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ON SEMI-RECONSTRUCTION OF GRAPHS OF CONNECTIVITY 2

Abstract. Recall that *the deck* of a graph G is the collection of subgraphs $G-v$ for all vertices v of the graph G. We prove that at most two graphs of connectivity 2 and minimal degree at least 3 can have the same deck. Let $\mathcal{D}(G)$ be a deck of a 2-connected graph G. We describe an algorithm which construct by the deck $\mathcal{D}(G)$ of a 2-connected graph G with minimal degree at least 3 two graphs G_1, G_2 such that $G \in \{G_1, G_2\}$.

§1. Introduction

1.1. Definitions and notation. We consider graphs without loops and multiple edges and use the standard notation. For a graph G, we denote the set of its vertices by $V(G)$ and the set of its edges by $E(G)$. We use the notation $v(G)$ for the number of vertices of G.

For a vertex $x \in V(G)$, we denote by $d_G(x)$ its degree in the graph G. The minimal vertex degree of G is denoted by $\delta(G)$.

Let $\mathrm{N}_G(w)$ denote the *neighborhood* of a vertex $w \in V(G)$ (i.e. the set of all vertices of the graph G, adjacent to w).

We say that a vertex $u \in V(G)$ is *adjacent* to a set $W \subset V(G)$ if $u \notin W$ and u is adjacent to a vertex of W.

For a subset W of $V(G)$, we denote by $G(W)$ the *induced subgraph* of G on the set W.

A *xy-path* is a simple path between the vertices x and y. We say that k different xy-paths are *independent* if any two of them have no common inner vertex.

Definition 1. Let $R \subset V(G) \cup E(G)$.

1) We denote by $G-R$ the graph obtained from G upon deleting all vertices and edges of the set R and all edges incident to vertices of R. The set R is a *cutset* if the graph $G-R$ is disconnected.

If R is a cutset, $R \subset V(G)$ and $|R| = k$ then R is called a *k-cutset*.

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2) The *connectivity* $\kappa(G)$ of a graph G is the minimal size of its vertex cutset. A graph G is *k-connected* if $v(G) > k$ and $\kappa(G) \geqslant k$ (i.e. G has no vertex cutset of size less than k).

3) Let $X, Y \subset V(G)$. We say that R *separates* X from Y if $X \not\subset R$, $Y \not\subset R$ and there is no path from $X \setminus R$ to $Y \setminus R$ in $G - R$.

4) For $X \subset V(G)$, we say that R *splits* X if $X \not\subset R$ and the set $X \setminus R$ is disconnected in graph $G - R$.

For $x, y \in V(G)$, we denote by $G + xy$ the graph obtained from G by adding the edge xy. (If $xy \in E(G)$ then $G + xy = G$.)

Definition 2. Let G and H be two graphs with the same number of vertices. A *graph isomorphism* $\varphi\colon G \to H$ is a bijection $\varphi\colon V(G) \to V(H)$ such that

$$xy \in E(G) \quad \Longleftrightarrow \quad \varphi(x)\varphi(y) \in E(H)$$

for all $x, y \in V(G)$. In this case, we say that the graphs G and H are *isomorphic* and denote this by $G \simeq H$.

Definition 3. Let G be a graph with $V(G) = \{v_1, \dots, v_n\}$. Then *the deck* $\mathcal{D}(G)$ is the *collection* of graphs $G - v_1, \dots, G - v_n$.

Remark 1. 1) Note that some graphs in $\mathcal{D}(G)$ may coincide.

2) In this paper, dealing with collections (not sets) of objects, we will use notations like the following: $\mathcal{D}(G) = \{G - v \,:\, v \in V(G)\}$.

1.2. The history and the main results. The *Graph Reconstruction Conjecture* formulated by Kelly [1] and Ulam [2] is well known.

Conjecture. *If both graphs G and H have at least* 3 *vertices and* $\mathcal{D}(G) = \mathcal{D}(H)$ *then* $G \simeq H$.

Note that several graph parameters can be reconstructed from $\mathcal{D}(G)$ for graphs on at least 3 vertices: the number of vertices and the number of edges, the collection of vertex degrees of G, the connectivity $\kappa(G)$.

The Reconstruction Conjecture is rather simple for disconnected graphs. In 1957, Kelly [1] proved this conjecture for trees. In 1969, Bondy [4] proved the Reconstruction Conjecture for graphs of connectivity 1 without pendant vertices. Finally, in 1988, Yongzhi [5] proved the Conjecture for all graphs which are not 2-connected. No results for 2-connected graphs are known now.

This paper can be considered as a beginning of studying reconstruction of graphs of connectivity 2. We will prove that at most two graphs of connectivity 2 and minimal degree at least 3 can have the same deck.

In general, the proof of our result follows Bondy's way [4]. However, instead of the classic tree of blocks and cutpoints we need a similar tree $\mathrm{BT}(G)$ which shows the structure of decomposition of a 2-connected graph G by its 2-cutsets. For the first time, such a tree was presented in 1966 by Tutte [3]. We will define the tree $\mathrm{BT}(G)$ in detail in Section 2.2 and list its properties that we need. This tree is an important characteristics of graphs of connectivity 2 and also can be reconstructed from $\mathcal{D}(G)$.

The following theorem is the main result of our paper.

Theorem 1. *Let G be a graph with $\kappa(G) = 2$ and $\delta(G) \geqslant 3$. Then, having $\mathcal{D}(G)$, we can find a pair of graphs G_1, G_2 such that* $\mathrm{BT}(G_1) = \mathrm{BT}(G_2)$ *and $G \in \{G_1, G_2\}$.*

Let's discuss the obstacle in reconstruction of graphs of connectivity 2.

Definition 4. Let G', G'' be two graphs such that $V(G') \cap V(G'') = T$ and let $\psi\colon T \to T$ be a bijection. To *glue the graphs G' and G'' by the set T* is to identify each vertex $a \in T$ of the graph G_1 with the vertex $\psi(a) \in T$ of the graph G_2.

Let induced subgraphs H' and H'' of G be such that $V(H') \cup V(H'') = V(G)$ and $V(H') \cap V(H'') = T$. Assume that we know the graphs H' and H'' and the set T is marked in both these graphs. If $|T| = 1$ then we can easily glue the graph G from H' and H''. In the case $|T| = 2$, there are two ways of gluing together H' and H'' by the set T (see figure 1). Unfortunately, the problem of how two distinguish two such graphs G_1 and G_2 by their decks is not trivial.

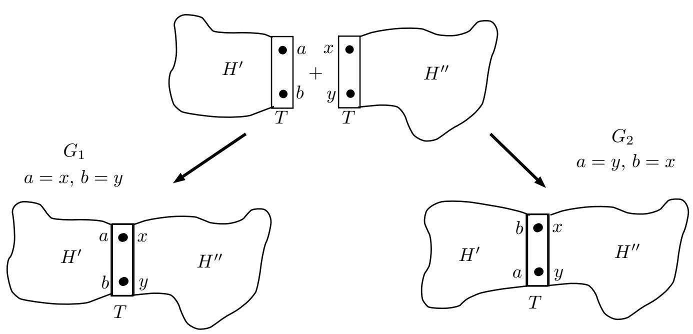


Figure 1. Two ways of gluing by a 2-vertex cutset.

At the end of this paper, we will formulate a theorem which describes the non-uniqueness which probably can appear in the reconstruction of graphs of connectivity 2. The formulation of this Theorem is not very elegant, but we hope that it will help to prove that, really, every graph of connectivity 2 can be uniquely reconstructed from its deck.

§2. Necessary tools

We need to describe the structure of decomposition of a 2-connected graph by its 2-cutsets. We define the *decomposition tree* of a 2-connected graph as in [9]. In general, this structure is similar to Tutte's one [3]. Let's start with the *decomposition of a graph by a set of cutsets* [7].

2.1. The decomposition of a graph by a set of cutsets. In this section, $k \geqslant 2$ and G is a k-connected graph. Denote by $\mathfrak{R}_k(G)$ the set of all k-cutsets of G.

Definition 5. Let $\mathfrak{S} \subset \mathfrak{R}_k(G)$.

1) A set $A \subset V(G)$ is a *part of decomposition* of G by $\mathfrak{S}$ if no cutset of $\mathfrak{S}$ splits A and A is a maximal up to inclusion set with this property. By $\mathrm{Part}(G;\mathfrak{S})$, we denote the set of all parts of decomposition of G by $\mathfrak{S}$.

2) Let $A \in \mathrm{Part}(G;\mathfrak{S})$. A vertex of A is *inner* if it does not belong to any cutset of $\mathfrak{S}$. The set of all inner vertices of the part A is called the *interior* of A, which is denoted by $\mathrm{Int}(A)$.

The *boundary* of A is the set $\mathrm{Bound}(A) = A \setminus \mathrm{Int}(A)$.

3) For a set $S \in \mathfrak{R}_k(G)$, we will write simply $\mathrm{Part}(G;S)$ instead of $\mathrm{Part}(G;\{S\})$.

It is clear that if two parts of $\mathrm{Part}(G;\mathfrak{S})$ have nonempty intersection then their intersection is a subset of a certain cutset of $\mathfrak{S}$.

Lemma 1 ([8, Theorem 2 and Corollary 2])**.** *Let G be a k-connected graph and $\mathfrak{S}, \mathfrak{T} \subset \mathfrak{R}_k(G)$.*

1) *Let $A \in \mathrm{Part}(G;\mathfrak{S})$. Then $\mathrm{Bound}(A)$ consists of all vertices of the part A which are adjacent to $V(G) \setminus A$. If $\mathrm{Int}(A) \neq \varnothing$ then $\mathrm{Bound}(A)$ separates $\mathrm{Int}(A)$ from $V(G) \setminus A$.*

2) *Assume that $A \in \mathrm{Part}(G;\mathfrak{S})$ and $A \in \mathrm{Part}(G;\mathfrak{T})$. Then the boundary of A as a part of $\mathrm{Part}(G;\mathfrak{S})$ coincides with the boundary of A as a part of $\mathrm{Part}(G;\mathfrak{T})$.*

Thus, the notions of the boundary and the interior of a part of decomposition do not depend on the set of cutsets $\mathfrak{S}$. Hence, the notation $\mathrm{Bound}(A)$ and $\mathrm{Int}(A)$ without referring to the set of cutsets is correct.

A part of $\mathrm{Part}(G;\mathfrak{S})$ can be represented as an intersection of parts of decomposition of G by cutsets of $\mathfrak{S}$.

Lemma 2 ([8, Theorem 1])**.** *Let* $\mathfrak{S} = \{S_1, \dots S_n\} \subset \mathfrak{R}_k(G)$. *Then* $\mathrm{Part}(G;\mathfrak{S})$ *consists of maximal up to inclusion sets of type* $\bigcap\limits_{i=1}^{n} A_i$ *where* $A_i \in \mathrm{Part}(G;S_i)$.

Lemma 3 ([8, Corollary 1])**.** *Let* $\mathfrak{S}, \mathfrak{T} \subset \mathfrak{R}_k(G)$ *and let a part* $A \in \mathrm{Part}(\mathfrak{S})$ *be such that no cutset of* $\mathfrak{T}$ *splits* A. *Then* $A \in \mathrm{Part}(\mathfrak{S} \cup \mathfrak{T})$.

Definition 6. Let G be a k-connected graph, $T \in \mathfrak{R}_k(G)$ and let U be a union of several (maybe, one, but not all) parts of $\mathrm{Part}(G;T)$. Then U is a *T-fragment* and $G(U)$ is a *T-subgraph* of G.

The *interior* $\mathrm{Int}(U)$ is the union of the interiors of all parts of $\mathrm{Part}(G;T)$ which union is U.

If $T = \{a, b\}$ then, for a T-subgraph H, we will use the notation $H^+ = H + ab$.

Remark 2. Let G be a k-connected graph and let U be a T-fragment for $T \in \mathfrak{R}_k(G)$. Then the set T is uniquely determined by U: by Lemma 1, T consists of all vertices of U which are adjacent to $V(G) \setminus U$.

Definition 7. Two cutsets $S, T \in \mathfrak{R}_k(G)$ are *independent* if S does not split T and T does not split S. Otherwise, these sets are *dependent*.

In [6], it is proved that, for a k-connected graph G and cutsets $S, T \in \mathfrak{R}_k(G)$, only two variants are possible: either S and T are independent, or each of them splits the other.

Lemma 4 ([8, Lemma 1])**.** *Assume that* $S, T \in \mathfrak{R}_k(G)$, $A \in \mathrm{Part}(G;S)$ *and* $B \in \mathrm{Part}(G;T)$ *are such that* $A \supset T$ *and* $B \supset S$. *Then* S *and* T *are independent and* A *contains the union of all parts of* $\mathrm{Part}(G;T)$, *except for* B.

2.2. The decomposition of a 2-connected graph and its properties. In this section, G is a 2-connected graph. We will list some definitions and results proved before and, after that, we will prove several new lemmas.

Definition 8. 1) A cutset $S \in \mathfrak{R}_2(G)$ is *single* if S is independent with all other cutsets of $\mathfrak{R}_2(G)$. Denote by $\mathfrak{O}(G)$ the set of all single cutsets of G.

2) We will write simply $\mathrm{Part}(G)$ instead of $\mathrm{Part}(G; \mathfrak{O}(G))$ and will call these parts simply *parts of* G.

Lemma 5 ([9, Lemma 6])**.** *Let* $S = \{a, b\} \in \mathfrak{R}_2(G)$ *be a non-single cutset. Then* $|\mathrm{Part}(S)| = 2$ *and, for every part* $A \in \mathrm{Part}(S)$*, the graph* $G(A)$ *has a cutpoint which separates* a *from* b*.*

Lemma 6. *Let* $S = \{a, b\} \in \mathfrak{R}_2(G)$*. Then* $S \in \mathfrak{O}(G)$ *if and only if there exist three independent* ab*-paths in* G*. In particular,* $d_G(a) \geqslant 3$ *and* $d_G(b) \geqslant 3$*.*

Proof. Clearly, the existence of three independent ab-paths in G is equivalent to the fact that no cutset of $\mathfrak{R}_2(G)$ separates a from b i.e. is dependent with S. □

Definition 9. The *decomposition tree* $\mathrm{BT}(G)$ of a 2-connected graph G is a bipartite graph with bipartition $(\mathfrak{O}(G), \mathrm{Part}(G))$, where a single cutset S and a part A are adjacent if and only if $S \subset A$.

The following lemma is a particular case of Theorem 1 of [9].

Lemma 7. *For a* 2*-connected graph* G*, the following statements hold.*

1) $\mathrm{BT}(G)$ *is a tree. Every leaf of* $\mathrm{BT}(G)$ *corresponds to a part of* $\mathrm{Part}(G)$*.*

2) *For any* $S \in \mathfrak{O}(G)$*,* $d_{\mathrm{BT}(G)}(S) = |\mathrm{Part}(G; S)|$*. Moreover, for any part* $A \in \mathrm{Part}(G; S)$*, there exists exactly one part* $B \in \mathrm{Part}(G)$ *such that* $B \subset A$ *and* B *is adjacent to* S *in* $\mathrm{BT}(G)$*.*

3) *Let* $B, B' \in \mathrm{Part}(G)$*. Then a cutset* $S \in \mathfrak{O}(G)$ *separates* B *from* B' *in* G *if and only if* S *separates* B *from* B' *in* $\mathrm{BT}(G)$*.*

Definition 10. A part $A \in \mathrm{Part}(G)$ is *pendant* if it corresponds to a leaf of the tree $\mathrm{BT}(G)$.

Remark 3. 1) If $A \in \mathrm{Part}(G)$ is a pendant part then $\mathrm{Bound}(A) \in \mathfrak{O}(G)$.

2) Interiors of two distinct parts of $\mathrm{Part}(G)$ are disjoint.

Definition 11. 1) For a 2-connected graph G, we denote by G' the graph obtained from G upon adding all edges of type ab where $\{a, b\} \in \mathfrak{O}(G)$.

2) Let $A \in \mathrm{Part}(G)$. If $G'(A)$ is a 3-connected graph then A is called a 3-*block*. If the graph $G'(A)$ is a cycle then A is called a *cycle* and $|A|$ is the *length* of A.

Lemma 8 ([10, Lemma 2])**.** *For a* 2*-connected graph* G*, the following statements hold.*

1) *Every part of* $\mathrm{Part}(G)$ *is either a cycle or a* 3*-block.*

2) *If* $A \in \mathrm{Part}(G)$ *is a cycle then all vertices of* $\mathrm{Int}(A)$ *have degree* 2 *in the graph* G*. If* $\delta(G) \geqslant 3$ *then all pendant parts of* $\mathrm{Part}(G)$ *are* 3*-blocks.*

3) *Let* $A \in \mathrm{Part}(G)$ *be a cycle of length at least* 4*. Then any pair of its non-neighboring vertices form a non-single cutset of the graph* G*. All non-single cutsets of* G *are of such type.*

Lemma 9 ([11, Lemma 3])**.** *If* $B \in \mathrm{Part}(G)$ *is a* 3*-block and* $w \in \mathrm{Int}(B)$ *then the graph* $G - w$ *is* 2*-connected.*

Lemma 10 ([12, Lemma 5])**.** *Let* $S = \{a, b\} \in \mathfrak{R}_2(G)$ *and* $D \in \mathrm{Part}(G; S)$*. Then one of the two following statements holds.*

1°. $G(D)$ *is an* ab*-path.*

2°. *There exists a pendant part* $A \in \mathrm{Part}(G)$ *such that* $\mathrm{Int}(A) \subset \mathrm{Int}(D)$*.*

Corollary 1. *Let* $S \in \mathfrak{O}(G)$ *and* $D \in \mathrm{Part}(G; S)$*. Then there exists a pendant part* $A \in \mathrm{Part}(G)$ *such that* $\mathrm{Int}(A) \subset \mathrm{Int}(D)$*.*

Proof. If statement 1° of Lemma 10 holds then D is a pendant part of $\mathrm{Part}(G)$. If statement 2° of Lemma 10 holds then we are done. □

Lemma 11. *A set* $B \subset V(G)$ *is a pendant* 3*-block of* G *if and only if there exists a set* $T \subset B$ *such that* $T \in \mathfrak{R}_2(G)$*,* B *is a* T*-fragment and the graph* $G(B)^+$ *is* 3*-connected.*

Proof. $\Rightarrow$. A consequence of definitions of a 3-block and a pendant part.

$\Leftarrow$. By Lemma 5, $T \in \mathfrak{O}(G)$. Since $G(B)^+$ is 3-connected, $B \in \mathrm{Part}(G; T)$ and no cutset $S \in \mathfrak{O}(G)$ splits B. By Lemma 3, then $B \in \mathrm{Part}(G)$. Thus, B is a 3-block. By Lemma 1, $\mathrm{Bound}(B) = T$. Hence, B is a pendant 3-block. □

We need to study how pendant 3-blocks of a 2-connected graph with minimal degree 3 are changed after deleting an inner vertex of one of them.

Lemma 12. *Assume that* $\delta(G) \geqslant 3$*,* B *is a pendant* 3*-block of* G*,* $T = \mathrm{Bound}(B)$ *and* $x \in \mathrm{Int}(B)$*. Then the following statements hold.*

1) $T \in \mathfrak{R}_2(G - x)$ *and* $\mathrm{Part}(G; T) \setminus \{B\} \subset \mathrm{Part}(G - x; T)$*. The set* $B \setminus \{x\}$ *is a* T*-fragment of* $G - x$*.*

2) *Pendant* 3*-blocks of* $G - x$ *are all pendant* 3*-blocks of* G *which are different from* B *and, probably, some subsets of* $B \setminus \{x\}$*.*

Proof. 1) Let $B' \in \mathrm{Part}(G; T)$ and $B' \neq B$. Then T separates B' from $V(G - x - B')$ in $G - x$ and the graph $(G - x)(\mathrm{Int}(B')) = G(\mathrm{Int}(B'))$ is connected. Therefore, $B' \in \mathrm{Part}(G-x;T)$. Hence, clearly, $B \setminus \{x\}$ is a T-fragment of $G-x$.

2) Let D be a pendant 3-block of G different from B. Then the graph $(G-x)(D)^+ = G(D)^+$ is 3-connected by Lemma 11, and, by the same lemma, D is a pendant 3-block of $G-x$.

Assume that $A \in \mathrm{Part}(G-x)$ is a pendant 3-block of $G-x$ but is not a pendant 3-block of G. Let $S = \mathrm{Bound}(A)$. By Lemma 11, the graph $(G-x)(A)^+$ is 3-connected. If S separates A from $V(G) \setminus A$ in G then, by Lemma 11, A is a pendant 3-block of G, a contradiction. Hence, S does not separate A from $V(G) \setminus A$ in G. Therefore, x is adjacent to a vertex $y \in \mathrm{Int}(A)$.

Let's prove that there exists a part $B' \in \mathrm{Part}(G-x;T)$ such that $A \subset B'$. If $S = T$ this is clear. Let $S \neq T$. Then the cutsets $S, T \in \mathfrak{R}_2(G-x)$ are independent. If $A \supset T$ then, by Lemma 4, there exists a part $A' \in \mathrm{Part}(G-x;T)$ such that $A \supset A'$. Clearly, T separates A' from $A \setminus A'$ in $(G-x)(A)^+$. Hence, $(G-x)(A)^+$ is not 3-connected, a contradiction. Thus, $A \not\supset T$ and, by Lemma 4, there exists a part $B' \in \mathrm{Part}(G-x;T)$ such that $A \subset B'$.

The vertex $y \in \mathrm{Int}(A)$ cannot belong to $T \in \mathfrak{R}_2(G-x)$. Hence, $y \in \mathrm{Int}(B')$. Since $x \in \mathrm{Int}(B)$ and y is adjacent to x, we have $y \in \mathrm{Int}(B)$. Now, by item 1, we obtain $B' \subset B \setminus \{x\}$. □

Lemma 13. *Let $T \in \mathfrak{R}_2(G)$, $D \in \mathrm{Part}(G;T)$ and $H = G(D)^+$. Then the following statements hold.*

1) *Let $x, y \in D$ and $T \neq \{x, y\}$. Then k independent xy-paths exist in G if and only if k independent xy-paths exist in H. In particular, H is 2-connected.*

2) *Let $x, y \in D$ and $S \subset D$. Then S separates x from y in G if and only if S separates x from y in H.*

3) *The set $\mathfrak{R}_2(H)$ consists of all cutsets of $\mathfrak{R}_2(G)$ lying in D and different from T. The set $\mathfrak{O}(H)$ consists of all cutsets of $\mathfrak{O}(G)$ lying in D and different from T.*

4) *Let $S \in \mathfrak{R}_2(H)$. Then $\mathrm{Part}(H;S)$ consists of all sets of type $A \cap D$ where $A \in \mathrm{Part}(G;S)$.*

5) *$\mathrm{Part}(H)$ consists of all maximal up to inclusion sets of type $A \cap D$ where $A \in \mathrm{Part}(G)$.*

Proof. Let $T = \{a, b\}$, $D' \in \mathrm{Part}(G;T)$, $D' \neq D$.
1) At most one xy-path in G can be not contained in D: such path must

contain both vertices of the set $T = \{a, b\}$. In H, we substitute the ab-part of this path by the edge ab. Conversely, at most one xy-path in H contains the edge ab which can be replaced by an ab-path in G through the part D'. Thus, since G is 2-connected, H is also 2-connected.
2) Let's prove that a xy-path P_G exists in $G - S$ if and only if a xy-path P_H exists in $H - S$. Indeed, the part of P_G outside D can be substituted by the edge ab. Conversely, if P_H contains the edge ab then this edge can be replaced by an ab-path through D'.
3) Let $S \in \mathfrak{R}_2(G)$, $S \subset D$. Then S is independent with T and, by Lemma 4, there exists a part $A \in \mathrm{Part}(G; S)$ such that $A \subsetneq D$. By item 2, S separates $\mathrm{Int}(A)$ from $T \setminus S$ in H and, therefore, $S \in \mathfrak{R}_2(H)$. Conversely, if $S \in \mathfrak{R}_2(H)$ then it is clear that $S \neq T$ and, by item 2, $S \in \mathfrak{R}_2(G)$.

Let $S = \{x, y\}$. By Lemma 6, $S \in \mathfrak{O}(G)$ if and only if $S \in \mathfrak{R}_2(G)$ and there exist 3 independent xy-paths in G. Similarly, $S \in \mathfrak{O}(H)$ if and only if $S \in \mathfrak{R}_2(H)$ and there exist 3 independent xy-paths in H. By item 1 and proved above, these two statements are equivalent.
4) A straightforward consequence of items 2 and 3.
5) By Lemma 2, $\mathrm{Part}(G)$ consists of maximal up to inclusion sets of type $\cap_{S\in\mathfrak{O}(G)} A_S$ where $A_S \in \mathrm{Part}(G; S)$. Single cutsets of G which do not lie in D are independent with T and, by Lemma 4, do not split D. T also does not split D. By item 3, single cutsets of G which lie in D and are different from T form the set $\mathfrak{O}(H)$. Therefore, by item 4, $\mathrm{Part}(H)$ consists of maximal up to inclusion sets of type $\cap_{S\in\mathfrak{O}(G)}(A_S \cap D)$ where $A_S \in \mathrm{Part}(G; S)$. These sets are exactly maximal up to inclusion sets of type $A \cap D$ where $A \in \mathrm{Part}(G)$. □

Definition 12. Consider a cycle $C \in \mathrm{Part}(G)$. Let its vertices follow $c_1, \ldots, c_k$ in the cyclic order (we suppose that $c_{k+m} = c_m$).

1) If $S_i = \{c_i, c_{i+1}\} \in \mathfrak{O}(G)$ then there exists unique part $C'_i \in \mathrm{Part}(G; S_i)$ which contains C. Then the *weight* $w(c_ic_{i+1})$ is equal to $v(G - C'_i)$ (i.e. to the sum of sizes of interiors of all parts of $\mathrm{Part}(G; S_i)$ different from C'_i). If $S_i \notin \mathfrak{O}(G)$ then we set $w(c_ic_{i+1}) = 0$.

2) An *arc* c_pCc_q is the path $c_pc_{p+1}\ldots c_q$ along the cycle C. *The weight* of this arc is $w(c_pCc_q) = q - p + 1 + \sum_{i=p}^{q-1} w(c_ic_{i+1})$ (i.e. the number of vertices of the arc plus the sum of weights of its edges).

Remark 4. If $w(c_ic_{i+1}) = 0$ then it is clear that $\{c_i, c_{i+1}\} \notin \mathfrak{R}_2(G)$ (two neighboring vertices of a cycle of $\mathrm{Part}(G)$ cannot form a non-single cutset of $\mathfrak{R}_2(G)$ by Lemma 5).

Lemma 14. *Let $C \in \mathrm{Part}(G)$ be a cycle. Let vertices $u, v \in C$ be non-neighboring in the cyclic order and $T = \{u, v\}$. Then one can set the notation $\mathrm{Part}(G;T) = \{D, D'\}$ such that D contains all vertices of the arc uCv and D' contains all vertices of the arc vCu. Moreover, $|D| = w(uCv)$ and $|D'| = w(vCu)$.*

Proof. By Lemma 8, T is a non-single cutset of G. By Lemma 5, $|\mathrm{Part}(G;T)| = 2$. Clearly, T separates inner vertices of the arc uCv from inner vertices of the arc vCu. Therefore, we can set the notation $\mathrm{Part}(G;T) = \{D, D'\}$ such that D contains all inner vertices of the arc uCv and D' contains all inner vertices of the arc vCu.

Consider an edge $e = xy$ of the arc uCv such that $S = \{x, y\} \in \mathfrak{O}(G)$. Let $C_S \in \mathrm{Part}(G;S)$ be the part which contains C. Then $C_S \supset T$ and, by Lemma 4, the part D contains the union of all parts of $\mathrm{Part}(G;S)$ different from C_S. The sum of sizes of interiors of all these parts is exactly $w(e)$. Thus, we have found $w(uCv)$ vertices in D. Since a similar reasoning is valid for the part D' and the arc vCu, the part D cannot contain other vertices. Hence, $|D| = w(uCv)$ and, similarly, $|D'| = w(vCu)$. $\square$

Definition 13. We will say that the part D from Lemma 14 *corresponds* to the arc $L = uCv$ and denote this part by D_L. And, conversely, we will say that the arc L *corresponds* to the part D_L.

§3. Reconstruction of pendant 3-blocks

Let's recall that, since $\delta(G) \geqslant 3$, all pendant parts of G are 3-blocks. Let $B_1, \dots, B_k \in \mathrm{Part}(G)$ be all pendant 3-bocks of G, $T_i = \mathrm{Bound}(B_i)$ and $H_i = G(B_i)$.

By the definition of a 3-block, each graph H_i^+ is 3-connected.

Denote by n_1 the minimal number of vertices in a pendant 3-block of G. Clearly, $n_1 \geqslant 4$. First, we will show how to determine all graphs H_i (in each H_i the set T_i will be marked).

Definition 14. Let $\mathcal{D}^2(G)$ be the subcollection of $\mathcal{D}(G)$ consisting of all 2-connected graphs and $\mathcal{D}^1(G)$ be the subcollection of $\mathcal{D}(G)$ consisting of all graphs which are not 2-connected.

Remark 5. 1) The collection $\mathcal{D}(G)$ can be easily divided into $\mathcal{D}^2(G)$ and $\mathcal{D}^1(G)$.

2) Clearly, any graph $G - x \in \mathcal{D}^1(G)$ is connected. A vertex $y \in V(G-x)$ is a cutpoint of $G - x$ if and only if $\{x, y\} \in \mathfrak{R}_2(G)$.

First, we extract some information from the collection $\mathcal{D}^1(G)$.

Lemma 15. *Having the collection $\mathcal{D}(G)$, one can determine n_1 and the number of pendant 3-blocks of size n_1 in G.*

Proof. Consider all graphs $G-x \in \mathcal{D}^1(G)$ and all connected components of all graphs $G-x-y$ where y is a cutpoint of $G-x$. Let $\mathcal{C}$ be the collection of vertex sets of all such components. By Remark 5, these vertex sets are interiors of parts of decomposition of G by one set of $\mathfrak{R}_2(G)$. By Lemma 10, for $\delta(G) \geqslant 3$, a minimal part of decomposition of G by a set of $\mathfrak{R}_2(G)$ is a pendant 3-block. Therefore, the minimal size of a component in $\mathcal{C}$ is equal to $n_1 - 2$. Thus, we determine n_1.

Let $B \in \mathrm{Part}(G)$ be a pendant 3-block of size n_1 and $\mathrm{Bound}(B) = \{a, b\}$. Then $\mathrm{Int}(B)$ occurs in $\mathcal{C}$ exactly two times: from the graph $G - a$ and from the graph $G - b$. Hence, the number of pendant 3-blocks of size n_1 is equal to the number of sets of size $n_1 - 2$ in $\mathcal{C}$ divided by 2. □

Definition 15. 1) Pendant 3-blocks B_i and B_j are *isomorphic* if there exists a graph isomorphism $\varphi\colon H_i \to H_j$ such that $\varphi(T_i) = T_j$.

2) Let B_ℓ be a pendant 3-block of G.

We denote by $\mathcal{D}(B_\ell)$ the collection of all graphs $G - x$, where $x \in \mathrm{Int}(B_i)$ and the pendant 3-block B_i is isomorphic to B_ℓ.

We denote by $\mathcal{D}'(B_\ell)$ the collection of all graphs $G - x$, where $x \in \mathrm{Int}(B_i)$ and $|B_i| = |B_\ell|$.

Lemma 16. *Having the collection $\mathcal{D}(G)$, one can determine all graphs $H_1,\dots, H_k$ and, in each graph H_i, mark the set T_i.*

Proof. By Lemma 15, we know the minimal size n_1 of a pendant 3-block of G and the number s of 3-blocks of size n_1 in G. Let $B_1, \dots, B_s$ be all pendant 3-blocks of size n_1 in G. By Lemma 12, any graph $G-x \in \mathcal{D}^2(G)$ has at least $s-1$ pendant 3-blocks of size n_1. Moreover, $G-x$ has exactly $s-1$ pendant 3-blocks of size n_1 if and only if $x \in \mathrm{Int}(B_i)$ where $1 \leqslant i \leqslant s$. Thus, $\mathcal{D}'(B_1)$ consists of all graphs of the collection $\mathcal{D}^2(G)$ which have $s-1$ pendant 3-blocks of size n_1.

By Lemma 12, in any graph $G - x \in \mathcal{D}'(B_1)$, all pendant 3-blocks of size greater than n_1 are exactly all pendant 3-blocks of G of size greater than n_1. Thus, we can determine all graphs $H_{s+1}, \dots, H_k$ and, in each of them, mark the corresponding set T_i. It remains to determine the graphs $H_1, \dots, H_s$ (induced on pendant 3-blocks of size n_1). Consider two cases.

1. $s \geqslant 2$.
Clearly, $|\mathcal{D}'(B_1)| = s(n_1 - 2)$. For $1 \leqslant i \leqslant s$ and all $n_1 - 2$ vertices $x \in \mathrm{Int}(B_i)$, the graph $G - x$ has exactly $s - 1$ pendant 3-blocks of size n_1 (all such 3-blocks of G except for B_i). Consider all subgraphs of each $G - x \in \mathcal{D}'(B_1)$ induced on their pendant 3-blocks of size n_1. In this collection of subgraphs, each $H_1, \ldots, H_s$ occurs exactly $(s - 1)(n_1 - 2)$ times. Thus, we can determine all graphs $H_1, \ldots, H_s$. In each H_i, it is easy to mark the set T_i.

2. $s = 1$. We have to determine the only pendant 3-block B_1 of size n_1. In this case, G has another pendant 3-block, say, B_2. Let $|B_2| = n_2$ and $B_2, \ldots, B_{t+1}$ be all pendant 3-blocks of G with n_2 vertices. We already know all graphs $H_2, \ldots, H_{\ell+1}$. Then the collection $\mathcal{D}'(B_2)$ consists of all graphs $G - x \in \mathcal{D}^2(G)$ which have all pendant 3-blocks of G of size greater than n_2 and exactly $t-1$ pendant 3-blocks with n_2 vertices. Clearly, $|\mathcal{D}'(B_2)| = t(n_2 - 2)$.

In each graph $G - x \in \mathcal{D}'(B_2)$, consider all pendant 3-blocks of size n_1 and put to the collection $\mathcal{B}$ all subgraphs of G induced on them (we mark all boundaries of corresponding 3-blocks in these graphs). What subgraphs occur in $\mathcal{B}$? First, the subgraph H_1 occurs in $\mathcal{B}$ exactly $t(n_2-2)$ times (once for each graph of the collection $\mathcal{D}'(B_2)$). Other subgraphs in $\mathcal{B}$ (we call them *surplus subgraphs*) are subgraphs of graphs $H_i - x$ (where $x \in \mathrm{Int}(B_i)$ and $2 \leqslant i \leqslant t + 1$). All surplus subgraphs can be easily found. For this purpose, we are to consider all graphs $H_i^+ - x$ where $x \in \mathrm{Int}(B_i)$ and $2 \leqslant i \leqslant t + 1$ and, in each graph $H_i^+ - x$, find all pendant 3-blocks of size n_1, which interiors do not intersect T_i. Subgraphs induced on the blocks found above are exactly all surplus subgraphs and each of them occurs in $\mathcal{B}$ once. After deleting all of them from $\mathcal{B}$, we find here only several subgraphs H_1 with T_1 marked. □

Lemma 17. *Having the collection $\mathcal{D}(G)$, one can determine collections $\mathcal{D}(B_\ell)$ for all $\ell \in \{1, \ldots, k\}$.*

Proof. Let $|B_\ell| = n'$. By Lemma 16, we know the number t of pendant 3-blocks of G which are isomorphic to B_ℓ. Then $\mathcal{D}(B_\ell)$ consists of all graphs $G - x \in \mathcal{D}^2(G)$ which have all pendant 3-blocks of G of size greater than n' and exactly $t - 1$ pendant 3-blocks isomorphic to B_ℓ. □

§4. Proofs of main theorems

4.1. Properties of graphs of the collection $\mathcal{D}(B_1)$. Let $\{B_1, \dots, B_s\}$ be a maximal up to inclusion set of pairwise isomorphic pendant 3-blocks of the minimal size n_1 of the 2-connected graph G. We know the collection $\mathcal{D}(B_1)$. With the help of this collection we will construct the desired graphs G_1 and G_2 (one of which coincides with G). Consider a graph $G - x \in \mathcal{D}(B_1)$. Assume that $x \in \mathrm{Int}(B_1)$. Recall that the graph $G - x$ is 2-connected.

Remark 6. 1) By Lemma 12, $B_1 \setminus \{x\}$ is a T_1-fragment of $G - x$ and any part $A \in \mathrm{Part}(G - x; T_1)$ not contained in $B_1 \setminus \{x\}$ belongs to $\mathrm{Part}(G; T_1)$. By Corollary 1, A contains a pendant part of $\mathrm{Part}(G)$. Hence, $|A| \geqslant n_1$.

2) Let U be a T_1-fragment of $G - x$ which contains all parts of $\mathrm{Part}(G - x; T_1)$ except for one of them. By item 1, $|U| \geqslant n_1 - 1$.

3) Let U be a T_1-fragment of $G - x$ such that $|U| = n_1 - 1$. Then U cannot contain a part of $\mathrm{Part}(G - x; T_1)$ which is not a subset of $B_1 \setminus \{x\}$. Hence, $U = B_1 \setminus \{x\}$.

We will prove several claims describing properties of the graph $G - x$.

Claim 1. *Assume that $T \in \mathfrak{O}(G - x)$, and a part $B \in \mathrm{Part}(G - x; T)$ is such that $|B| < n_1$. Then $\mathrm{Int}(B) \cap T_1 = \varnothing$.*

Proof. Assume the converse. Since T_1 is independent with T, we have $T_1 \subset B$. Let $B' \in \mathrm{Part}(G - x; T_1)$ be the part which contains T and let A be the union of all parts of $\mathrm{Part}(G - x; T_1)$ different from B'. By Lemma 4, $B \supsetneq A$. However, by Remark 6, $|A| \geqslant n_1 - 1 \geqslant |B|$. We obtain a contradiction. □

Claim 2. *Let a pendant part $B \in \mathrm{Part}(G - x)$ be such that $|B| < n_1$. Then $B \subset (B_1 \setminus \{x\})$ and $\mathrm{Int}(B) \subset \mathrm{Int}(B_1 \setminus \{x\})$.*

Proof. By item 2 of Lemma 12, either $B \subset (B_1 \setminus \{x\})$ or B is a pendant 3-block of G. The latter is impossible due to $|B| < n_1$. By Lemma 1, $\mathrm{Int}(B) \subset \mathrm{Int}(B_1 \setminus \{x\})$. □

Claim 3. *Assume that $T \in \mathfrak{O}(G - x)$ and a T-fragment U of the graph $G - x$ is such that $|U| = n_1 - 1$. Then $T = T_1$ and $U = B_1 \setminus \{x\}$.*

Proof. If $T = T_1$ then, by Remark 6, $U = B_1 \setminus \{x\}$. Further, let $T \neq T_1$. Then these cutsets are independent. By Corollary 1, there exists a pendant part $B \in \mathrm{Part}(G-x)$ such that $\mathrm{Int}(B) \subset \mathrm{Int}(U)$. Then $|B| < n_1$ and, by Claim 2, we obtain $\mathrm{Int}(B) \subset \mathrm{Int}(B_1 \setminus \{x\})$.

By Claim 1, $T_1 \cap \mathrm{Int}(U) = \varnothing$. By Lemma 4, then $U \subsetneq B'$ where $B' \in \mathrm{Part}(G-x;T_1)$. In our case, $\mathrm{Int}(U) \cap \mathrm{Int}(B_1 \setminus \{x\}) \neq \varnothing$. Since $B_1 \setminus \{x\}$ is a T_1-fragment, $B' \subset B_1 \setminus \{x\}$. Then $|U| < |B_1 \setminus \{x\}| = n_1 - 1$, a contradiction. □

Now assume that there is no cutset $T \in \mathfrak{O}(G-x)$ and T-fragment U of the graph $G-x$ such that $|U| = n_1 - 1$. However, $T_1 \in \mathfrak{R}_2(G-x)$ and the T_1-fragment $B_1 \setminus \{x\}$ of the graph $G-x$ has exactly $n_1 - 1$ vertices. Therefore, $T_1 \in \mathfrak{R}_2(G-x)$ is a non-single cutset. Let's study $\mathrm{Part}(G-x;T_1)$ in this case.

Claim 4. *Let* $T_1 = \{a,b\} \notin \mathfrak{O}(G-x)$. *Then* $\mathrm{Part}(G;T_1) = \{B_1, D_1\}$ *and* $\mathrm{Part}(G-x;T_1) = \{B_1 \setminus \{x\}, D_1\}$ *where* $|D_1| \geqslant n_1$.

Proof. Since T_1 is a non-single cutset of $G-x$, $|\mathrm{Part}(G-x;T_1)| = 2$ by Lemma 5. By Lemma 12, hence, $|\mathrm{Part}(G;T_1)| = 2$. Let $\mathrm{Part}(G;T_1) = \{B_1, D_1\}$. Then, clearly, $\mathrm{Part}(G-x;T_1) = \{B_1 \setminus \{x\}, D_1\}$. By Remark 6, $|D_1| \geqslant n_1$. □

By Lemma 5, there exists a part $A \in \mathrm{Part}(G-x)$ such that A is a cycle and T_1 is its diagonal (i. e. T_1 consists of two non-neighboring in the cyclic order vertices of A).

Definition 16. Let $C \in \mathrm{Part}(G-x)$ be a cycle and $u, v \in C$. The arc uCv is *proper* if $w(uCv) = n_1 - 1$.

Remark 7. By the definitions, an arc uCv is proper if and only if the part of $\mathrm{Part}(G-x;\{u,v\})$ which corresponds to the arc uCv contains exactly $n_1 - 1$ vertices.

Clearly, the cycle A has a proper arc: it is the arc which corresponds to the part $B_1 \setminus \{x\} \in \mathrm{Part}(G-x;T_1)$. We will prove that other cycles of $\mathrm{Part}(G-x)$ has no proper arcs and study the structure of proper arcs of the cycle A.

Claim 5. *Let $N = aAb$ be the proper arc corresponding to the part $D_N = B_1 \setminus \{x\}$. Then the following statements hold.*

1) *If $L = uCv$ is a proper arc and $L \neq N$ then $\{u,v\}$ is dependent with T_1.*

2) *If $C \in \mathrm{Part}(G-x)$ is a cycle and $C \neq A$ then C has no proper arc.*

3) *Assume that an edge uv of the cycle A does not lie on the arc N and $w(uv) > 0$. Then uv cannot be contained in a proper arc.*

4) *Let vertices of A be cyclically enumerated such that $N = a_1Aa_k$ where $k \geqslant 3$. Assume that L is another proper arc. Then $L = a_2Aa_{k+1}$ or $L = a_0Aa_{k-1}$. If $L = a_2Aa_{k+1}$ then $w(a_1a_2) = w(a_ka_{k+1}) = 0$. If $L = a_0Aa_{k-1}$ then $w(a_0a_1) = w(a_{k-1}a_k) = 0$.*

Proof. 1) Let $T = \{u, v\} \in \mathfrak{R}_2(G - x)$ and $|D_L| = n_1 - 1$. Assume that T is independent with T_1. Note that cases $D_N \supset D_L$ and $D_L \supset D_N$ are impossible (in this cases, $|D_N| \neq |D_L|$).

By Lemma 4, the only case remaining is where $D_N \cap \mathrm{Int}(D_L) = \varnothing$. Let us prove that this is also impossible and obtain a contradiction. By Lemma 10, either $(G - x)(D_L) = G(D_L)$ is a simple uv-path or there exists a pendant part $B \in \mathrm{Part}(G - x)$ such that $\mathrm{Int}(B) \subset \mathrm{Int}(D_L)$. In the first case, all vertices of $\mathrm{Int}(D_L)$ have degree 2 in $G - x$. Hence, all vertices of $\mathrm{Int}(D_L)$ are adjacent in G with the vertex $x \in \mathrm{Int}(B_1)$ and, therefore, $\mathrm{Int}(D_L) \subset D_N$. In the second case, clearly, $|B| < n_1$ and, by Claim 2, $\mathrm{Int}(B) \subset \mathrm{Int}(D_N)$. In both cases we have a contradiction with $D_N \cap \mathrm{Int}(D_L) = \varnothing$.

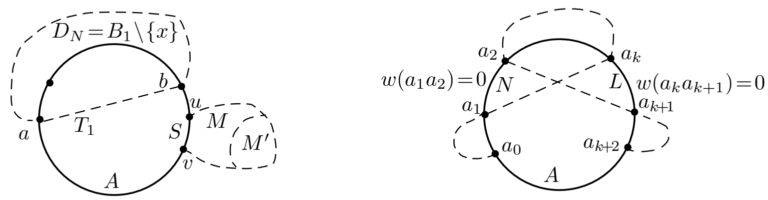
2) Assume the converse, let uCv be a proper arc and $T = \{u, v\}$. There exists a cutset $S \in \mathfrak{O}(G - x)$ which is adjacent to A in the tree $\mathrm{BT}(G - x)$ and separates A from C. By Lemma 7, S separates A from C in $G - x$. Hence, there exist distinct parts $M_A, M_C \in \mathrm{Part}(G - x; S)$ such that $T_1 \subset A \subset M_A$ and $T \subset C \subset M_C$. Then T_1 does not split T. Thus, T and T_1 are independent, a contradiction with item 1.

3) By the condition, $S = \{u, v\} \in \mathfrak{O}(G - x)$. Let $M \in \mathrm{Part}(G - x; S)$ be a part which does not contain A. By Corollary 1, there exists a pendant part $M' \in \mathrm{Part}(G - x)$ such that $\mathrm{Int}(M') \subset \mathrm{Int}(M)$. Since the edge uv does not lie on the arc N, we have $\mathrm{Int}(M) \cap D_N = \varnothing$ (see figure 2a). Then $|M'| \geqslant n_1$ by Claim 2. Therefore, $w(uv) \geqslant |\mathrm{Int}(M')| \geqslant |M'| - 2 \geqslant n_1 - 2$. If uv lies on a proper arc L then $w(uv) \leqslant w(L) - 2 = n_1 - 3$, a contradiction.

4) By item 2, $L = a_iAa_j$. By item 1, $T = \{a_i, a_j\}$ is dependent with $T_1 = \{a_1, a_k\}$ i.e. T corresponds to a diagonal of the cycle A which intersects a_1a_k in an inner point. Hence, we have one of the two following cases:

(a) L does not contain the vertex a_1 and the edge a_1a_2 but contains the vertex a_{k+1} and the edge a_ka_{k+1};

(b) L does not contain the vertex a_k and the edge $a_{k-1}a_k$ but contains the vertex a_0 and the edge a_0a_1.



a b

Figure 2. Proper arcs of the cycle A.

Consider the case (a). By item 3, $w(a_k a_{k+1}) = 0$ (see figure 2b). Assume that L also contains the edge $a_{k+1}a_{k+2}$. Then we also have $w(a_{k+1}a_{k+2}) = 0$, whence it follows $d_{G-x}(a_{k+1}) = 2$. Therefore, $a_{k+1} \in D_N$, we have a contradiction.

Thus, $w(a_k a_{k+1}) = 0$ and a_{k+1} is an end of L. Taking into account $w(N) = w(L)$ we obtain that $L = a_2 A a_{k+1}$ and $w(a_1 a_2) = 0$. Since $1 \leqslant |\mathrm{Int}(B \setminus \{x\})| = w(a_1 A a_k)$, we have $k \geqslant 3$.

In the similar case (b), we obtain $L = a_0 A a_{k-1}$ and $w(a_0 a_1) = w(a_{k-1} a_k) = 0$. $\square$

Claim 6. *Let $N = aAb$ be the proper arc corresponding to the part $B_1 \setminus \{x\}$ and let $F = G - \mathrm{Int}(B_1)$. Assume that N cannot be distinguished among proper arcs of the cycle A. Then the following statements hold.*

1) *A has exactly two proper arcs. The vertices of A can be cyclically enumerated such that proper arcs are $a_1 A a_k$ and $a_2 A a_{k+1}$. The arc $a_1 A a_{k+1}$ can be uniquely determined.*

2) *One of the degrees $d_F(a)$ and $d_F(b)$ is equal to 1 and the other is greater than 1. If $d_F(a) > 1$ then $d_{H_1 - x}(a) = 1$.*

Proof. 1) Let $A = a_0 a_1 \dots a_\ell$. We may assume that both arcs $L_1 = a_1 A a_k$ and $L_2 = a_2 A a_{k+1}$ are proper and $w(a_1 a_2) = w(a_k a_{k+1}) = 0$. (By item 4 of Claim 5, the cycle A has two such proper arcs.)

Assume that $w(a_0 a_1) = 0$. Then $d_{G-x}(a_1) = 2$ and, therefore, a_1 is adjacent to x in G, whence it follows $a_1 \in B_1$. Then the arc $a_1 A a_k$ corresponds to the part $B_1 \setminus \{x\}$ i.e. $N = a_1 A a_k$, a contradiction. Hence, $w(a_0 a_1) > 0$ and, similarly, $w(a_{k+1} a_{k+2}) > 0$.

We claim that N must coincide with one of the arcs L_1 and L_2. Indeed, assume the converse. Then, applying item 4 of Claim 5 to N and L_1, we obtain $N = a_0 A a_{k-1}$. At the same time, applying item 4 of Claim 5 to

N and L_2, we obtain $N = a_3Aa_{k+2}$. Since the arc N is unique, we have $a_0 = a_3$ i.e. $|A| = 3$ and, by Claim 5, $k = 3$. Thus, $L_1 = a_1Aa_0$ is a proper arc. Hence, by proved above, $w(a_1a_2) = 0$ and, at the same time, $w(a_1a_2) > 0$ (since $a_1a_2 = a_{k+1}a_{k+2}$), a contradiction.

Now N coincides with L_1 or L_2. Hence, the edges a_0a_1 and $a_{k+1}a_{k+2}$ do not belong to N. By item 3 of Claim 5, a_0a_1 and $a_{k+1}a_{k+2}$ cannot belong to proper arcs. However, if A has one more proper arc a_iAa_j then $S = \{a_i, a_j\}$ is dependent with T_1 by Claim 5 i.e. S is dependent either with $\{a_1, a_k\}$ or with $\{a_2, a_{k+1}\}$. Therefore, the arc a_iAa_j must contain one of the edges a_0a_1 and $a_{k+1}a_{k+2}$, a contradiction.

2) Assume that $aAb = a_1Aa_k$ (the case $aAb = a_2Aa_{k+1}$ is similar). Then $w(a_1a_2) = 0$ implies that $a = a_1$ is adjacent to exactly one vertex of $B_1 \setminus \{x\}$ — namely, to a_2. Since $w(a_0a_1) > 0$, we have $\{a_0, a_1\} \in \mathfrak{O}(G-x)$. Hence, $d_{G-x}(a_1) \geqslant 3$, whence it follows $d_F(a_1) \geqslant 2$. On the other side, $w(a_ka_{k+1}) = 0$ implies that $b = a_k$ is adjacent to exactly one vertex of D_1 (to a_{k+1}). Thus, $d_F(a) > 1$, $d_{H_1-x}(a) = 1$ and $d_F(b) = 1$. □

Claim 7. *Assume that, for all $y \in \mathrm{Int}(B_1)$, the set $B_1 \setminus \{y\}$ cannot be distinguished in the graph $G - y$. Then $n_1 = 4$ and H_1 is a complete graph on 4 vertices without the edge ab.*

Proof. Let $F = G - \mathrm{Int}(B_1)$. Consider a vertex $z \in \mathrm{Int}(B_1)$. By Claim 3, T_1 is a non-single 2-cutset of the graph $G - z$. By Lemma 5, we may assume that T_1 is a diagonal of a cycle $A_z \in \mathrm{Part}(G - z)$. In our case, we cannot determine the proper arc of A_z corresponding to $B_1 \setminus \{z\}$. Hence, by Claim 6, the cycle A_z has exactly two proper arcs, one of the degrees $d_F(a)$ and $d_F(b)$ is equal to 1 and the other is greater than 1. Without loss of generality, assume that $d_F(a) \geqslant 2$.

By Claim 6, for any vertex $y \in \mathrm{Int}(B_1)$, at least one of the following two statements hold: $d_{H_1-y}(a) = 1$ and $d_{H_1-y}(b) = 1$. Since H_1^+ is 3-connected, we obtain $ab \notin H_1$ and $d_{H_1}(a) = d_{H_1}(b) = 2$.

Assume that $n_1 \geqslant 5$. Then $|\mathrm{Int}(B_1)| \geqslant 3$. Therefore, there exists a vertex $x \in \mathrm{Int}(B_1)$ not adjacent to a. Consider the graph $G - x \in \mathcal{D}(B_1)$. Clearly, $d_{H_1-x}(a) = 2$. Since, at the same time, $d_F(a) \geqslant 2$, we have a contradiction with item 2 of Claim 6.

Thus, $n_1 = 4$ and $|\mathrm{Int}(B_1)| = 2$. Since $\delta(G) \geqslant 3$, two vertices of $\mathrm{Int}(B_1)$ must be adjacent to each other and to both vertices a and b. It was proved above that $ab \notin E(G)$. Hence, H_1 is a complete graph on 4 vertices without the edge ab. □

4.2. Proof of Theorem 1. Now we are able to finish the proof of our main theorem.

Proof of Theorem 1. We will consider two cases.

1. *There exists a graph $G - x \in \mathcal{D}(B_1)$, such that $x \in \mathrm{Int}(B_i)$ and the set $B_i \setminus \{x\}$ can be distinguished in $G - x$.*

Without loss of generality, we may assume that $x \in \mathrm{Int}(B_1)$. Let $T_1 = \{a, b\}$. Recall that $B_1 \setminus \{x\}$ is a T_1-fragment of the graph $G - x$. The set T_1 can be easily marked in a T_1-fragment. Hence, we know the graph $H' = G - \mathrm{Int}(B_1) = G - x - \mathrm{Int}(B_1 \setminus \{x\})$ with vertices of the set T_1 marked. By Lemma 16, we also know the graph $H_1 = H(B_1)$ with vertices of T_1 marked. Two ways of gluing together the graphs H' and H_1 by the set T_1 give us two graphs G_1 and G_2 such that $G \in \{G_1, G_2\}$.

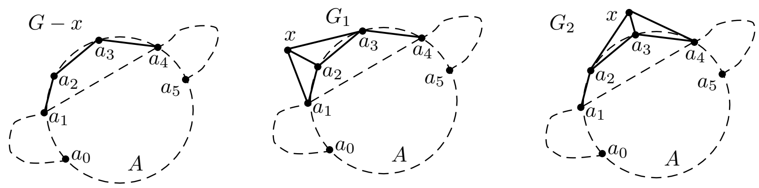
Let's prove that $\mathrm{BT}(G_1) = \mathrm{BT}(G_2)$. The graph H_1^+ is 3-connected. Hence, H_1 has no cutpoint separating a from b. Then, by Lemma 5, $T_1 \in \mathfrak{O}(G_1)$ and $T_1 \in \mathfrak{O}(G_2)$. Since H_1^+ is 3-connected, B_1 is a pendant 3-block of both graphs G_1 and G_2 by Lemma 11. Since $G_1 - \mathrm{Int}(B_1) = H' = G_2 - \mathrm{Int}(B_1)$, by Lemma 13, all single cutsets and parts of G_1 lying outside B_1 are all single cutsets and parts of G_2 lying outside B_1.

2. *For all $G - x \in \mathcal{D}(B_1)$ where $x \in \mathrm{Int}(B_i)$ and $1 \leqslant i \leqslant s$, the set $B_i \setminus \{x\}$ cannot be distinguished in the graph $G - x$.*

By Claim 7, then $n_1 = 4$ and, for every $i \in \{1, \dots, s\}$, H_i is a clique on 4 vertices without the edge between vertices of T_i.

Consider a graph $G - x \in \mathcal{D}(B_1)$, say $x \in \mathrm{Int}(B_1)$. Let $T_1 = \{a, b\}$. By Claim 3, T_1 is a non-single cutset of $G - x$. By Claim 5, we can find the unique cycle $A \in \mathrm{Part}(G - x)$ which has proper arcs. Then T_1 is a diagonal of A which consists of ends of the proper arc of A corresponding to the part $B_1 \setminus \{x\}$. In the case we are considering, $H_1 - x$ is a simple path $ax'b$ where $\mathrm{Int}(B) = \{x, x'\}$. Hence, any proper arc of A is a path of length 2. By item 2 of Claim 6, the cycle A has exactly two proper arcs. Moreover, vertices of A can be cyclically enumerated such that proper arcs are $N = a_1a_2a_3$ and $L = a_2a_3a_4$ (see figure 3a) and the arc a_1Aa_4 can be determined.

If the part $B_1 \setminus \{x\}$ corresponds to the arc N then x is adjacent in G to a_1, a_2 and a_3 (see figure 3b). Denote this graph by G_1. If the part $B_1 \setminus \{x\}$ corresponds to the arc L then x is adjacent in G to a_2, a_3 and a_4 (see figure 3c). Denote this graph by G_2. Then $G \in \{G_1, G_2\}$. Let $H = G(\{x, a_1, a_2, a_3, a_4\})$ and $H^* = G - \{x, a_2, a_3\}$. Clearly, both G_1 and G_2 are results of gluing together H and H^* by the set $\{a_1, a_4\}$.



a b c

Figure 3. The graphs G_1 and G_2.

Similarly to Case 1, $T_1 \in \mathfrak{O}(G_1)$, $T_1 \in \mathfrak{O}(G_2)$ and B_1 is a pendant 3-block of both graphs G_1 and G_2. Lemma 13 implies $\mathrm{BT}(G_1) = \mathrm{BT}(G_2)$ (this tree can be obtained from $\mathrm{BT}(G - x)$ by adding vertices T_1, B_1 and edges AT_1, T_1B_1). □

4.3. On possible non-uniqueness of the reconstruction. In what follows, we will formulate and prove a more detailed theorem on possible non-uniqueness in the reconstruction of graphs of connectivity 2. We hope this theorem will help to prove the full version of Reconstruction Conjecture for graphs of connectivity 2.

Definition 17. Let G be a 2-connected graph. We say that a graph G^* is obtained from G by *inverting* a subgraph H if there exists $T \in \mathfrak{R}_2(G)$ such that H is a T-subgraph of G and G^*, G are two graphs obtained from H and $G - \mathrm{Int}(H)$ upon gluing them together by the set T.

Remark 8. Let G^* is obtained from G by inverting a subgraph H. It is easy to see that then H is a T-subgraph of G^* and G is obtained from G^* by inverting H.

Theorem 2. *Assume that G_1 and G_2 are non-isomorphic graphs on the vertex set V such that $\kappa(G_1) = \kappa(G_2) = 2$, $\delta(G_1) \geqslant 3$, $\delta(G_2) \geqslant 3$ and $\mathcal{D}(G_1) = \mathcal{D}(G_2)$. Then there exists a set $A \subset V$ such that G_2 is obtained from G_1 by inverting $H = G_1(A)$. Moreover, one of the two following conditions holds.*

(a) *A is a minimal pendant 3-block of both graphs G_1 and G_2.*

(b) *$|A| = 5$, $A = B_1 \cup \{u\}$ where $u \notin B_1$ and B_1 is a minimal pendant 3-block of both graphs G_1 and G_2. Moreover, $d_H(u) = 1$ (u is adjacent in H to one vertex of* $\mathrm{Bound}(B_1)$*).*

Proof. Assume that $G = G_1$ is a graph we know. Let's consider the collection $\mathcal{D}(G)$. Having $\mathcal{D}(G)$, we construct by the algorithm of Theorem 1 two graphs G_1^*, G_2^* such that $G \in \{G_1^*, G_2^*\}$.

At least one of the collections $\mathcal{D}(G_1^*)$ and $\mathcal{D}(G_2^*)$ coincides with the known collection $\mathcal{D}(G)$. If $G_1^* \simeq G_2^*$ or $\mathcal{D}(G_1^*) \neq \mathcal{D}(G_2^*)$ then we can uniquely reconstruct the graph $G = G_1$ from $\mathcal{D}(G_1)$. However, it is impossible in our case (since $\mathcal{D}(G_1) = \mathcal{D}(G_2)$ and $G_1 \not\simeq G_2$). If there exists a graph G^* with $\mathcal{D}(G^*) = \mathcal{D}(G_1)$ which is isomorphic to neither G_1^* nor G_2^* then the statement $G \in \{G_1^*, G_2^*\}$ is wrong, a contradiction with Theorem 1. Thus, the only case remaining is $\{G_1^*, G_2^*\} = \{G_1, G_2\}$.

By the construction of G_1 and G_2 (see the proof of Theorem 1), G_2 is obtained from G_1 by inverting a subgraph $H = G_1(A) = G_2(A)$. In Case 1 of the proof, A is a minimal pendant 3-block i.e. statement (a) holds. In Case 2 of the proof, statement (b) holds. □

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