S^{m-1} \subset S^{m+k-1}$ of $\tilde{f}$ extends, after an equivariant homotopy, to an equivariant map $\check{X} \to S^{m+k-1}_\circ$ if and only if the projection $\check{\Delta}_f \times \partial D^m_\circ \to \partial D^m_\circ = S^{m-1}_\circ \subset S^{m+k-1}_\circ$ extends to an equivariant map $\check{\Delta}_f \times D^m_\circ \to S^{m+k-1}_\circ$. Or, in other words, if and only if the inclusion $S^{m-1}_\circ \hookrightarrow S^{m+k-1}_\circ$ extends to an equivariant map $\check{\Delta}_f * S^{m-1}_\circ \to S^{m+k-1}_\circ$. But since $k > 0$, the latter is equivalent to the existence of an equivariant map $\check{\Delta}_f * S^{m-1}_\circ \to S^{m+k-1}_\circ$. Taking into account Theorem 2.4(a), this completes the proof of the case $M = \mathbb{R}^m$.

Now let us return to the case where M is smoothable. Let $j \colon M \hookrightarrow M \times \mathbb{R}^k$ be the inclusion, and let ε^k be the trivial equivariant PL k-disc bundle over Δ_M (with fibers $D^k_\circ$). It is easy to see that $(jf) \times (jf)$ is C^0-approximable by isovariant maps if and only if the canonical map $d_{\varphi,\tau} \colon D(\varphi^*\tau) \to D(\tau) \subset D(\tau \oplus \varepsilon^k)$ is equivariantly homotopic, keeping the total space $S(\varphi^*\tau)$ of the boundary sphere bundle fixed, to a map $D(\varphi^*\tau) \to S(\tau \oplus \varepsilon^k)$. Since the bundle projection $S(\tau \oplus \varepsilon^k) \to \Delta_M$ is an equivariant fibration, the latter holds if and only if the canonical map $s_{\varphi,\tau} \colon S(\varphi^*\tau) \to S(\tau) \subset S(\tau \oplus \varepsilon^k)$ extends to an equivariant map $D(\varphi^*\tau) \to S(\tau \oplus \varepsilon^k)$ lying over the map $\varphi \colon \check{\Delta}_f \to \Delta_M$. Thus, taking into account Theorem 2.4(b), it remains to show that $s_{\varphi,\tau}$ extends to an equivariant map $D(\varphi^*\tau) \to S(\tau \oplus \varepsilon^k)$ lying over φ if and only if $\check{\Delta}_f * S^{m-1}_\circ$ admits an equivariant map to $S^{m+k-1}_\circ$.

Let ν be a normal PL m-disc bundle of M, regarded as an equivariant PL disc bundle over Δ_M (with fiber $D^m_\circ$), so that $\tau \oplus \nu$ is equivariantly isomorphic to the trivial PL $2m$-disc bundle ε^{2m}. If $s_{\varphi,\tau}$ extends to an equivariant map $D(\varphi^*\tau) \to S(\tau \oplus \varepsilon^k)$ lying over φ, then $s_{\varphi,\tau\oplus\nu}$ extends to an equivariant map $D\big(\varphi^*(\tau\oplus\nu)\big)\to S(\tau\oplus\nu\oplus\varepsilon^k)$ lying over φ (by considering each fiber of $\tau\oplus\nu$ as the join of $D^m_\circ$ and $S^{m-1}_\circ$). But the latter is equivalent to the existence of an equivariant map $\check\Delta_f * S^{2m-1}_\circ\to S^{2m+k-1}_\circ$. Since $\dim\Delta_f\leqslant 2n-m$ and $2n\leqslant 2(m+k)-3$ (using that $2(m+k)\geqslant 3(n+1)$), the latter is in turn equivalent to the existence of an equivariant map $\check\Delta_f * S^{m-1}_\circ\to S^{m+k-1}_\circ$ by Lemma 2.7(a) below.

Conversely, suppose that there exists an equivariant map $\check\Delta_f * S^{m-1}_\circ\to S^{m+k-1}_\circ$. Then $s_{\varphi,\varepsilon^m}$ extends to an equivariant map $D(\varphi^*\varepsilon^m)\to S(\varepsilon^{m+k})$ lying over φ. Therefore, $s_{\varphi,\tau\oplus\varepsilon^m}$ extends to an equivariant map

$$D\big(\varphi^*(\tau\oplus\varepsilon^m)\big)\to S(\tau\oplus\varepsilon^{m+k})$$

lying over φ (by considering each fiber of $\tau\oplus\varepsilon^m$ as the join of $S^{m-1}_\circ$ and $D^m_\circ$). By Lemma 2.7(b) below, $s_{\varphi,\tau\oplus\varepsilon^i}$ extends to an equivariant map $D\big(\varphi^*(\tau\oplus\varepsilon^i)\big)\to S(\tau\oplus\varepsilon^{i+k})$ lying over φ for each $i=m-1,\dots,0$, due to the inequality $2n\leqslant 2(m+k)-3$.

In the case where M is not smoothable, we can use the equivariant normal block bundle of Δ_M in $M\times M$ in place of τ (see [8, 35]) and the normal block bundle of some PL immersion of M in $\mathbb{R}^{2m}$ in place of ν. The proof is similar, except that we can no longer use the homotopy lifting property. Because of this, we have to keep track of additional homotopies, but it is straightforward; we leave the details to the reader. $\square$

Lemma 2.7. (a) (Conner–Floyd [10]). *If $\mathbb{Z}/2$ acts freely on a polyhedron K^k, the suspension map*

$$[K,\,S^{q-1}_\circ]_{\mathbb{Z}/2}\xrightarrow{\Sigma}[S^0_\circ * K,\,S^q_\circ]_{\mathbb{Z}/2}$$

is onto for $k\leqslant 2q-3$ and one-to-one for $k\leqslant 2q-4$.

(b) *Let $\pi\colon K\to P$ be a $\mathbb{Z}/2$-equivariant PL map between polyhedra, where the action on K^k is free, and let $\Sigma_\pi(K)$ be the double mapping cylinder of π, i.e., the adjunction space of the partial map $K\times[-1,1]\supset K\times\{-1,1\}\xrightarrow{f\sqcup f}P\sqcup P$, with $\mathbb{Z}/2$ acting antipodally on $[-1,1]$. Let L be an invariant subpolyhedron of K, and let $\Sigma_\pi(L)$ be the double mapping cylinder of $\pi|_L$ (in particular, if $L=\varnothing$, it is $P\sqcup P$). Let M be a polyhedron with the trivial action of $\mathbb{Z}/2$, and let $\varphi\colon P\to M$ be an equivariant map. Let τ be a vector q-bundle over M, and let ε be the trivial line bundle $M\times\mathbb{R}\to M$, where $\mathbb{Z}/2$ acts antipodally on the fibers of τ and ε. Let $S(\tau)$ denote the associated sphere bundle (in particular, $S(\tau\oplus\varepsilon)$ is the double mapping cylinder of the bundle projection $\Pi\colon S(\tau)\to M$), let $\lambda\colon L\to S(\tau)$*

be an equivariant map lying over φ, and let $\Sigma\lambda\colon \Sigma_\pi(L) \to S(\tau \oplus \epsilon)$ be the "double mapping cylinder" of λ. Let $[K,\, S(\tau);\, \lambda]^{\varphi}_{\mathbb{Z}/2}$ denote the set of equivariant extensions $K \to S(\tau)$ of λ lying over φ up to homotopy through such extensions. Then

$$[K,\, S(\tau);\, \lambda]^{\varphi}_{\mathbb{Z}/2} \xrightarrow{\Sigma} [\Sigma_\pi(K),\, S(\tau \oplus \varepsilon);\, \Sigma\lambda]^{\varphi}_{\mathbb{Z}/2}$$

is onto if $k \leqslant 2q-3$ and one-to-one if $k \leqslant 2q-4$.

Conner and Floyd require $q \geqslant 3$ in (a), but this is not needed, as shown, for instance, by the usual geometric arguments.

Namely, Lemma 2.7(a) can be proved by a version of Pontryagin's proof of the Freudenthal suspension theorem [33]. In more detail, the assertion follows by a general position argument when the homotopy sets in question are rewritten in their geometric form. Such a geometric description is given by the absolute case ($L = \varnothing$) of Lemma 3.2 below along with the Pontryagin–Thom construction 4.1(a).

Alternatively, Lemma 2.7(a) can be viewed as a special case (with $P = M = \mathrm{pt}$ and $L = \varnothing$) of Lemma 2.7(b), which we now prove by adapting the geometric proof of the Freudenthal suspension theorem in [12] (see also [26, Lemma 7.7(c)]).

Proof of (b). We will prove the surjectivity; the injectivity can be proved similarly.

We are given an equivariant extension $\kappa\colon \Sigma_\pi(K) \to S(\tau \oplus \varepsilon)$ of $\Sigma\lambda$ lying over φ. Let $M_+ \sqcup M_-$ be the two copies of M in the double mapping cylinder $\Sigma_\Pi\big(S(\tau)\big) = S(\tau \oplus \varepsilon)$. We may assume that $\kappa|_{K\times(-1,1)}$ is PL transverse to $M_+ \sqcup M_-$ (see [8]). Let $Q_+ = \kappa^{-1}(M_+)$ and $Q_- = \kappa^{-1}(M_-)$. Since $\Sigma_\pi(L)$ contains the two copies $P_+ \sqcup P_-$ of P, we have $P_+ \subset Q_+$ and $P_- \subset Q_-$. Thus, by tranversality, $Q_+ \setminus P_+$ and $Q_- \setminus P_-$ have disjoint product neighborhoods in $K \times (-1,1)$ equivariantly PL homeomorphic to their products with $D^q_\circ$. Since $k \leqslant 2q-3$, after an arbitrarily small equivariant isotopy of $\Sigma_\pi(K)$ keeping $\Sigma_\pi(L)$ fixed we may assume that $Q_+ \setminus P_+$ and $Q_- \setminus P_-$ have disjoint images under the projection $K \times (-1,1) \to K$. Then after an equivariant isotopy of $\Sigma_\pi(K)$ rel $\Sigma_\pi(L)$ which restricts to a vertical isotopy of $K \times (-1,1)$ (lying over the identity on K) we may assume that $Q_+ \setminus P_+$ lies in $K \times (0,1)$ and $Q_- \setminus P_-$ lies in $K \times (-1,0)$. Hence κ is equivariantly homotopic keeping $\Sigma_\pi(L)$ fixed to a map κ_1 such that $\kappa_1^{-1}(M_+) \subset K \times (0,1)$ and $\kappa_1^{-1}(M_-) \subset K \times (-1,0)$.

Then $\kappa_1(K \times \{0\})$ is disjoint from $M_+ \sqcup M_-$, and hence also from some disjoint neighborhoods N_+ and N_- of M_+ and M_-. Since $\mathrm{id}_{S(\tau\oplus\varepsilon)}$ is equivariantly homotopic to a map sending N_+ onto $M_+ \cup S(\tau) \times [0,1)$ and N_- onto $M_- \cup S(\tau) \times (-1,0]$, κ_1 is equivariantly homotopic to a map κ_2 sending $K \times \{0\}$ into $M \times \{0\}$, $P_+ \cup K \times [0,1)$ into $M_+ \cup S(\tau) \times [0,1)$, and $P_- \cup K \times (-1,0]$ into $M_- \cup S(\tau) \times (-1,0]$. With some work, this homotopy can be amended so as to fix $\Sigma_\pi(L)$. Since κ_2 is equivariantly homotopic rel $\Sigma_\pi(L)$ to κ, which lies over φ, and Π is an equivariant fibration, $\kappa_2|_{K\times\{0\}}\colon K \times \{0\} \to M \times \{0\}$ is equivariantly homotopic rel $\Sigma_\pi(L)$ to a map which lies over φ. By similar arguments, κ_2 is equivariantly homotopic rel $\Sigma_\pi(L)$ to a map κ_3 which lies over φ and sends $K \times \{0\}$ into $M \times \{0\}$, $P_+ \cup K \times [0,1)$ into $M_+ \cup S(\tau) \times [0,1)$, and $P_- \cup K \times (-1,0]$ into $M_- \cup S(\tau) \times (-1,0]$. Then, by the fiberwise Alexander trick (performed by induction over simplices of K, in the order of increasing dimension), κ_3 is equivariantly homotopic rel $\Sigma_\pi(L)$ and over φ to the "double mapping cylinder" of a map $K \to S(\tau)$ (which must extend λ and lie over φ). $\square$

By combining Theorem 2.1, Theorem 2.6, and Lemma 2.7(a), we obtain the following.

Corollary (= Theorem 2)**.** *Let N^n be a compact smooth manifold (respectively, a compact polyhedron), M^m a smooth (PL) manifold, and $f\colon N \to M$ a stable smooth (PL) map, where $m \geqslant n$ and $2n-m \leqslant 2k-3$. In the smooth case, assume additionally that either f is a fold map or $3n-2m \leqslant k$. Then f is k-realizable if and only if it is a smooth (PL) k-prem.*

2.8. Proofs of Theorems 1 and 3.

Example 2.9. Let $f\colon S^3 \to S^2$ be a PL map with Hopf invariant $H(f)$ even but nonzero. Consider $K := \mathrm{Cone}(f) \cup_{S^2} \mathrm{Cone}(-f)$, where $-$ denotes the antipodal involution on S^2. This K is endowed with the involution interchanging the two copies of the mapping cone of f. Then $S^0_\circ * K$ is $\mathbb{Z}/2$-homotopy equivalent to $\mathrm{Cone}(\Sigma f) \cup_{S^3} \mathrm{Cone}(-\Sigma f)$. Since $\Sigma f\colon S^4 \to S^3$ is null-homotopic, there exists an equivariant map $S^0_\circ * K \to S^3_\circ$.

At the same time, there exists no equivariant map $K \to S^2_\circ$. Indeed, every equivariant map $\varphi\colon S^2_\circ \to S^2_\circ$ has an odd degree, since its first obstruction to homotopy with id_{S^2} is, on the one hand, $\deg(\varphi) - 1 \in \mathbb{Z} \simeq H^2(S^2)$ and, on the other hand, the image of the first obstruction to equivariant homotopy with id_{S^2} in $H^2_{\mathbb{Z}/2}(S^2_\circ) \simeq \mathbb{Z}$ under the forgetful map, which is onto $2\mathbb{Z}$. Now $H(\varphi f) = \deg(\varphi)^2 H(f) \neq 0$, which is easy to see from the definition of the Hopf invariant as the total linking number between the point-inverses of two regular values.

A $k\lambda$*-framing* of a bundle ξ over a base X is an (unstable) isomorphism $\xi \simeq k\lambda$ with k copies of the line bundle λ over X. A smooth manifold is called *stably* $k\lambda$*-parallelizable* if it admits a k-dimensional normal bundle isomorphic to $k\lambda$.

Recall from [22, §1] that if $f\colon S^n \looparrowright \mathbb{R}^m$ is a smooth immersion whose normal bundle is trivial over its non-one-to-one points $\Delta_f \looparrowright S^n$, then the normal bundle of $\Delta_f/t \looparrowright \mathbb{R}^m$ is $(m-n)\lambda$-framable by the line bundle λ associated to the double covering $\Delta_f \to \Delta_f/t$. Specifically, the normal bundle of $\Delta_f \to \Delta_f/t \looparrowright \mathbb{R}^m$ is equivariantly diffeomorphic to $\Delta_f \times \bigoplus_{m-n} \mathbb{R}[\mathbb{Z}/2]$ endowed with the diagonal action of $\mathbb{Z}/2$, but each copy of $\mathbb{R}[\mathbb{Z}/2]$ splits into the direct sum of $\mathbb{R}$ with the trivial action and $\mathbb{R}$ with the sign action of $\mathbb{Z}/2$.

This corresponds to the $m\lambda$-framing of the normal bundle of Δ_f/t in $\tilde{S}^n/t$, which is implicit in the proof of Theorem 2.6, under an $n\lambda$-framing of the tangent bundle of $\tilde{S}^n/t$, constructed in [4, Lemma 3.1].

Theorem 2.10 (realization principle)**.** *Let λ be the line bundle associated to a double covering $\bar{M} \to M$ over a closed smooth $(m-n)\lambda$-parallelizable manifold of dimension $2n-m$. If $2m \geqslant 3(n+1)$, then there exists a stable smooth immersion $f\colon S^n \looparrowright \mathbb{R}^m$ such that Δ_f is equivariantly diffeomorphic to $\bar{M}$.*

It is very easy to construct a stable immersion φ of *some* closed stably parallelizable n-manifold into $\mathbb{R}^m$ with trivial normal bundle such that Δ_φ is equivariantly diffeomorphic to $\bar{M} \sqcup \bar{M}$. Specifically, φ is the double of the "figure 8" proper immersion $\bar{M} \times D^{n-m} \looparrowright \mathbb{R}^{m-1} \times [0,\infty)$, depicted in [22, §4].

Proof. By Koschorke's framed version [21, proof of Theorem 1.15] of the Whitney–Haefliger trick [16, démonstration du Théorème 2, a], [1, §VII.4], there exists a self-transverse regular homotopy $F\colon S^{n-1} \times I \looparrowright \mathbb{R}^{m-1} \times I$ between the standard embedding $g\colon S^{n-1} \hookrightarrow \mathbb{R}^{m-1}$ and some embedding $g'\colon S^{n-1} \hookrightarrow \mathbb{R}^{m-1}$ such that the conclusion of Theorem 2.10 holds for F in place of f. (Koschorke deals with a slightly more general situation: his g is a self-transverse immersion, whose double points he wants to eliminate by a regular homotopy, and his M is an $(m-n)\lambda$-framed null-bordism of Δ_g/t.) Now g' is smoothly isotopic to the standard embedding (see [16]), so F can be capped off to a stable smooth immersion $f\colon S^n \looparrowright \mathbb{R}^m$ without adding new double points. $\square$

Example (= Theorem 1). *For each $n = 4k+3 \geqslant 15$ there exists a stable smooth immersion $S^n \looparrowright \mathbb{R}^{2n-7}$ that is 3-realizable but is not a 3-prem.*

Proof. Let K be as in Example 2.9, with the involution t. Then K/t is the mapping cone of the composition $S^3 \xrightarrow{f} S^2 \xrightarrow{p} \mathbb{R}P^2$, where p is the double covering. The cylinder of pf properly embeds, via the graph of pf, into $\mathbb{R}P^2 \times D^4$. Hence K minus two symmetric small 4-balls properly equivariantly embeds into $S^2_\circ \times D^4$, where D^4 has the trivial action of $\mathbb{Z}/2$. Therefore, K equivariantly embeds into $S^3_\circ \times D^4$. Consider an equivariant regular neighborhood R of K in $S^3_\circ \times D^4$, and let M be its double $M = \partial(R \times I)$. Since M equivariantly retracts onto a copy of K, by Example 2.9 it does not admit an equivariant map to $S^2_\circ$, but $S^0_\circ * M$ admits an equivariant map to $S^3_\circ$. Since $M/t = \partial((R/t) \times I)$ embeds with trivial line normal bundle into the parallelizable manifold $\mathbb{R}P^3 \times D^5$, it is stably parallelizable. Since the tangent bundle of $\mathbb{R}P^k$ is stably equivalent to the sum of $k+1$ copies of the canonical line bundle γ, the bundle 4γ is stably trivial over $\mathbb{R}P^3$. Hence M/t admits a skew $4k\lambda$-framed normal bundle for each $k \geqslant 0$, where λ is associated with the double covering $M \to M/t$ and so is the pullback of γ. Thus, by Theorem 2.10, for each $n = 4k+3 \geqslant 15$, M is the double point locus of some stable smooth immersion $S^n \looparrowright \mathbb{R}^{2n-7}$. $\square$

Theorem (= Theorem 3). *Let N^n be a compact smooth manifold (respectively, a compact polyhedron), M^m a smooth (PL) manifold, and suppose that $2m > 3n$. A stable smooth (PL) map $f\colon N \to M$ is 1-realizable if and only if it is a smooth (PL) 1-prem.*

Proof. If f is 1-realizable, by the proof of Theorem 2.6 there exists a stable equivariant map $\Delta_f \to S^0_\circ$, i.e., an equivariant map $S^\infty_\circ * \Delta_f \to S^{\infty+1}_\circ$. Consider the Euler class $e(\lambda) \in H^1(\Delta_f/t; \mathbb{Z}_\lambda)$ of the line bundle λ associated to the double covering $\Delta_f \to \Delta_f/t$, where $\mathbb{Z}_\lambda$ is the integral local coefficient system associated with λ. The Euler class is cohomological, hence, in particular, a stable invariant, due to the natural isomorphism $H^i(S^0_\circ * K;\, P) \simeq H^{i-1}(K;\, P \otimes \mathbb{Z}_\lambda)$ for any local coefficient system P. So $e(\lambda) = 0$, whence λ is orientable and so Δ_f admits an equivariant map to S^0. By Theorem 2.1, f is a 1-prem. $\square$

§3. Equivariant stable cohomotopy

Suppose that P is a pointed polyhedron and G is a finite group acting on P and fixing the basepoint $*$. We also assume that P is G-homotopy equivalent to a compact polyhedron. If V is a finite-dimensional $\mathbb{R}G$-module, let S^V be the one-point compactification of the Euclidean space V with the obvious action of G. If G acts trivially on V, we follow the standard convention of denoting V by the integer $\dim V$. The equivariant stable cohomotopy group

$$\omega_G^{V-W}(P) := [S^{W+V_\infty} \wedge P, S^{V+V_\infty}]_G^*$$

is well defined, where V_∞ denotes a sufficiently large (with respect to the partial ordering by inclusion) finite-dimensional $\mathbb{R}G$-submodule of the countable direct sum $\mathbb{R}G \oplus \mathbb{R}G \oplus \dots$ (see [25]).

Our main interest here lies in the case where $G = \mathbb{Z}/2$, V is the sum of m copies of the nontrivial one-dimensional representation T of $\mathbb{Z}/2$ (i.e., V is $\mathbb{R}^m$ with the sign action of $\mathbb{Z}/2$), and $P = K_+$ (i.e., the union of K with a disjoint basepoint), where the action of $\mathbb{Z}/2$ on K is free. In this case, the following lemma guarantees that V_∞ from the above definition can be taken to be any $\mathbb{R}[\mathbb{Z}/2]$-module of sufficiently large dimension.

Lemma 3.1 (Hauschild; see [25, IX.I.4]). *The basepoint-preserving homotopy set $[P, S^{mT}]_{\mathbb{Z}/2}^*$ surjects to $\omega_{\mathbb{Z}/2}^{mT}(P)$ if $\dim P \leqslant 2m-1$ and $\dim P^{\mathbb{Z}/2} \leqslant m-1$, and injects there if $\dim P \leqslant 2m-2$ and $\dim P^{\mathbb{Z}/2} \leqslant m-2$.*

In the above-mentioned case $P = K_+$, the lemma follows by a general position argument when the Pontryagin–Thom construction 4.1(a) is taken into account.

As before, by $S_\circ^k$ we denote the k-sphere with the antipodal involution, i.e., the unit sphere in $(k+1)T$. For $k \geqslant m$, we have that $S_\circ^k \setminus S_\circ^{m-1}$ is $\mathbb{Z}/2$-homeomorphic to $S_\circ^{k-m} \times mT$. Shrinking to points $S_\circ^{m-1}$ and each fiber $S_\circ^{k-m} \times \{\mathrm{pt}\}$, we get an equivariant map $\rho_m^k \colon S_\circ^k \to S^{mT}$.

For any k-polyhedron K with a free $\mathbb{Z}/2$-action and a $\mathbb{Z}/2$-invariant subpolyhedron $L \subset K$, any equivariant map $\ell \colon L \to S_\circ^{m-1}$ extends to an equivariant map $\varphi_K^\ell \colon K \to S_\circ^\infty$, which is unique up to equivariant homotopy $\operatorname{rel} L$. Here ∞ may be thought of as a sufficiently large natural number (specifically, $k+1$ will do). If $f, g \colon K \to S_\circ^{m-1}$ are equivariant extensions of ℓ, they are joined by a $\operatorname{rel} L$ equivariant homotopy

$$\varphi_{f,g}^\ell \colon K \times I \to S_\circ^\infty,$$

which is unique up to equivariant homotopy $\operatorname{rel} K \times \partial I \cup L \times I$.

Lemma 3.2 (Melikhov [28])**.** *Let $\mathbb{Z}/2$ act freely on a k-polyhedron K and trivially on I. Let L be a $\mathbb{Z}/2$-invariant subpolyhedron of K and $\ell\colon L \to S^{m-1}_\circ$ be an equivariant map. Suppose that $k \leqslant 2m-3$ for* (a), (c) *and $k \leqslant 2m-4$ for* (b).

(a) *The map ℓ extends to an equivariant map $K \to S^{m-1}_\circ$ if and only if $\rho^\infty_m \varphi^\ell_K\colon K \to S^{mT}$ is equivariantly null-homotopic* $\operatorname{rel} L$.

(b) *Equivariant extensions $f, g\colon K \to S^{m-1}_\circ$ of ℓ are equivariantly homotopic* $\operatorname{rel} L$ *if and only if $\rho^\infty_m \varphi^\ell_{f,g}\colon K \times I \to S^{mT}$ is equivariantly null-homotopic* $\operatorname{rel} K \times \partial I \cup L \times I$.

(c) *For any equivariant extension $f\colon K \to S^{m-1}_\circ$ of ℓ and any equivariant* $\operatorname{rel} L$ *self-homotopy $H\colon K \times I \to S^{mT}$ of the constant map $K \to * \subset S^{mT}$ there exists an equivariant extension $g\colon K \to S^{m-1}_\circ$ of ℓ such that $\rho^\infty_m \varphi^\ell_{f,g}$ and H are equivariantly homotopic* $\operatorname{rel} K \times \partial I \cup L \times I$.

Example 3.3. Lemma 3.2(a) with $L = \varnothing$ does not hold for $k = 2m-2 = 4$.

Indeed, let K be the 4-dimensional $\mathbb{Z}/2$-polyhedron from Example 2.9. Thus there exists an equivariant map $f\colon S^0_\circ * K \to S^3_\circ$. Then, by the trivial impliciation in Lemma 3.2(a), the composition $S^0_\circ * K \xrightarrow{\varphi_{S^0_\circ * K}} S^\infty_\circ \xrightarrow{\rho^\infty_4} S^{4T}$ is equivariantly null-homotopic. Consequently, the composition

$$S^0_\circ * K \xrightarrow{S^0_\circ * \varphi_K} S^0_\circ * S^\infty_\circ \xrightarrow{\rho^{\infty+1}_4} S^{4T}$$

is also equivariantly null-homotopic by a homotopy H. The latter composition sends $S^0_\circ$ into $S^{4T} \setminus \mathbb{R}^{4T} = \{\infty\}$, and we may assume that H sends $S^0_\circ \times I$ into ∞ (otherwise, it can be amended by shrinking the loop $H(S^0_\circ \times I)$). Since the diagram

$$\begin{array}{ccccc}
S^0_\circ * K & \xrightarrow{S^0 * \varphi_K} & S^0 * S^\infty_\circ & \xrightarrow{\rho^{\infty+1}_4} & S^{4T} \\
\downarrow & & \downarrow & & \| \\
S^T \vee K_+ & \xrightarrow{S^T \vee \varphi_K} & S^T \vee (S^\infty_\circ)_+ & \xrightarrow{S^T \vee \rho^\infty_3} & S^T \wedge S^{3T}
\end{array}$$

commutes, we get an equivariant null-homotopy of the bottom line. Hence, by Lemma 3.1, the composition $K \xrightarrow{\varphi_K} S^\infty_\circ \xrightarrow{\rho^\infty_3} S^{3T}$ is equivariantly null-homotopic. If Lemma 3.2(a) with $L = \varnothing$ were true for $k = 2m-2 = 4$, this would imply that there exists an equivariant map $K \to S^2_\circ$, contradicting Example 2.9.

3.4. Proof of Theorem 4. Given a compact n-polyhedron X and a map $f\colon X \to M$ into a PL m-manifold, we let

$$\Theta(f) := [\rho_m^\infty \varphi_{\tilde f}] \in [\tilde X_+, S^{mT}]_{\mathbb{Z}/2}^{\mathrm{rel}\,\tilde X_+ \setminus \Delta_f} = \omega_{\mathbb{Z}/2}^{mT}(\tilde X_+, \tilde X_+ \setminus \Delta_f).$$

Theorem 3.5. *Let X^n be a compact polyhedron, M^m a PL manifold, and let $f\colon X \to M$ be a stable PL map. Suppose that $2(m+k) \geqslant 3(n+1)$ in* (a), (c) *and $2(m+k) > 3(n+1)$ in* (b).

(a) *f is k-realizable if and only if $\Theta(f) = 0 \in \omega_{\mathbb{Z}/2}^{kT}(\Delta_{f_+})$.*

(b) *k-realizations $g, g'\colon X \hookrightarrow M \times \mathbb{R}^k$ are isotopic through k-realizations of f if and only if $\Theta_f(g, g') = 0 \in \omega_{\mathbb{Z}/2}^{kT-1}(\Delta_{f_+})$.*

(c) *If g is a k-realization of f, for each $\alpha \in \omega_{\mathbb{Z}/2}^{kT-1}(\Delta_{f_+})$ there exists a k-realization g' of f such that $\Theta_f(g, g') = \alpha$.*

Proof. Let R be as in the proof of Theorem 2.6, so

$$\omega_{\mathbb{Z}/2}^{(m+k)T}(\tilde X_+, \tilde X_+ \setminus \Delta_f) \simeq \omega_{\mathbb{Z}/2}^{(m+k)T}(\tilde X_+, \overline{\check X_+ \setminus R}).$$

By the proof of Theorem 2.6, $R/\operatorname{Fr} R$ is equivariantly homotopy equivalent with $\Delta_{f_+} \wedge S^{mT}$. Therefore,

$$\omega_{\mathbb{Z}/2}^{(m+k)T}(\tilde X_+, \overline{\check X_+ \setminus R}) \simeq \omega_{\mathbb{Z}/2}^{(m+k)T}(R_+, \operatorname{Fr} R_+) \simeq \omega_{\mathbb{Z}/2}^{kT}(\Delta_{f_+}).$$

Assertion (a) now follows from Theorem 2.4 and Lemma 3.2(a). Similarly, (b) and (c) follow (see [28] for some details) from the "moreover" part of Theorem 2.4 and its parametric version below (Theorem 3.6) using Lemma 3.2(b,c). ☐

Theorem 3.6 (Melikhov [26, 7.9(a)])**.** *Let X^n be a compact polyhedron, M^m a PL manifold, and $f\colon X \to M$ a PL map, where $2m > 3(n+1)$. If embeddings $g, g'\colon X \hookrightarrow M$ are C^0-close to f, then they are isotopic through C^0-approximations of f if and only if $g \times g$ and $g' \times g'$ are isovariantly homotopic through C^0-approximations of $f \times f$.*

Combining Theorem 3.5(a,b), Lemma 3.1 (applied to Δ_{f_+} and $\Delta_{f_+} \wedge S^1$), the absolute case ($L = \varnothing$) of Lemma 3.2(a,c), and Theorem 2.1, we get a slightly different proof of Theorem 2, and also its relative version.

Theorem 3.7. *Let N^n be a compact smooth manifold (respectively, a compact polyhedron), M^m a smooth (PL) manifold, and $f\colon N \to M$ a stable smooth (PL) map, where $m \geqslant n$, $2n - m \leqslant 2k - 3$, and also $3n - 2m < 2k - 3$. In the smooth case, assume additionally that either f is a fold* *map or $3n-2m \leqslant k$. Then every k-realization of f is isotopic through k-realizations of f to a smooth (PL) vertical lift of f.*

Example (= Theorem 4)**.** *For $n \geqslant 7$ there exists a stable smooth immersion $f\colon S^n \looparrowright \mathbb{R}^{2n-2}$ and its 2-realization $g\colon S^n \hookrightarrow \mathbb{R}^{2n}$ not isotopic through 2-realizations of f to any vertical lift of f.*

Proof. Let f be such that Δ_f is $\mathbb{Z}/2$-homeomorphic to $S^2 \times S^0_\circ$ (cf. Theorem 2.10). Clearly, f lifts to an embedding $g_0\colon S^n \hookrightarrow \mathbb{R}^{2n-1} \subset \mathbb{R}^{2n}$ (cf. Remark 2.2). Let α be the generator of $\omega^{2T-1}_{\mathbb{Z}/2}(\Delta_{f+}) \simeq \pi_{2+\infty}(S^{1+\infty}) \simeq \mathbb{Z}/2$. By Theorem 3.5(c), there exists a 2-realization g of f with $\Theta_f(g_0, g) = \alpha$. If $g_1\colon S^n \hookrightarrow \mathbb{R}^{2n}$ is an embedding projecting onto f, then $\tilde g_1|_{\Delta_f}\colon \Delta_f \to S^1_\circ$ is equivariantly homotopic to $\tilde g_0$ due to $\pi_2(S^1) = 0$. Hence $\Theta_f(g_1, g_0) = 0$ and $\Theta_f(g_1, g) = \alpha$, so g cannot be isotopic to any such g_1 through 2-realizations of f. $\square$

3.8. Proof of Theorem 5. The following lemma improves on the absolute case ($L = \varnothing$) of Lemma 3.2(a). As shown by the proof of Theorem 4, the analogous strengthening of the absolute case of Lemma 3.2(c) is not true.

Lemma 3.9. *Let K be a polyhedron with a free action of $\mathbb{Z}/2$, and suppose that $m = 2, 4,$ or 8. In the case $m = 8$ assume additionally that $\dim K \leqslant 22$. There exists an equivariant map $K \to S^{m-1}_\circ$ if and only if the composition $K \xrightarrow{\varphi_K} S^\infty_\circ \xrightarrow{\rho^\infty_m} S^{mT}$ is equivariantly null-homotopic.*

Proof. Note that ρ^∞_m is the composition of the quotient map q of $S^\infty_\circ$ onto the "cosphere" $S_m := S^\infty_\circ / S^{m-1}_\circ$ and a fibration $S_m \to S^{mT}$ (with all fibers S^∞ except for one singleton fiber). Hence the composition $K \xrightarrow{\varphi_K} S^\infty_\circ \xrightarrow{q} S_m$ is equivariantly null-homotopic. In the case $m = 8$ the additional hypothesis allows us to assume that the null-homotopy of $q\varphi_K$ lies in $S^{23}_\circ / S^7_\circ$. Now consider the diagram

$$\begin{tikzcd} S^{m-1}_\circ \arrow[r, phantom, "\subset"] & S^\infty_\circ \arrow[r, "q"] \arrow[d, "h"'] & S_m \arrow[dl] \\ & \mathbb{K}P^\infty & \end{tikzcd}$$

where $\mathbb{K}P^\infty$ denotes $\mathbb{C}P^\infty$ if $m = 2$, the quaternionic infinite projective space QP^∞ if $m = 4$, and the octonionic (also known as Cayley) projective plane OP^2 if $m = 8$, and h is the standard Hopf bundle with fiber S^{m-1}

(regarded as a partial map defined on $S^{23} \subset S^\infty$ in the case $m = 8$). Since h is equivariant with respect to the antipodal involution on S^∞ and the identity on $\mathbb{K}P^\infty$, we may identify the distinguished $S^{m-1}_\circ \subset S^\infty_\circ$ with one of its fibers. Then h factors through S_m, so the composition $K \xrightarrow{\varphi_K} S^\infty \xrightarrow{q} S_m \to \mathbb{K}P^\infty$ is null-homotopic. Since h is a fibration, this null-homotopy lifts to an equivariant homotopy of φ_K to a map $K \to S^{m-1}$. $\square$

Lemma 3.9 fits to replace the absolute case of Lemma 3.2(a) in the above alternative proof of Theorem 2, thus proving the following theorem.

Theorem (= Theorem 5)**.** *Let N^n be a compact smooth manifold (respectively, a compact polyhedron), $M^{2n-2k+2}$ a smooth (PL) manifold, and $f\colon N \to M$ a stable smooth (PL) map, where $n \geqslant 2k-1$. In the smooth case, assume additionally that either f is a fold map or $n \geqslant 3k-4$. If f is k-realizable and $k \in \{2, 4, 8\}$, then f is a smooth (PL) k-prem.*

§4. Skew-co-oriented skew-framed comanifolds

In this section, we use geometric methods to analyze the failure of injectivity of the map $[K_+, S^{mT}]^*_{\mathbb{Z}/2} \to \omega^{mT}_{\mathbb{Z}/2}(K_+)$ from Lemma 3.1 beyond the stable range.

Let $\bar X$ be a compact polyhedron with a free PL action of $\mathbb{Z}/2 = \langle t \mid t^2 \rangle$. Write $X = \bar X/t$, and let λ be the line bundle associated with the double covering $\bar X \to X$. Given an equivariant PL map $f\colon \bar X \to S^{qT+r}$, after an equivariant homotopy we may assume that it is PL transverse to the origin O of the Euclidean space $qT + r \subset S^{qT+r}$, and so a $\mathbb{Z}/2$-invariant regular neighborhood of $\bar Q := f^{-1}(O)$ in the polyhedron $\bar X$ is equivariantly PL homeomorphic to $Q \times I^{q+r}$ [35, § II.4]. A fixed orientation of S^{qT+r} induces an orientation of this bundle, and hence a $\lambda^{\otimes q}$*-co-orientation* of $Q := \bar Q/t$ in X, that is an isomorphism $H^{q+k}(X, X \setminus Q; \mathbb{Z}_{\lambda^{\otimes q}}) \simeq \mathbb{Z}$, where $\mathbb{Z}_\lambda$ denotes the integral local coefficient system associated with λ (cf. [8, §IV.1]). The action of $\mathbb{Z}/2$ on S^{qT+r} fixes O and each vector of a fixed r-frame at O, and inverts each vector of a fixed complementary q-frame. Hence Q has a normal PL disc bundle ξ in X, endowed with a $(q\lambda+r)$*-framing* (as a $\lambda^{\otimes q}$-oriented bundle), i.e., a $\lambda^{\otimes q}$-orientation-preserving isomorphism $\xi \simeq q\lambda|_Q \oplus r\varepsilon$ of PL disc bundles, where ε denotes the trivial line bundle. This isomorphism also makes sense when q and r are not necessarily nonnegative, by transferring any negative terms to the left-hand side of the equation.

For brevity, we shall call a $\lambda^{\otimes q}$-co-oriented subpolyhedron $Q \subset X$ with a $(q\lambda + r)$-framed normal PL disc bundle in X a *$(q\lambda + r)$-comanifold* in X. A *$(q\lambda + r)$-cobordism* between two $(q\lambda + r)$-comanifolds Q_0, Q_1 in X is a $(q\lambda + r)$-comanifold in $X \times I$ meeting $X \times \{i\}$ in Q_i for $i = 0, 1$. (For the reader who wonders what object we might call just "comanifold," it is an embedded mock bundle in the sense of [8, p. 34], or, equivalently, a subpolyhedron of X that has a normal block bundle in X.)

The usual Pontryagin–Thom argument shows that the pointed equivariant homotopy set $[\bar{X}, S^{qT+r}]_{\mathbb{Z}/2}$ is in pointed bijection with the pointed set $\mathrm{Emb}^{q\lambda+r}(X)$ of $(q\lambda + r)$-comanifolds in X up to $(q\lambda + r)$-cobordism. Further, it follows that the equivariant stable cohomotopy group $\omega^{qT+r}_{\mathbb{Z}/2}(\bar{X}_+)$ is isomorphic to the group $\mathrm{Imm}^{q\lambda+r}(X)$ of singular $(q\lambda + r)$-comanifolds in X up to singular $(q\lambda + r)$-cobordism. Here a *singular $(q\lambda + r)$-comanifold* in X is the projection $Q \to X$ of a $(q\lambda + r + \infty)$-comanifold Q in $X \times \mathbb{R}^\infty$ (cf. [8, §IV.2]).

The dual bordism group $\Omega^{\mathrm{sf}(k)}_n(X; \lambda)$ consists of stably $k\lambda$-parallelized $\lambda^{\otimes k}$-oriented singular PL n-manifolds $f\colon N \to X$ up to stably $k\lambda$-parallelized $\lambda^{\otimes k}$-oriented singular bordism (cf. [4]). Here a *$\lambda^{\otimes k}$-orientation* of N is an isomorphism $H^n(N; \mathbb{Z}_{f^*\lambda^{\otimes k}}) \simeq \mathbb{Z}$. A *stable $k\lambda$-parallelization* of N (as a $\lambda^{\otimes k}$-oriented manifold) is a $\lambda^{\otimes k}$-orientation-preserving isomorphism between $f^*(k\lambda)$ and a k-dimensional normal bundle of N.

Let us summarize.

Lemma 4.1. *Let $\bar{X} \to X$ be a double covering of compact polyhedra, and let λ be the associated line bundle.*

(a) (Pontryagin–Thom construction) [35, 3.3], [8, IV.2.4]. *There exist a natural pointed bijection $[\bar{X}, S^{qT+r}]_{\mathbb{Z}/2} \leftrightarrow \mathrm{Emb}^{q\lambda+r}(X)$ and a natural isomorphism $\omega^{qT+r}_{\mathbb{Z}/2}(\bar{X}_+) \simeq \mathrm{Imm}^{q\lambda+r}(X)$.*

(b) (Poincaré duality) [8, II.3.2, IV.2.4]. *If X is a closed stably $p\lambda$-parallelizable m-manifold, there exists an isomorphism*

$$\mathrm{Imm}^{q\lambda}(X) \simeq \Omega^{\mathrm{sf}(p+q)}_{m-q}(X; \lambda),$$

which gets natural once X is endowed with a $\lambda^{\otimes p}$-orientation and a stable $p\lambda$-parallelization.

Remark 4.2. Let us indicate a relation to a more traditional approach. Let $\mathrm{Imm}^{q\bigstar}(X)$ be defined similarly to $\mathrm{Imm}^{q\lambda}(X)$, except that λ is not globally defined, but is a part of the data of the $q\lambda$-comanifold. It is well known that this group is isomorphic to $[S^\infty * X; S^\infty * (\mathbb{R}P^\infty/\mathbb{R}P^{q-1})]$, cf. [5]. Here $\mathbb{R}P^\infty/\mathbb{R}P^{q-1}$ is the Thom space of the bundle $q\gamma$ over $\mathbb{R}P^{\infty-q}$, so the point-inverse Q of the basepoint of this space is $q\lambda$-framed in X, where λ is the pullback of γ under the map $Q \to \mathbb{R}P^{\infty-q}$.

The notation “Imm” is partially justified by the following lemma.

Lemma 4.3. *Suppose that $q \geqslant 0$, $r \geqslant 0$, and $q + r \geqslant 1$.*

(a) (Hirsch lemma). *Every element of* $\mathrm{Imm}^{q\lambda+r}(X)$ *admits an immersed representative, unique up to immersed $(q\lambda + r)$-cobordism.*

(b) (compression theorem). *For large n, every embedded $(q\lambda + r + n)$-comanifold Q in $X \times \mathbb{R}^n$ is isotopic by an arbitrarily C^0-small ambient isotopy to one whose last n vectors of framing are standard. The ambient isotopy can be chosen to fix $Y \times \mathbb{R}^n$ where Y is a subpolyhedron of X such that Q is PL transverse to $Y \times \mathbb{R}^n$ and the last n vectors of the framing of Q are already standard over $Q \cap Y \times \mathbb{R}^n$.*

More specifically, “large n” means $q + r + n \geqslant \dim X + 1$. Using [16], this can be weakened to $q + r + n \geqslant \frac{1}{2}(\dim X + 3)$.

If the ambient isotopy is replaced by regular homotopy, (b) holds in codimension one [19].

Proof. Clearly, (a) follows from (b).

By PL transversality [8, II.4.4], given a triangulation of X, any singular $(q\lambda + r)$-comanifold $f\colon Q \to X$ can be altered by an arbitrarily C^0-small homotopy so that every simplex Δ^i of this triangulation meets Q in a singular $(q\lambda + r)$-comanifold $f|_{f^{-1}(\Delta^i)}\colon f^{-1}(\Delta^i) \to \Delta^i$. If f is the projection of $Q \subset X \times \mathbb{R}^n$, this homotopy can be chosen to fix $f^{-1}(Y)$ and lift to an arbitrarily C^0-small ambient isotopy of $X \times I$ fixing $Y \times I$. By [8, Lemma II.1.2], a singular $(q\lambda + r)$-comanifold in the PL manifold Δ^i is a singular PL $(i - q - r)$-manifold with boundary $\partial f^{-1}(\Delta^i) = f^{-1}(\partial\Delta^i)$. Picking a sufficiently fine triangulation of X, we may assume that each such PL manifold $Q \cap (\Delta^i, \partial\Delta^i) \times \mathbb{R}^n$ is contained in an $(i-q-r)$-PL ball contained in Q. Since n is large, it follows that each such intersection is smoothable (with corners) as a submanifold in the standard smooth structure on $\Delta^i \times \mathbb{R}^n$, so that the restriction of the smoothing to the boundary agrees with those constructed earlier over the faces of Δ^i. Part (b) now follows by induction on the simplices of X from its smooth case, which was proved in [36]. $\square$

Addendum 4.4 (density of the h-principle for PL immersions)**.** *Let P be an n-polyhedron, $f\colon P \to \mathbb{R}^{2n}$ a PL map, and $g\colon P \looparrowright \mathbb{R}^{2n}$ a PL immersion. Then f is arbitrarily C^0-close to a PL immersion PL regularly homotopic to g.*

Proof. Let $h\colon P \times I \to \mathbb{R}^{2n} \times I$ be a generic homotopy between g and f. By general position, it fails to be an immersion only at finitely many points (p,t). Let L be the link of p in P. The restriction of h to $(p*L)\times\{0\}\cup L\times I$ is a regular homotopy $(p*L)\times\{0\}\cup L\times I \xrightarrow{h_p} B^{2n}\cup S^{2n-1}\times I \xrightarrow{g_p} \mathbb{R}^{2n}\times I$. We redefine h on $(p*L)\times I$ by $h(x,s)=g_p((\pi h_p\varphi(x,s),s))$, where $\varphi\colon (p*L)\times I \to (p*L)\times\{0\}\cup L\times I$ is a collapse and $\pi\colon B^{2n}\times I \to B^{2n}$ is the projection. Doing this for every cusp (p,t) converts h to a regular homotopy, which is sufficiently close to h as long as each $(p*L)$ is chosen to be sufficiently small. $\square$

Let us recall the geometric definition of the cup product in $\mathrm{Imm}^*(X)$. Let $\varphi\colon Q \to X$ be a singular $(q\lambda+r)$-comanifold representing some $\Phi \in \mathrm{Imm}^{q\lambda+r}(X)$. The *transfer* $\varphi_!\colon \mathrm{Imm}^{q'\lambda+r'}(Q) \to \mathrm{Imm}^{(q+q')\lambda+(r+r')}(X)$ of the normal bundle of Q sends a representative $\psi\colon M \to Q$ to the class of $\varphi\psi\colon M \to Q \to X$, endowed with the skew framing obtained by combining those of φ and ψ, and the λ-co-orientation obtained by combining those of φ and ψ. For any $\Phi' \in \mathrm{Imm}^{q'\lambda+r'}(X)$, the cup product $\Phi\smile\Phi'$, defined originally in terms of the cross product, equals $\varphi_!\varphi^*(\Phi')$ [8].

Let ξ be a $q\lambda$-framed PL disc bundle over X. By PL transversality [8], the zero set $X'\cap X$ of a generic cross-section X' (that is the image of X under a fiber-preserving self-homeomorphism of the total space) is a $q\lambda$-comanifold in X. We denote its class in the set $\mathrm{Emb}^{q\lambda}(X)$ by $\mathcal{E}(\xi)$ and its image in the group $\mathrm{Imm}^{q\lambda}(X)$ by $E(\xi)$. The latter can also be defined as $i^*i_!([\mathrm{id}_X])$, where $i\colon X \hookrightarrow \xi$ is the inclusion of the zero cross-section into the total space, and $[\mathrm{id}_X]\in \mathrm{Imm}^0(X)$ is the fundamental class. The Hurewicz homomorphism (cf. [28]) obviously sends $E(\xi)$ to the usual (twisted) Euler class $e(\xi)\in H^q(X;\mathbb{Z}_{\lambda^{\otimes q}})$, where $\mathbb{Z}_\lambda$ denotes the integral local coefficient system associated with λ.

Lemma 4.5. *Let $\bar X \to X$ be a double covering between compact polyhedra, and let λ be the associated line bundle. The Pontryagin–Thom correspondence* 4.1(a) *sends*

- *$\mathcal{E}(q\lambda)$ to the equivariant homotopy class of the composition $\bar X \xrightarrow{\varphi_{\bar X}} S^\infty_\circ \xrightarrow{\rho_q^\infty} S^{qT}$;*

- *$E(q\lambda)$ to $[\rho_q^\infty \varphi_{\bar X}] \in \omega^{qT}_{\mathbb{Z}/2}(\bar X_+)$.*

Proof. We have $E(q\lambda) = E(\lambda)^q = \varphi_X^* E(\gamma)^q = \varphi_X^* E(q\gamma)$, where $\varphi_X\colon X \to \mathbb{R}P^\infty$ classifies λ and γ is the universal line bundle. By similar considerations, $\mathcal{E}(q\lambda) = \varphi_X^*\mathcal{E}(q\gamma)$ as well. By a direct geometric construction, $\mathcal{E}(q\gamma)$ is represented by $\mathbb{R}P^{\infty-q}$ with the canonical $q\lambda$-framing. On the other hand, $(\rho_q^\infty)^{-1}(0) = S_\circ^{\infty-q}$. □

Remark 4.6. By Lemma 4.5, the Pontryagin–Thom isomorphism 4.1(a) sends the obstruction $\Theta(f) \in \omega^{kT}_{\mathbb{Z}/2}(\Delta_{f+})$ from Theorem 3.5 to $E(k\lambda) \in \mathrm{Imm}^{k\lambda}(\Delta_f/t)$, where λ is the line bundle associated with the double covering $\Delta_f \to \Delta_f/t$. On the other hand, the Poincaré duality 4.1(b) obviously sends $E(k\lambda)$ to the obstruction $O(f) \in \Omega_d^{\mathrm{sf}(n-d)}(\Delta_f/t;\lambda)$, defined in [4] in the case of a map $f\colon S^n \to \mathbb{R}^{2n-d}$.

Let $\varphi\colon Q \looparrowright X$ be an immersed $q\lambda$-comanifold representing some element $\Phi \in \mathrm{Imm}^{q\lambda}(X)$. By PL transversality, the double point immersion $d\varphi\colon \Delta_\varphi \looparrowright Q$ is an immersed $q\lambda$-comanifold in Q, hence the composition $\Delta_\varphi \xrightarrow{d\varphi} Q \xrightarrow{\varphi} X$ is an immersed $2q\lambda$-comanifold in X. Let Δ_Φ denote its class in $\mathrm{Imm}^{2q\lambda}(X)$.

Lemma 4.7 (Herbert's formula)**.** $\Delta_\Phi + \varphi_! E(\nu_\varphi) = \Phi^2$.

Compare [11, 18, 20]. We will be slightly sloppy about notation in this proof, since spelling out all the conventions would only make it less readable it seems.

Proof. By the geometric definition of the cup product, $\Phi \smile \Phi = \varphi_!\varphi^*(\Phi)$. We can think of Φ as $\varphi_!([\mathrm{id}_Q])$. Now φ factors into the composition $Q \xrightarrow{i} \nu_\varphi \xrightarrow{\bar\varphi} X$, so $\varphi^*\varphi_! = i^*\bar\varphi^*\bar\varphi_! i_!$, where the domain of $\varphi_!$ and the range of $i_!$ is $\mathrm{Imm}^{q\lambda}(\nu_\varphi, \partial\nu_\varphi)$. Since $\bar\varphi$ is a codimension zero immersion, $\bar\varphi^*\bar\varphi_! = 1 + (d\bar\varphi)_! T (d\bar\varphi)^*$, where

$$T = t^* = t_!\colon \mathrm{Imm}^{q\lambda}(\Delta_{\bar\varphi}, \Delta_{\bar\varphi} \cap \partial\nu_\varphi) \to \mathrm{Imm}^{q\lambda}(\Delta_{\bar\varphi}, \overline{\partial\Delta_{\bar\varphi} \setminus \partial\nu_\varphi})$$

is induced by the involution $t\colon \Delta_{\bar\varphi} \to \Delta_{\bar\varphi}$. Now $i^* i_!([\mathrm{id}_Q])$ is, by definition, $E(\nu_\varphi)$, so

$$\varphi^*\varphi_!([\mathrm{id}_Q]) = E(\nu_\varphi) + i^*(d\bar\varphi)_! T (d\bar\varphi)^* i_!([\mathrm{id}_Q]).$$

We have $\Delta_{\bar\varphi} \simeq \nu_{d\varphi} \oplus \nu_{d\varphi}$ as PL disc bundles over Δ_φ. Let $j\colon \nu_{d\varphi} \hookrightarrow \Delta_{\bar\varphi}$ be the inclusion onto one of the factors, and let $\overline{d\varphi}\colon \nu_{d\varphi} \looparrowright Q$ be the extension of $d\varphi$. Then $(d\bar\varphi)^*i_! = j_!\overline{d\varphi}^*$ and $i^*(d\bar\varphi)_! = \overline{d\varphi}_!j^*$. Hence $i^*(d\bar\varphi)_!T(d\bar\varphi)^*i_! = \overline{d\varphi}_!j^*Tj_!\overline{d\varphi}$. Finally, $\varphi\overline{d\varphi} = \overline{d\bar\varphi}j$, so we obtain

$$\varphi_!\overline{d\varphi}_!j^*Tj_!\overline{d\varphi}([\mathrm{id}_Q]) = \overline{d\bar\varphi}_!j_!j^*([tj]) = \overline{d\bar\varphi}_!([j]\smile[tj]) = \Delta_\Phi,$$

where

$$\smile\colon \mathrm{Imm}^{q\lambda}(\Delta_{\bar\varphi}, \Delta_{\bar\varphi}\cap\partial\nu_\varphi)\otimes\mathrm{Imm}^{q\lambda}(\Delta_{\bar\varphi}, \overline{\partial\Delta_{\bar\varphi}\setminus\partial\nu_\varphi}) \to \mathrm{Imm}^{2q\lambda}(\Delta_{\bar\varphi}, \partial\Delta_{\bar\varphi}).$$

$\square$

4.8. Secondary obstruction. Let again λ be the line bundle associated to a double covering $\bar X \to X$. Assuming that $E(q\lambda) = 0$, we will construct a secondary obstruction to the vanishing of $\mathcal{E}(q\lambda)$. Let $Q \hookrightarrow X$ be an embedded $q\lambda$-comanifold representing $\mathcal{E}(q\lambda) \in \mathrm{Emb}^{q\lambda}(X)$. Let $\varphi\colon R \looparrowright X \times I$ be a generic immersed $q\lambda$-null-cobordism of Q given by the hypothesis $E(q\lambda) = 0$. The immersed $2q\lambda$-manifold $\Delta_\varphi \looparrowright R \looparrowright X \times I$ represents an element of $\mathrm{Imm}^{2q\lambda}(X \times I, X \times \partial I)$. The Thom isomorphism sends it to an element

$$F(q\lambda) \in \mathrm{Imm}^{2q\lambda-1}(X) \simeq \omega^{2qT-1}_{\mathbb{Z}/2}(\bar X_+).$$

Proposition 4.9. *$F(q\lambda)$ is well-defined if $E(q\lambda) = 0$, and vanishes if $\mathcal{E}(q\lambda) = 0$.*

Proof. The second assertion holds by construction. To prove the first, let us temporarily denote $F(q\lambda)$ by $F(R)$, and let us consider another immersed $q\lambda$-null-cobordism $\varphi'\colon R' \looparrowright X \times I$ of Q. Let $\tau\colon I = [0,1] \to [-1,0]$ be defined by $x \mapsto -x$, and let $\varphi'' = (\mathrm{id}_X \times\tau)\varphi'\colon R' \looparrowright X \times [-1,0]$. Let $\psi\colon W \looparrowright X \times [-1,1]$ be the immersed $q\lambda$-comanifold $\varphi \cup (-\varphi'')$. Then $F(W) = F(R) - F(R')$. We may assume that W is contained in $X \times [-1, 1-\varepsilon]$ for some $\varepsilon > 0$. Since W is regularly homotopic into $X \times [1-\varepsilon, 1]$ by an ambient isotopy, $[\psi]^2 = 0$ and so $F(W) + \psi_! E(\nu_\psi) = 0$ by Herbert's formula (Lemma 4.7). Now W is $q\lambda$-framed, so ν_ψ is isomorphic to $q(\psi^*\lambda) = \psi^*(q\lambda)$. Hence $E(\nu_\psi) = \psi^*E(q\lambda) = 0$ by the hypothesis. Thus $F(W) = 0$. $\square$

Example 4.10. Let M be a closed $(2k-1)$-manifold and λ a line bundle over M with $E(k\lambda) = 0$. Then $F(k\lambda) = 0$.

A stronger result will be obtained in Theorem 4.13 by a different method.

Proof. Note that $F(k\lambda)$ lies in $\mathrm{Imm}^{2k\lambda-1}(M) \simeq H^{2k-1}(M)$. On the other hand, if $F(k\lambda)$ is represented by $\Delta_\varphi \looparrowright M \times I$ for some generic immersed $k\lambda$-null-cobordism $\varphi\colon R \looparrowright M \times I$ of a representative of $\mathcal{E}(k\lambda)$, the double covering $p\colon \Delta_\varphi \to \Delta_\varphi/t$ is trivial, so $F(k\lambda) = 2[\Delta_\varphi/t]$ if t preserves the orientation of the bundle $p^*(2k\lambda)$, and 0 otherwise. If k is odd, the orientation is reversed. If M is nonorientable, $H^{2k-1}(M) \simeq \mathbb{Z}/2$ and again $F(k\lambda) = 0$.

If k is even and M is orientable, R is also orientable, and $F(k\lambda)$ is twice the algebraic number of double points of φ. Let $\bar{M}$ be the double cover of M corresponding to λ and $\bar{R} \looparrowright \bar{M} \times I$ the double cover over φ. Since the coverings $\bar{R} \to R$ and $\bar{M} \to M$ preserve orientations, $2F(k\lambda)$ is twice the algebraic number of double points of $\bar{R}$. Since $\bar{M} \to S^\infty$ is nonequivariantly null-homotopic, by the nonequivariant version of Lemma 4.5, $\partial\bar{R}$ admits an embedded framed null-cobordism $\bar{R}'$. By the nonequivariant version of the proof of Proposition 4.9 (which is easier, since all normal bundles are trivial), $\bar{R}$ has as many double points as $\bar{R}'$, which is algebraically zero. So again $F(k\lambda) = 0$. □

Using Lemma 4.9, we will now prove that $F(k\lambda)$ vanishes identically whenever it is defined.

Lemma 4.11. *Let λ be a line bundle over a polyhedron X such that $E(k\lambda) = 0$. There exists a singular 0-comanifold $f\colon X' \to X$ such that $[f] = [\mathrm{id}_X] \in \mathrm{Imm}^0(X)$ and the double cover $\bar{X}'$ corresponding to $f^*\lambda$ admits an equivariant map to $S^{k-1}_\circ$.*

Remark 4.12. In the case where $\bar{X} = \Delta_f$ for some k-realizable stable $f\colon N \to \mathbb{R}^m$, we can take $\bar{X}' = \Delta_{\pi g}$, where $g\colon N \hookrightarrow \mathbb{R}^{m+k}$ is an embedding such that the composition $N \xrightarrow{g} \mathbb{R}^{m+k} \xrightarrow{\pi} \mathbb{R}^m$ is stable and C^0-close to f. A bordism between id : $\bar{X} \to \bar{X}$ and the projection $\bar{X}' \to \bar{X}$ is given by the projection of Δ_H, where H is a generic homotopy between f and πg.

Proof. By Lemma 4.5, the composition $\bar{X} \xrightarrow{\varphi_{\bar{X}}} S^\infty_\circ \xrightarrow{\rho^\infty_k} S^{kT}$ is stably equivariantly null-homotopic. That is, for some m there exists an equivariant homotopy $H_t\colon S^{mT} \wedge \bar{X}_+ \to S^{(m+k)T}$ between the suspension of $\rho^\infty_k \varphi_{\bar{X}}$ and the constant map to the basepoint b. By using the composition

$$S^{m-1}_\circ * \bar{X} \xrightarrow{r_1} (S^{m-1}_\circ * \bar{X}) \cup_{S^{m-1}_\circ} (S^{m-1}_\circ \times I) \xrightarrow{r_2} (S^{mT} \wedge \bar{X}_+) \vee (S^{m-1}_\circ * b),$$

where r_1 is the obvious surjection and r_2 shrinks $S^{m-1}_\circ$ to the basepoint, it follows that the composition $S^{m-1}_\circ * \bar{X} \xrightarrow{\varphi_{S^{m-1}_\circ * \bar{X}}} S^{m+\infty}_\circ \xrightarrow{\rho^{m+\infty}_{m+k}} S^{(m+k)T}$ is equivariantly homotopic to the join $j\colon S^{m-1}_\circ * \bar{X} \to S^{m-1}_\circ * b$ of the identity and the constant map to b by a homotopy h_t which is *suspension sphere* *preserving*, i.e., for each t sends the suspension sphere $S_\circ^{m-1}$ identically onto $S_\circ^{m-1} \subset S^{(m+k)T} \setminus S^{kT}$. Since its final map h_1 is null-homotopic, by Lemma 3.2(a) with $L = \varnothing$, if m is large enough, there exists an equivariant map $S_\circ^{m-1} * \bar{X} \to S_\circ^{m+k-1} \subset S^{(m+k)T}$. By general position, we may assume that this map is suspension sphere preserving, and is homotopic to j by a suspension sphere preserving homotopy h'_t. Combining h_t and h'_t, we get an equivariant suspension sphere preserving homotopy $H\colon (S_\circ^{m-1} * \bar{X}) \times I \to S^{(m+k)T}$ from $\rho_{m+k}^{m+\infty} \varphi_{S_\circ^{m-1} * \bar{X}}$ to a map into $S_\circ^{k+m-1} \subset S^{(k+m)T}$. Assuming that this homotopy is generic, let $\bar{W} := H^{-1}(S^{kT})$. Since $\rho_{m+k}^{m+\infty}$ restricts to ρ_k^∞ over S^{kT}, and $\varphi_{S_\circ^{m-1} * \bar{X}}$ may be taken to be the suspension of $\varphi_{\bar{X}}$, $\bar{W}$ meets $(S_\circ^{m-1} * \bar{X}) \times \{0\}$ in $\bar{X}$. By construction, the intersection $\bar{X}'$ of $\bar{W}$ with $(S_\circ^{m-1} * \bar{X}) \times \{1\}$ admits an equivariant map to $S_\circ^{k-1}$. Finally, $\bar{W}$ lies in $(mT \times \bar{X}) \times I \subset (S_\circ^{m-1} * \bar{X}) \times I$, so its projection onto $\bar{X} \times I$ is an equivariant singular 0-cobordism between $[\bar{X}]$ and $[H_1\colon \bar{X}' \to \bar{X}]$. ☐

Theorem 4.13. *Let λ be a line bundle over a polyhedron X such that $E(k\lambda) = 0$. Then $F(k\lambda) = 0$.*

Proof. Let f be given by Lemma 4.11. Then we have $\mathcal{E}(kf^*\lambda) = 0$, so $F(kf^*\lambda) = 0$. On the other hand, if $w\colon W \to X \times I$ is a singular 0-cobordism between $[\mathrm{id}_X]$ and $[f]$, then $w_! F(kw^*\lambda)$ is represented by a $k\lambda$-cobordism between some representative of $F(k\lambda)$ and $f\varphi$, where φ is some representative of $F(kf^*\lambda)$. Since $F(kf^*\lambda)$ is well defined, φ is $kf^*\lambda$-null-cobordant, hence $f\varphi$ is $k\lambda$-null-cobordant. ☐

Remark 4.14. By Lemma 4.5 and Lemma 3.2(a), $E(k\lambda) = 0$ is equivalent to the existence of an equivariant map $g\colon S_\circ^{\infty-1} * \bar{X} \to S_\circ^{\infty+k-1}$. We do not know if the suspension of the equivariant map $\bar{X}' \to S_\circ^{k-1}$ given by Lemma 4.11 can be chosen homotopic to the composition $S_\circ^{\infty-1} * \bar{X}' \xrightarrow{\Sigma f} S_\circ^{\infty-1} * \bar{X} \xrightarrow{g} S_\circ^{\infty+k-1}$. If this were the case, the proof given by Theorem 4.13 that $\mathcal{E}(k\lambda)$ bounds an immersed $k\lambda$-null-cobordism $\varphi\colon R \looparrowright X \times I$ with $[\Delta_\varphi] = 0$ would not require the use of Proposition 4.9. The same argument would then work for $[\Delta_\varphi/t] \in \mathrm{Imm}^{k\lambda \otimes (\bigstar+1)-1}(X)$ in place of $[\Delta_\varphi]$, where $\bigstar$ stands for a line bundle, which is a part of the data of the immersed comanifold Δ_φ/t (namely, it is associated to the double covering $\Delta_\varphi \to \Delta_\varphi/t$).

4.15. Proof of Theorem 6.

Proposition 4.16. *Let M be a closed nonorientable $(2k-1)$-manifold and λ a line bundle over M. If k is even, then* $\mathrm{Imm}^{k\lambda}(M) = \mathrm{Emb}^{k\lambda}(M)$*.*

Proof. Let O be a connected codimension one submanifold in $\mathbb{R}^{k-1}$ (e.g., a point when $k = 2$ and S^{k-2} when $k > 2$). Let ℓ denote S^1, then $\ell \times O \subset \ell \times \mathbb{R}^{k-1} \subset \ell \times \mathbb{R}^{2k-2}$ is k-framed in $\ell \times \mathbb{R}^{2k-2}$. This extends to a k-framing of $\mu \times O$ in $\mu \times \mathbb{R}^{2k-2}$, where μ is the mapping cylinder of the nontrivial double covering $\bar{\ell} \to \ell$. Since $\bar{\ell} \times O \to \ell \times O$ is a double covering, if we twist the (integer) framing of $\ell \times O$ in $\ell \times \mathbb{R}^{2k-2}$ by one full twist, the (integer) framing of $\bar{\ell} \times O$ in $\bar{\ell} \times \mathbb{R}^{2k-2}$ will differ from the original one by two full twists. These two framed embeddings $\bar{\ell} \times O \hookrightarrow \bar{\ell} \times \mathbb{R}^{2k-2}$ can be joined by a framed regular homotopy with one transverse double point in $\bar{\ell} \times \mathbb{R}^{2k-2} \times I$. Combining the two framed embeddings $\mu \times O \hookrightarrow \mu \times \mathbb{R}^{2k-2}$ with this regular homotopy, we obtain a framed immersion $K \times O \looparrowright K \times \mathbb{R}^{2k-2}$ with one double point, where $K = \mu \cup \bar{\ell} \times I \cup \mu$ is the Klein bottle. Since K is the boundary of the nontrivial 2-disk bundle over ℓ, $K \times \mathbb{R}^{2k-2}$ embeds into the nontrivial $(2k-1)$-vector bundle over ℓ. Thus, if ℓ is an orientation-reversing loop in $M \times I$, we obtain a k-framed immersion with one double point $K \times O \looparrowright K \times \mathbb{R}^{2k-2} \subset \nu_\ell$ in the regular neighborhood of ℓ. Since k is even, $k\lambda$ restricts to the trivial bundle over ℓ, so this immersion is also $k\lambda$-framed in $M \times I$.

By general position, every element of $\mathrm{Imm}^{k\lambda}(M)$ can be represented by an embedded $k\lambda$-framed manifold Q in M. Suppose that it admits an immersed $k\lambda$-null-cobordism $R \looparrowright M \times I$. If R has an odd number of double points, we replace it by $R \cup K \times O$. Since M is nonorientable, the double points of R carry no global signs and so can now be paired up so as to match the local signs. More precisely, each double point lifts to a pair of double points of opposite signs of the immersion $\bar{R} \looparrowright \bar{M} \times I$ in the orientation double cover. Since the number of pairs is even, we can pick one double point from each pair so that the total algebraic number of the picked points is zero. Then the picked points can be paired up with signs and cancelled along framed 1-handles in $\bar{M}$. These project to $k\lambda$-framed handles in M, killing all double points of R. $\square$

Theorem 4.17. *Let X be a $(2k-1)$-polyhedron and λ a line bundle over X with $E(k\lambda) = 0$. If k is even, then $\mathcal{E}(k\lambda) = 0$.*

Proof. By Theorem 4.13, $F(k\lambda) = 0$. Let $\varphi\colon R \looparrowright X \times I$ be an immersed $k\lambda$-null-cobordism of a representative of $\mathcal{E}(k\lambda)$. Since k is even, $F(k\lambda) = 2[\Delta_\varphi/t] \in H^{2k-1}(X)$ (cf. the proof of Example 4.10).

First assume that $H^{2k-1}(X)$ contains no elements of order 2. Then the algebraic number of double points of R is zero. So they can be paired up with signs and cancelled by a surgery along $k\lambda$-framed 1-handles (cf. the proof of Proposition 4.16; if $H^{2k-1}(X) = 0$ say, the double points can be pushed off the boundary). Thus $Q = \partial R$ bounds an embedded $k\lambda$-null-cobordism.

Now in the general case it suffices to prove that for each generator $[D] \in H^{2k-1}(X)$ of order $2n$ there is an immersed $k\lambda$-comanifold in $X \times I$ with n double points each representing $\delta^*[D] \in H^{2k}(X \times I, X \times \partial I)$. Since $2n[D] = 0$, there is a $(2k-1)$-framed comanifold $C \looparrowright X \times I$ with boundary the constant map $\{1, \dots, 2n\} \to D \subset X \times \{0\}$. Let us pick a free involution t on the $2n$ points ∂C. It extends to a free involution on a double cover $\bar{\ell}$ of $\ell := C/t$. An embedded perturbation $\ell \subset X \times I$ of C/t is clearly $(2k - 2 + \mu)$-framed in X, where μ is the line bundle associated with the double covering $\bar{\ell} \to \ell$. The remainder of the construction repeats that in the proof of Proposition 4.16. $\square$

Corollary (= Theorem 6)**.** *Let N^n be a compact smooth manifold (respectively, a compact polyhedron), $M^{2n-2k+1}$ a smooth (PL) manifold, and $f\colon N \to M$ a stable smooth (PL) map, where $n \geqslant 2k+1$. In the smooth case, assume additionally that either f is a fold map or $n \geqslant 3k-2$. If $f\colon N \to M$ is k-realizable and $k \in \{2, 4, 8\}$, then f is a smooth (PL) k-prem.*

Proof. Let λ be the line bundle associated with the 2-cover $\Delta_f \to \Delta_f/t$ and set $X = \Delta_f/t$. By the trivial direction of Theorem 3.5(a), we have $\Theta(f) = 0$, hence, by Lemma 4.5, $E(k\lambda) = 0$. Then, by Theorem 4.17, $\mathcal{E}(k\lambda) = 0$. Therefore, by Lemma 4.5, the composition $X \xrightarrow{\varphi_X} S^\infty_\circ \xrightarrow{\rho_k^\infty} S^{kT}$ is equivariantly null-homotopic. By Lemma 3.9, X admits an equivariant map to $S^{k-1}_\circ$. By Theorem 2.1, f is a k-prem. $\square$

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