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PROBABILITY MEASURE NEAR THE BOUNDARY OF TENSOR POWER DECOMPOSITION FOR $\mathfrak{so}_{2n+1}$

ABSTRACT. Character measure is a probability measure on irreducible representations of a semisimple Lie algebra. It appears from the decomposition into irreducibles of tensor power of a fundamental representation. In this paper we calculate the asymptotics of character measure on representations of $\mathfrak{so}_{2n+1}$ in the regime near the boundary of weight diagram. We find out that it converges to a Poisson-type distribution.

§1. INTRODUCTION

The probability measures that appear from a decomposition of representation into irreducibles are actively studied for many decades. One of the most famous asymptotic results is Vershik–Kerov–Logan–Shepp limit shape of Young diagrams with respect to Plancherel measure for S_N as $N \to \infty$ [1, 2]. This result is connected to Ulam's problem [3] on the length of the maximal increasing subsequence in a uniform random sequence. The same limit shape is obtained from the decomposition of N-th tensor power of the $\mathfrak{sl}_n$-fundamental representation into irreducibles in the limit $N, n \to \infty$ with N/n in the paper [4], where the case $N \to \infty, n = \text{const}$ was also considered. The case $\mathfrak{g} = \mathfrak{so}_{2n+1}, N \to \infty, n = \text{const}$ was studied in [5] and [6].

Character measure is a generalization of Plancherel-type measures that is introduced by substituting characters of representations instead of their dimensions. The central limit regime and the regime of large deviations for character measure that appears in the decomposition of N-th tensor power of a fundamental representation of a simple Lie algebra of rank n, with n fixed, $N \to \infty$, were considered in [7] by O. Postnova and N. Reshetikhin. They also suggested that in the regime near the boundary the character measure would converge to some Poisson type process. They

demonstrated that for $\mathfrak{g} = \mathfrak{sl}_2$ it converges to Poisson distribution; in this
paper we generalize this result to all $\mathfrak{so}_{2n+1}$ algebras.

This paper is organized as follows: in Section 2 we recall the definitions of
Plancherel and character measures, fix the notations and state Theorem 1
on convergence of the character measure near the boundary. In Sections 3,4
we derive the asymptotics for the tensor power decomposition multiplicities
and the characters correspondingly. In Section 5 we complete the proof of
Theorem 1, discuss the connection of this result to random walks and
possible future work.

§2. RESULT AND NOTATIONS

Consider a simple finite-dimensional complex Lie algebra $\mathfrak{g}$ of rank n and
its irreducible finite-dimensional highest-weight representation L^λ. Denote
simple roots of $\mathfrak{g}$ by $\alpha_1, \ldots, \alpha_n$ and fundamental weights by $\omega_1, \ldots \omega_n$:
$(\alpha_i, \omega_j) = \delta_{ij}$. Let us take a fundamental representation L^ω of $\mathfrak{g}$, $\omega \in
\{\omega_1, \ldots, \omega_n\}$, and consider its tensor power. Tensor power of L^ω is a com-
pletely reducible representation and can be decomposed as:

$$(L^\omega)^{\otimes N} = \bigoplus_\lambda M_\lambda^{\omega,N} L^\lambda, \tag{1}$$

where $M_\lambda^{\omega,N}$ is a multiplicity of L^λ. The sum is taken over all irreducible
components of the tensor product. This means that for the dimensions
holds the equality

$$\dim\left((L^\omega)^{\otimes N}\right) = \sum_\lambda M_\lambda^{\omega,N} \dim(L^\lambda). \tag{2}$$

This formula can be used to introduce Plancherel-type probability mea-
sure on the set of dominant integral weights:

$$P_\lambda^N = \frac{M_\lambda^{\omega,N} \dim L^\lambda}{(\dim L^\omega)^N}. \tag{3}$$

We can generalize this formula by replacing $\dim(L^\lambda)$, which is equal to
$\chi_{L^\lambda}(0)$, by a character $\chi_\lambda(e^t)$ of L^λ taken at any item e^t from maximal
torus of the Lie group or item t from the Cartan subalgebra $\mathfrak{h}$ of the Lie
algebra $\mathfrak{g}$. The character of a representation L^λ of a Lie algebra is equal to

$$\chi_\lambda(e^t) = \sum_{\mu \in \mathcal{N}(\lambda)} \dim V_\mu \cdot e^{(\mu,t)}. \tag{4}$$

Representation $L^\lambda = \bigoplus_\mu V_\mu$ is decomposed into direct sum of weight subspaces and $\dim V_\mu = m_\mu^\lambda$ is the multiplicity of weight μ in L^λ. $\mathcal{N}(\lambda)$ is the weight diagram. So we get the character measure:

$$P_\lambda^N(t) = \frac{M_\lambda^{\omega,N} \chi_\lambda(e^t)}{(\chi_\omega(e^t))^N}. \tag{5}$$

Theorem 1. *Consider Lie algebra $\mathfrak{so}_{2n+1}$ with root system B_n, n is fixed. Let us take the tensor power N of its irreducible representation with the highest weight $\lambda = \omega_n$ (it is known as last fundamental representation or spinor representation). Consider the character measure $P_\lambda^N(t)$ given by the formula* (5), *taken at e^t, where t is an element of Cartan subalgebra of $\mathfrak{so}_{2n+1}$.*

Then as $N \to \infty$ and $t \to \infty$ such that $\Theta_i = Ne^{-2t_i}$ is fixed, the probability measure $P_\lambda^N(t)$ converges pointwise to

$$P_s(\Theta) = \prod_{i<j}^n (-s_i - i + s_j + j) \cdot \prod_{k=1}^n \frac{\Theta_k^{\frac{s_1 + \cdots + s_n}{n}} \cdot e^{-\Theta_k}}{(s_k + k - 1)!} \chi_\gamma^{\mathfrak{sl}_n}(e^\tau), \tag{6}$$

where $N - 2s_i$ are coordinates of the highest weight λ, $\chi_\gamma^{\mathfrak{sl}_n}(e^\tau)$ is a character of the $\mathfrak{sl}_n$-subalgebra representation of the highest weight γ with the coordinates $\gamma_i = \frac{2s_1 + \cdots + 2s_n}{n} - 2s_i$ and τ is an element of Cartan subalgebra in $\mathfrak{sl}_n \subset \mathfrak{so}_{2n+1}$ with the coordinates $\tau_i = -t_i + t_{i+1}$.

§3. COORDINATES AND MULTIPLICITIES

We will use the standard orthogonal basis such that all the elements of Cartan subalgebra would be diagonal. In this basis simple roots are

$$\alpha_i = e_i - e_{i+1}, \ i = 1, \ldots, n-1; \quad \alpha_n = e_n, \tag{7}$$

and fundamental weights are

$$\omega_1 = e_1; \ \omega_2 = e_1 + e_2; \ \ldots \ \omega_{n-1} = e_1 + \ldots + e_{n-1}; \ \omega_n = \frac{1}{2}(e_1 + \ldots + e_n). \tag{8}$$

For the highest weight λ we denote its coordinates by $\widetilde{\lambda}_i$ so that $\lambda = \sum_{i=1}^n \widetilde{\lambda}_i e_i$. For convenience we rescale the coordinates: $\lambda_i = 2\widetilde{\lambda}_i$. The multiplicity formula [8] can be written easily using the coordinates shifted by the Weil vector $\rho = \omega_1 + \cdots + \omega_n$, $a_i = \lambda_i + \rho_i$, where $\rho_i = 2(n-i) + 1$

are the coordinates of ρ:

$$M_\lambda^{\omega_n,N} = \prod_{k=0}^{n-1} \frac{(N+2k)!}{2^{2k}\left(\frac{N+a_{k+1}+2n-1}{2}\right)!\left(\frac{N-a_{k+1}+2n-1}{2}\right)!} \prod_{l=1}^{n} a_l \prod_{i<j}^{n}(a_i^2-a_j^2). \quad (9)$$

We study regime near the boundary, which means λ should be not far from the highest weight of $(L^{\omega_n})^{\otimes N}$ representation. So we take

$$\lambda_i = N - 2s_i, \quad s_i = \mathcal{O}(1). \quad (10)$$

Let us rewrite the formula (9) using the parameters $\{s_i\}$:

$$M_\lambda^{\omega_n,N} = \prod_{k=0}^{n-1} \frac{(N+2k)!}{2^{2k}\left(N+\frac{-2s_{k+1}+\rho_{k+1}+2n-1}{2}\right)!\left(\frac{2s_{k+1}-\rho_{k+1}+2n-1}{2}\right)!} \cdot \widetilde{M}, \quad (11)$$

where we have denoted by $\tilde{M}$ the following expression:

$$\widetilde{M} = \prod_{l=1}^{n}(N-2s_l+\rho_l) \prod_{i<j}^{n}(-2s_i+\rho_i+2s_j-\rho_j)(2N-2s_i+\rho_i-2s_j+\rho_j). \quad (12)$$

Substituting the coordinates $\{\rho_i\}$ and using the identity $\prod_{k=0}^{n-1} 2^{2k} = 2^{n(n-1)}$, we get

$$M_\lambda^{\omega_n,N} = \frac{1}{2^{n(n-1)}} \prod_{k=1}^{n} \frac{(N+2(k-1))!}{(N-s_k+2n-k)!(s_k+k-1)!} \cdot \widetilde{M}, \quad (13)$$

with

$$\widetilde{M} = \prod_{l=1}^{n}(N-2s_l+2n-2l+1)$$
$$\times \prod_{i<j}^{n}(-2s_i-2i+2s_j+2j)(2N-2s_i-2i-2s_j-2j+4n+2). \quad (14)$$

First we write down the leading asymptotic in N for $\widetilde{M}$:

$$\widetilde{M} = N^n \cdot (2N)^{\frac{n(n-1)}{2}} \cdot \prod_{i<j}^{n}(-2s_i-2i+2s_j+2j)\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right). \quad (15)$$

Then we derive the leading asymptotic in N for another factor in the expression (13):

$$\frac{(N + 2(k-1))!}{(N - s_k + 2n - k)!} = N^{N+2k-2-N+s_k-2n+k}\left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right)$$

$$= N^{s_k+3k-2n-2}\left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right). \tag{16}$$

The whole expression is:

$$M_\lambda^{\omega_n,N} = \frac{1}{2^{n(n-1)}}\prod_{k=1}^{n}\left(\frac{N^{s_k+3k-2n-2}}{(s_k + k - 1)!}\right) \cdot N^n \cdot (2N)^{\frac{n(n-1)}{2}}$$

$$\times \prod_{i<j}^{n}(-2s_i - 2i + 2s_j + 2j)\left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right)$$

$$= N^{-\frac{3}{2}n(n+1)}\prod_{k=1}^{n}\frac{N^{s_k+3k}}{(s_k + k - 1)!} \cdot \prod_{i<j}^{n}(-s_i - i + s_j + j)\left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right). \tag{17}$$

Substitute $\prod\limits_{k=1}^{n} N^{3k} = N^{\frac{3}{2}n(n+1)}$ and obtain the asymptotic for the multiplicities:

$$M_\lambda^{\omega_n,N} = \left[\prod_{k=1}^{n}\frac{N^{s_k}}{(s_k + k - 1)!} \cdot \prod_{i<j}^{n}(-s_i - i + s_j + j)\right] \cdot \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right). \tag{18}$$

Now we need to derive the asymptotic of characters.

§4. Asymptotics of characters

First we consider the character of L^{ω_n}. Weights of L^{ω_n} are located at the vertices of an n-dimensional cube. Their coordinates are all combinations of 1 and -1, thus for the character in the N-th power we have

$$(\chi_{\omega_n}(e^t))^N = (e^{t_1+\cdots+t_n} + e^{-t_1+t_2+\cdots+t_n} + e^{t_1-t_2+\cdots+t_n} + \cdots + e^{-t_1-\cdots-t_n})^N. \tag{19}$$

Rewriting this expression using the parameters $\{\Theta_i\}$, in the limit we can neglect the terms like e^{-t_i} as $e^{-t_i} = \mathcal{O}\left(\frac{1}{N}\right)$:

$$\chi_{\omega_n}(e^t) = e^{t_1+\cdots+t_n} + e^{-t_1+t_2+\cdots+t_n} + \cdots + e^{t_1+\cdots-t_i+\cdots+t_n}$$

$$+ \cdots + e^{t_1+\cdots+t_{n-1}\cdots-t_n} + e^{t_1+\cdots+t_n} \cdot \mathcal{O}\left(\frac{1}{N^2}\right)$$

$$= e^{t_1+\cdots+t_n}\left(1 + e^{-2t_1} + \cdots + e^{-2t_n} + O\left(\frac{1}{N^2}\right)\right). \quad (20)$$

So, using the new parameters we get:

$$(\chi_{\omega_n}(\Theta))^N = \left(e^{t_1+\cdots+t_n}\left(1 + \frac{\Theta_1+\cdots+\Theta_n}{N} + \mathcal{O}\left(\frac{1}{N^2}\right)\right)\right)^N, \quad (21)$$

which is equal to

$$(\chi_{\omega_n}(\Theta))^N = e^{(t_1+\cdots+t_n)N} \cdot e^{(\Theta_1+\cdots+\Theta_n)\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right)}. \quad (22)$$

Then we calculate the leading contribution to the character of L^{N-2s}. Remember that terms of the same order as e^{-t_i} could be neglected. What weights contribute to this asymptotic? These are the weights with the maximal sum of coordinates, denote this set of weights by $\Omega \subset \mathcal{N}(\lambda)$. These weights belong to the plane that contains the highest weight λ and is orthogonal to $(1,\ldots,1)$. All these weights are obtained from the highest weight by subtracting the simple roots that lie in this plane. Simple roots lying in this plane are all the roots except α_n, since for $k \in \{1,\ldots,n-1\}$ we have:

$$(1,1,\ldots,1) \cdot \alpha_k = \frac{1}{2}(1,\ldots,1) \cdot (0,\ldots,1,-1,\ldots,0) = 0. \quad (23)$$

These roots make up the root system A_{n-1}, which corresponds to a regular subalgebra $\mathfrak{sl}_n$ in $\mathfrak{so}_{2n+1}$. Since the Weyl group $W_{\mathfrak{sl}_n}$ of the subalgebra $\mathfrak{sl}_n$ is a subgroup of the Weyl group $W_{\mathfrak{so}_{2n+1}}$ and leaves the plane invariant, the set Ω is also $W_{\mathfrak{sl}_n}$-invariant. The weight subspaces for the weights in Ω are obtained from the highest weight vector v_λ by a repeated action of the lowering generators $\{E^{-\alpha_k}, k = 1,\ldots,n-1\}$ corresponding to the $\mathfrak{sl}_n$ simple roots. Thus the set Ω with the corresponding weight multiplicities is the weight diagram $\mathcal{N}(\gamma)$ of the irreducible $\mathfrak{sl}_n$ representation with the highest weight γ, shifted by the vector p with the coordinates

$$p_k = N - \frac{2s_1+\cdots+2s_n}{n}, \quad k = 1,\ldots,n. \quad (24)$$

We can imagine this as a slice of an n-dimensional hypercube, which is normal to $(1, \ldots, 1)$ and has the center in p.

The highest weight γ is the projection of λ to the dual space $\mathfrak{h}^*_{\mathfrak{sl}_n}$ of the Cartan subalgebra of $\mathfrak{sl}_n$ and it has the coordinates:

$$\gamma_i = \frac{2s_1 + \cdots + 2s_n}{n} - 2s_i. \tag{25}$$

So we see that the leading asymptotic of $\chi_\lambda(e^t)$ is a character of the $\mathfrak{sl}_n$ irreducible representation with the highest weight γ, shifted by p:

$$\chi_\lambda(e^t) = \chi_\gamma^{\mathfrak{sl}_n}(e^\tau) \cdot e^{(t_1 + t_2 + \cdots + t_n) \cdot (N - \frac{2s_1 + \cdots + 2s_n}{n})} \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right). \tag{26}$$

Here τ is the item in Cartan subalgebra of $\mathfrak{sl}_n$ which is the projection of the item $t \in \mathfrak{h}$ in Cartan subalgebra of $\mathfrak{so}_{2n+1}$. Its coordinates are

$$\tau_i = -t_i + t_{i+1}, \quad i = 1, \ldots, n-1. \tag{27}$$

Changing the parameters from $\{t_i\}$ to $\{\Theta_i\}$ we obtain:

$$\chi_\lambda(e^t) = \chi_\gamma(e^\tau)$$
$$\times e^{(t_1 + t_2 + \cdots + t_n) \cdot N} \cdot \left(\frac{\Theta_1}{N} \cdot \ldots \cdot \frac{\Theta_n}{N}\right)^{\frac{s_1 + \cdots + s_n}{n}} \cdot \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right). \tag{28}$$

§5. Completion of the proof and discussion

We combine the equations (18), (22), (28) together and obtain:

$$P_\lambda^N(t) = \frac{M_\lambda^{\omega_n, N} \chi_\lambda(e^t)}{(\chi_\omega(e^t))^N} = \prod_{k=1}^{n} \frac{N^{s_k}}{(s_k + k - 1)!} \cdot \prod_{\substack{i<j}}^{n} (-s_i - i + s_j + j)$$
$$\times \left(\frac{\Theta_1}{N} \cdot \ldots \cdot \frac{\Theta_n}{N}\right)^{\frac{s_1 + \cdots + s_n}{n}} e^{-(\Theta_1 + \ldots + \Theta_n)\left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right)} \chi_\gamma(e^\tau) \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right). \tag{29}$$

Rearranging the factors we arrive to the asymptotic of character measure:

$$P_\lambda^N(t) = \prod_{\substack{i<j}}^{n} (-s_i - i + s_j + j) \cdot \prod_{k=1}^{n} \frac{N^{s_k} e^{-\Theta_k \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right)}}{(s_k + k - 1)!}$$
$$\times \frac{(\Theta_1 \ldots \Theta_n)^{\frac{s_1 + \cdots + s_n}{n}}}{N^{s_1 + s_2 + \cdots + s_n}} \chi_\gamma(e^\tau) \cdot \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right). \tag{30}$$

Taking the limit $N \to \infty$, we complete the proof of Theorem 1:

$$P_s(\Theta) = \prod_{i<j}^{n}(-s_i - i + s_j + j) \cdot \prod_{k=1}^{n} \frac{\Theta_k^{\frac{s_1+\cdots+s_n}{n}} \cdot e^{-\Theta_k}}{(s_k + k - 1)!} \chi_\gamma(e^\tau). \qquad (31)$$

This result can be interpreted as a random walk with a critical drift [7]. Connection to random walks is easy to understand in the $\mathfrak{sl}_2$-case for the probability measure defined on the weight diagram of N-th tensor power of fundamental representation L^ω: $p(\lambda) = \frac{\dim V_\lambda}{(\dim L^\omega)^N}$, where $\dim V_\lambda$ is the dimension of the weight subspace in the representation $(L^\omega)^{\otimes N}$. In this case the probability distribution $p(\lambda)$ is the binomial distribution and $\dim V_\lambda$ is the number of random walks of N steps on weight diagram ending in λ. Plancherel-type measures can be obtained from such a measure by taking an alternating sign combination of random walks [6]. The character measure has the expected value of the weight $\mathbb{E}[\lambda] = N\eta$, which is then interpreted as the expectation value of a random walk with a drift η, that is determined by the coordinates $\{t_i\}$. Our result appears in the critical drift regime $t_i \sim \ln N$, where random walks stay near the boundary and thus it generalizes Poisson distribution to the case of random walks on the weight diagram of $\mathfrak{so}_{2n+1}$.

In the upcoming publications we are going to consider a similar asymptotic for the $\mathfrak{sl}_n$-case and search for a general formula for all four series of classical Lie algebras: $\mathfrak{sl}_n, \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{so}_{2n}$.

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