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HEISENBERG XX0 CHAIN AND RANDOM WALKS ON A RING

Abstract. We obtain and investigate mean values of the exponential of the centroid operator for the periodic Heisenberg $XX0$ chain on a ring. The generating function of directed lattice paths is obtained in terms of circulant matrices which leads to generalizations of the Ramus's identity. The two-time correlation function of the exponential of the centroid operator is expressed in terms of the Cauchy determinant and thus explicitly calculated.

§1. Introduction

The Heisenberg XXZ 1/2-spin chain is one of the most popular fundamental models of theoretical physics [1–3], and it is tightly connected with the integrable enumerative combinatorics [4, 5]. The correlation functions of the $XX0$ spin chain, which is the zero anisotropy limit of the XXZ system, may be interpreted in terms of the directed lattice paths [6–8] generated by random walks. The considered periodic $XX0$ chain on the ring is based on the theory of the circulant matrices [9, 10] and lacunary sums of binomial coefficients [11, 12]. Our works [13–16] may be seen as the continuation of the present paper on the theory of lattice paths and calculation of correlation functions of the special type.

Organization of the paper is as follows. In Section 2 we give a short description of the $XX0$ chain subjected to the periodic boundary condition. The directed walks of a single walker on a cylinder are considered in Section 3. The exponential generating function of these walks corresponds to a thermal evolution of a single particle on a ring chain. The random turns walkers are considered in Section 4, and the exponential generation function of these walks is calculated. This generating function coincides with the multi-particles thermal correlation function of the $XX0$ chain

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over the ferromagnetic ground state. The *centroid operator*, i.e., the operator of the center of mass coordinate, is introduced in Section 4, and the time evolution of the exponential of this operator is calculated.

§2. Heisenberg $XX0$ chain

The periodic Heisenberg $XX0$ chain describing $\frac{1}{2}$-spins on sites of a ring chain is described by the Hamiltonian

$$\hat{H} = \sum_{n,m=0}^{M-1} \Delta_{nm} \sigma_n^- \sigma_m^+ = \sum_{n=0}^{M-1} \sigma_{n-1}^- \sigma_n^+ + \sigma_{n+1}^- \sigma_n^+ \,, \tag{1}$$

where the number of sites is $M = 0 \pmod 2$. The local spin operators $\sigma_n^\pm = \frac{1}{2}(\sigma_n^x \pm i\sigma_n^y)$ and σ_n^z depend on the lattice argument $n \in \mathcal{E} \equiv \{0, 1, \ldots, M-1\}$, act on the state space $\mathfrak{H}_M \equiv (\mathbb{C}^2)^{\otimes M}$ and satisfy the commutation relations:

$$[\sigma_n^+, \sigma_m^-] = \sigma_n^z \delta_{nm} \,, \quad [\sigma_n^z, \sigma_m^\pm] = \pm 2\sigma_n^\pm \delta_{nm} \,. \tag{2}$$

The coupling of spins is expressed by $M \times M$ *exchange matrix* $\boldsymbol{\Delta}$:

$$\Delta_{nm} = \delta_{|n-m|,1} + \delta_{|n-m|,M-1} \,, \tag{3}$$

where $\delta_{n,l} (\equiv \delta_{nl})$ is the Kronecker symbol. The periodic boundary conditions $\sigma_{n+M}^\# = \sigma_n^\#$, $\# \in \{\pm, z\}$, are imposed.

Spin "up" and "down" states on n^{th} site, $|\uparrow\rangle_n$ and $|\downarrow\rangle_n$, are defined so, that the rising/lowering operators $\sigma_n^\pm$ act on them as follows:

$$\sigma_n^+ \, |\downarrow\rangle_n = |\uparrow\rangle_n \,, \qquad \sigma_n^- \, |\uparrow\rangle_n = |\downarrow\rangle_n \,, \qquad \sigma_n^- \, |\downarrow\rangle_n = \sigma_n^+ \, |\uparrow\rangle_n = 0 \,. \tag{4}$$

The ferromagnetic state with all spins "up"

$$|\Uparrow\rangle \equiv \bigotimes_{n=0}^{M-1} |\uparrow\rangle_n \equiv \bigotimes_{n=0}^{M-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_n \,, \tag{5}$$

is an eigen-vector of σ_m^z: $\sigma_m^z | \Uparrow\rangle = | \Uparrow\rangle$, and it is chosen as the reference state (i.e., pseudovacuum [3]). The state (5) is annihilated by σ_m^+, $\sigma_m^+ | \Uparrow\rangle = 0$, and therefore, it is annihilated by the Hamiltonian (1):

$$\hat{H} \, |\Uparrow\rangle = 0 \,. \tag{6}$$

The state (5) is normalized $\langle \Uparrow | \Uparrow \rangle = 1$.

Let a set of strictly decreasing integers μ_k, $1 \leqslant k \leqslant N$ to constitute a *strict partition*, i.e., N-tuple $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_N)$, where $M - 1 \geqslant \mu_1 > \mu_2 > \ldots > \mu_N \geqslant 0$. Since the operators $\sigma_n^\pm$ act on the states $|\uparrow\rangle_n$ and

$|\downarrow\rangle_n$ according to (4), we define the state $|\boldsymbol{\mu}\rangle$ corresponding to N *flipped* (i.e., "down") spins on the sites labelled by the "coordinates" μ_k, and the corresponding conjugate state $\langle\boldsymbol{\nu}|$:

$$|\boldsymbol{\mu}\rangle \equiv |\mu_1, \mu_2, \ldots, \mu_N\rangle \equiv \left(\prod_{k=1}^{N} \sigma^-_{\mu_k}\right) |\Uparrow\rangle\,, \tag{7}$$

$$\langle\boldsymbol{\nu}| = \langle\nu_1, \nu_2, \ldots, \nu_N| \equiv \langle\Uparrow| \left(\prod_{k=1}^{N} \sigma^+_{\nu_k}\right)\,, \tag{8}$$

where $|\Uparrow\rangle$ is given by (5). The states (7), (8) provide a complete orthogonal base:

$$\langle\boldsymbol{\nu}|\boldsymbol{\mu}\rangle = \boldsymbol{\delta}_{\boldsymbol{\nu}\boldsymbol{\mu}} \equiv \prod_{n=1}^{N} \delta_{\nu_n \mu_n}\,, \tag{9}$$

and the decomposition of unity is of the form:

$$\begin{aligned} \mathbb{I} = & \sum_{M-1\geqslant\mu_1>\mu_2>\cdots>\mu_N\geqslant 0} |\mu_1, \mu_2, \ldots, \mu_N\rangle\langle\mu_1, \mu_2, \ldots, \mu_N| \\ \equiv & \sum_{\boldsymbol{\mu}_N \subseteq \{(M-1)^N\}} |\boldsymbol{\mu}_N\rangle\langle\boldsymbol{\mu}_N|\;. \end{aligned} \tag{10}$$

§3. Single walker

The continuous temporal evolution of the states, obtained by selective flipping of the spin $\sigma^-_m\,|\Uparrow\rangle$, is defined by the one-particle correlation function

$$G(j, m|t) \equiv \langle\Uparrow|\, \sigma^+_j e^{t\hat{H}} \sigma^-_m\, |\Uparrow\rangle\,. \tag{11}$$

The expansion of correlation function in powers of t gives:

$$G(j, m|t) = \sum_{K=0}^{\infty} \frac{t^K}{K!} \langle\Uparrow|\, \sigma^+_j (\hat{H})^K \sigma^-_m\, |\Uparrow\rangle\,. \tag{12}$$

Applying then the commutation relation

$$[\hat{H}, \sigma^-_m] = \sum_n \Delta_{nm} \sigma^-_n \sigma^z_m = \sigma^-_{m-1}\sigma^z_m + \sigma^-_{m+1}\sigma^z_m\,, \tag{13}$$

one obtains

$$\begin{aligned} \hat{H}^K \sigma^-_m\, |\Uparrow\rangle &= \hat{H}^{K-1} [\hat{H}, \sigma^-_m]\, |\Uparrow\rangle = \hat{H}^{K-1} \sum_{n_1} \Delta_{n_1, m}\, \sigma^-_{n_1}\, |\Uparrow\rangle \\ &= \sum_{n_1, \ldots, n_K} \Delta_{n_K, n_{K-1}} \ldots \Delta_{n_2, n_1} \Delta_{n_1, m}\, \sigma^-_{n_K}\, |\Uparrow\rangle\,. \end{aligned} \tag{14}$$

The multiplication of the equality (14) from the left by the state $\langle\Uparrow|\,\sigma_j^+$ leads to the expression:

$$\langle\Uparrow|\,\sigma_j^+(\hat{H})^K\sigma_m^-\,|\Uparrow\rangle \equiv \mathfrak{G}(j,m|K) = \sum_{n_1,n_2,\ldots,n_{K-1}} \Delta_{j,n_{K-1}}\cdots\Delta_{n_2,n_1}\Delta_{n_1,m} = (\boldsymbol{\Delta}^K)_{jm}. \tag{15}$$

It follows from (12) and (15), that

$$G(j,m|t) = (e^{t\boldsymbol{\Delta}})_{jm}\,. \tag{16}$$

The periodicity requirement implies that

$$G(j+M,m|t) = G(j,m+M|t) = G(j,m|t)\,. \tag{17}$$

Differentiating $G(j,m|t)$ with respect to t and taking into account the definition of the exchange matrix (3), we obtain the difference-differential equation

$$\frac{d}{dt}\,G(j,m|t) = G(j-1,m|t) + G(j+1,m|t) \tag{18}$$

for fixed m, and an analogous equation for fixed j. The initial condition is defined by the equality $G(j,m|0) = \delta_{jm}$.

To connect the model discussed with the theory of random walks, let us notice that moves of the single walker on a chain may be expressed by the matrix $\boldsymbol{\Delta}$ (3). This choice of $\boldsymbol{\Delta}$ means that the walker can step either up or down from an arbitrary m^{th} site. A random lattice path made by a walker is given by a sequence of edges Δ_{ij}, which joins a sequence of lattice sites:

$$\Delta_{j,n_{K-1}}\cdots\Delta_{n_2,n_1}\Delta_{n_1,m}\,. \tag{19}$$

The generating function of all admissible lattice paths of K steps running from m^{th} to j^{th} site is expressed as

$$\sum_{n_1,\ldots,n_{K-1}} \Delta_{j,n_{K-1}}\ldots\Delta_{n_2,n_1}\Delta_{n_1,m}\,, \tag{20}$$

and thus is given by the function $\mathfrak{G}(j,m|K)$ (15). The evaluation of $\mathfrak{G}(j,m|K)$ consists of listing all paths of K steps from m^{th} to j^{th} site.

The one dimensional walks introduced may be considered as random *directed* paths across the integer lattice on a cylinder defined by a sequence of steps of equal lengths. The paths made by a walker consist of forward steps $(1,1)$ and backward steps $(-1,1)$. The path of a length K runs from a site $(m, 0)$ to a site (j, K). The directed paths are defined by the fact, that for each step (x, y) one has $y \geqslant 0$ (see Figure (1)).

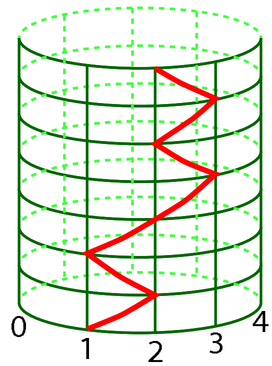

Figure 1. The directed path from $(1, 0)$ to $(2, 7)$.

The exchange matrix $\boldsymbol{\Delta}$ may be expressed in terms of the *circulant matrix*, [9], which is a square matrix of order $M \in \mathbb{N}$ of the form

$$\mathsf{C}_M = \begin{pmatrix} c_0 & c_1 & \dots & c_{M-2} & c_{M-1} \\ c_{M-1} & c_0 & c_1 & \dots & c_{M-2} \\ \vdots & c_{M-1} & c_0 & \ddots & \vdots \\ c_2 & & \ddots & \ddots & c_1 \\ c_1 & c_2 & \dots & c_{M-1} & c_0 \end{pmatrix}. \tag{21}$$

The first row of C_M (21), $(c_0, c_1, \dots, c_{M-1})$, is called the *generator* of C_M. The circulant matrix C_M (21) of order M is usually denoted as $\mathsf{C}_M = \mathrm{Circ}_M(c_0, c_1, \dots, c_{M-1})$. A special matrix

$$\begin{aligned} \mathsf{S}_M &\equiv \mathrm{Circ}_M(0, 0, \dots, 0, 1) \\ &= \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & 1 & 0 & \ddots & \vdots \\ 0 & & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \end{aligned} \tag{22}$$

is known as the *basic circulant permutation matrix*. With regard to the equivalnce $|\,j\rangle \sim |\,j'\rangle$ at $j-j'=0(\mathrm{mod}\,M)$, let us identify a spin "down" state on j^{th} site $|\,j\rangle$ as the "coordinate" column (cf. (7)):

$$|\,j\rangle \sim \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} (0) \\ \vdots \\ (j-1) \\ (j) \\ (j+1) \\ \vdots \\ (M-1) \end{matrix} \quad . \tag{23}$$

An l^{th} power of the basic matrix S_M (22), where $l=k(\mathrm{mod}\,M)$, generates k steps forward, since it acts on the column $|\,j\rangle$ (23) as follows:

$$\bigl(\mathsf{S}_M\bigr)^l\,|\,j\rangle \;=\; |\,j+k\rangle\,, \tag{24}$$

where k^{th} power of S_M is given by

$$\bigl(\mathsf{S}_M\bigr)^k = \mathrm{Circ}_M(0,0,\ldots,\underbrace{1,0,\ldots,0}_{k})\,, \tag{25}$$

and $\bigl(\mathsf{S}_M\bigr)^M$ is, therefore, a unit matrix of M^{th} order. Straightforwardly, the entries $\bigl((\mathsf{S}_M)^k\bigr)_{jm}$, $1\leqslant j,m\leqslant M$ take the form since, $1\leqslant k\leqslant M$:

$$\bigl((\mathsf{S}_M)^k\bigr)_{jm} = \begin{cases} 1\,, & m-j+k=0 \text{ or } M\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{26}$$

The *backward shift permutation matrix* $\bar{\mathsf{S}}_M$ is characterized by the relations:

$$\begin{aligned} &\bar{\mathsf{S}}_M \equiv \mathrm{Circ}_M(0,1,\ldots,0,0) = \bigl(\mathsf{S}_M\bigr)^{M-1} = \mathsf{S}_M^{-1} = \bigl(\mathsf{S}_M\bigr)^{\mathrm{T}}\,, \\ &\bigl(\bar{\mathsf{S}}_M\bigr)^l\,|\,j\rangle \;=\; |\,j-k\rangle\,, \end{aligned} \tag{27}$$

where $l=k(\mathrm{mod}\,M)$. The K^{th} power of the matrix S_M (22) is given by

$$\begin{aligned} \bigl((\mathsf{S}_M)^K\bigr)_{jm} &= \langle j|\bigl(\mathsf{S}_M\bigr)^K|m\rangle = \langle j|\bigl(\mathsf{S}_M\bigr)^K|j+\mathsf{sign}(m-j)|m-j|\rangle \\ &= \langle j|\bigl(\mathsf{S}_M\bigr)^{K'+\mathsf{sign}(m-j)|m-j|}|j\rangle\,, \end{aligned} \tag{28}$$

where $K' - K = 0(\mathrm{mod} M)$. Since $\langle\, j|m\rangle = \delta_{jm}$, it follows from (24), (27), (28) that

$$\big((\mathsf{S}_M)^K\big)_{jm} = \begin{cases} 1\,, & m - j + K = 0(\mathrm{mod} M)\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{29}$$

The exchange matrix $\boldsymbol{\Delta}$ is expressed in terms of the circulant matrices:

$$\boldsymbol{\Delta} = \mathsf{S}_M + \mathsf{S}_M^{-1}\,, \qquad \boldsymbol{\Delta}^{\mathrm{T}} = \boldsymbol{\Delta}\,, \tag{30}$$

and the K^{th} power of $\boldsymbol{\Delta}$ is given by the expression

$$\boldsymbol{\Delta}^K = \big(\mathsf{S}_M + \mathsf{S}_M^{-1}\big)^K = \sum_{s=0}^{K} \binom{K}{s} \big(\mathsf{S}_M\big)^{K-2s}\,. \tag{31}$$

The matrix $\boldsymbol{\Delta}$ is invariant under transposition, and non-zero entries of $\boldsymbol{\Delta}^K$ are given by $K - |m-j| = 0(\mathrm{mod} 2)$. From (31) we obtain:

$$(\boldsymbol{\Delta}^K)_{jm} = \sum_{s=0}^{K} \binom{K}{s} \langle j| \big(\mathsf{S}_M\big)^{2s-K} |m\rangle\,. \tag{32}$$

Right-hand side of (32) is non-trivial at $K - 2s + j - m = 0(\mathrm{mod} M)$. Let us assume that $L \equiv \frac{K-|m-j|\pm pM}{2}$ is chosen so that $0 \leqslant L < \frac{M}{2}$ and $p \in \mathbb{N}$. Then, (32) may be expressed, due to (29), as a lacunary sum [11]

$$(\boldsymbol{\Delta}^K)_{jm} = \binom{K}{L}_{M/2} \equiv \binom{K}{L} + \binom{K}{L + \frac{M}{2}\cdot 1} + \binom{K}{L + \frac{M}{2}\cdot 2} + \dots\,, \tag{33}$$

which is clearly truncated. The answer for the plane arises at $M \to \infty$ in (33): $(\boldsymbol{\Delta}^K)_{jm} = \binom{K}{\frac{K-|m-j|}{2}}$.

The matrix $\boldsymbol{\Delta}$ is diagonalized by the unitary matrix $\mathcal{U}$ with the entries

$$\mathcal{U}_{kj} = \frac{1}{\sqrt{M}} e^{i\frac{2\pi}{M}kj}\,, \quad k, j \in \mathcal{E}\,,$$

as follows:

$$\mathbf{E} = \mathcal{U}\,\boldsymbol{\Delta}\,\mathcal{U}^{\dagger}\,, \quad \mathbf{E} \equiv 2 \operatorname*{diag}_{0\leqslant m\leqslant M-1} \Big\{\cos\frac{2\pi m}{M}\Big\}\,. \tag{34}$$

Inverting (34) one obtains the entries of K^{th} power of the exchange matrix $\boldsymbol{\Delta}$:

$$(\boldsymbol{\Delta}^K)_{jm} = \frac{2^K}{M} \sum_{l=0}^{M-1} \cos^K\Big(\frac{2\pi l}{M}\Big) e^{i\frac{2\pi l}{M}(m-j)}\,. \tag{35}$$

Respectively we obtain for the one-particle correlation function (16):

$$G(j,m|t) = (e^{t\boldsymbol{\Delta}})_{jm} = \frac{1}{M}\sum_{l=0}^{M-1} e^{2t\cos(\frac{2\pi l}{M})}\, e^{i\frac{2\pi l}{M}(m-j)}\,. \tag{36}$$

Comparison of (33) and (35) gives the identity known as the *Ramus identity* [12]:

$$\frac{2^K}{M}\sum_{l=0}^{M-1}\cos^K\Big(\frac{2\pi l}{M}\Big)e^{i\frac{2\pi l}{M}(m-j)} \;=\; \binom{K}{L}_{M/2}\,, \tag{37}$$

where $0 \leqslant L \equiv \frac{K-|m-j|\pm pM}{2} < \frac{M}{2}$ at some $p \in \mathbb{N}$, and the lacunary sum in right-hand side is defined in (33).

§4. Random turns walkers

Consider now the average

$$\begin{aligned} &\mathfrak{G}(j_1, j_2, \dots, j_N;\, m_1, m_2, \dots, m_N|K) \\ &\quad\equiv \langle \Uparrow|\, \sigma^+_{j_1}\sigma^+_{j_2}\cdots\sigma^+_{j_N}\,(\hat{H})^K\sigma^-_{m_1}\sigma^-_{m_2}\cdots\sigma^-_{m_N}\,|\Uparrow\rangle \\ &\quad= \langle j_1, j_2, \dots, j_N|(\hat{H})^K|m_1, m_2\dots, m_N\rangle\,, \end{aligned} \tag{38}$$

or, using (7), (8),

$$\mathfrak{G}(\mathbf{j};\mathbf{m}|K) \;\equiv\; \langle \mathbf{j}|(\hat{H})^K|\mathbf{m}\rangle\,. \tag{39}$$

We shall consider the case, when $N \leqslant M/2$. Applying the commutation relation

$$[\hat{H}, \sigma^-_{l_1}\sigma^-_{l_2}\dots\sigma^-_{l_N}] = \sum_{k=1}^{N}\sigma^-_{l_1}\dots\sigma^-_{l_{k-1}}[\hat{H},\sigma^-_{l_k}]\sigma^-_{l_{k+1}}\dots\sigma^-_{l_N}\,, \tag{40}$$

we see that in each term the Hamiltonian acts on a single flipped spin only and moves it (if it is allowed by neighbours) forward (step $(1,1)$) or backward (step $(-1,1)$), while the remaining flipped spins keep their places (steps $(0,1)$). It means that the considered average gives the number of configurations traced by N random turns walkers being initially located on the lattice sites $m_1 > m_2 > \cdots > m_N$ and arrived after K steps at the positions $j_1 > j_2 > \cdots > j_N$. The *vicious walkers* condition, i.e., the condition that walkers do not touch each other up to K steps, is guaranteed by the property of the Pauli matrices $(\sigma^{\pm}_k)^2 = 0$. The number of forward and backward steps in the configuration is equal to K. The configuration of paths in Figure (2) may be expressed in terms of the matrix $\boldsymbol{\Delta}$:

$$(\Delta_{m_1,m_1+1}\Delta_{m_1+1,m_1+2}\Delta_{m_1+2,j_1})(\Delta_{m_2,m_2-1}\Delta_{m_2-1,m_2}\Delta_{m_2,j_2})$$
$$(\Delta_{m_3,m_3+1}\Delta_{m_3+1,j_3})\Delta_{m_4,j_4}\,.$$

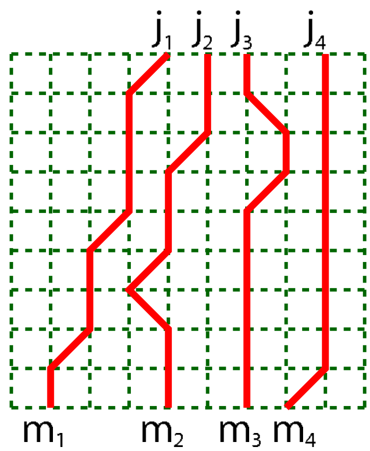

Figure 2. Random turns walk.

Consider now the multi-particles correlation function

$$G(j_1,j_2,\dots,j_N;m_1,m_2,\dots,m_N|t)=\langle j_1,j_2,\dots,j_N|e^{t\hat{H}}|m_1,m_2,\dots,m_N\rangle, \tag{41}$$

or

$$G_{\mathbf{j};\mathbf{m}}(t)\equiv\langle\mathbf{j}|\,e^{t\hat{H}}\,|\,\mathbf{m}\rangle\,. \tag{42}$$

This function is considered to be the *exponential generating function of the random turns walkers*. Differentiating (41), (42) by t and applying (40), we shall obtain the equation for the multi-particles correlation function

$$\begin{aligned}\frac{d}{dt}\,G(j_1,j_2,\dots,j_N;\,m_1,m_2,\dots,m_N|t)\\ =\sum_{k=1}^{N}G(j_1,j_2,\dots,j_k+1,\dots,j_N;\,m_1,m_2,\dots,m_N|t)\\ +\sum_{k=1}^{N}G(j_1,j_2,\dots,j_k-1,\dots,j_N;\,m_1,m_2,\dots,m_N|t)\,,\end{aligned}\tag{43}$$

where the indices m_k are fixed (and a similar equation for j_k fixed). The non-intersection condition means that

$$G(j_1,j_2,\dots,j_N;\,m_1,m_2,\dots,m_N|t)=0\,,$$

if $m_k=m_p$, $j_k=j_p$ for any $0\leqslant j,m\leqslant M-1$. The function G should be periodic in each j_k, m_k with all other j_s, m_s $(k\neq s)$ fixed:

$$\begin{aligned}G(j_1,j_2,\dots,j_k+M,\dots,j_N;\,m_1,m_2,\dots,m_N|t)\\ =G(j_1,j_2,\dots,j_N;\,m_1,m_2,\dots,j_k+M,\dots,m_N|t)\\ =G(j_1,j_2,\dots,j_N;\,m_1,m_2,\dots,m_N|t)\,.\end{aligned}$$

Equation (43) is supplied with the “initial” condition

$$G(j_1,j_2,\dots,j_N;\,m_1,m_2,\dots,m_N|0)=\prod_{m=1}^{N}\delta_{j_m,l_m}\,.$$

It is easy to verify that the solution of Eq. (43) may be represented in the determinantal form:

$$\begin{aligned}G(j_1,j_2,\dots,j_N;\,m_1,m_2,\dots,m_N|t)\\ =\det\big(G(j_r,m_s|t)\big)_{r,s=1,2,\dots,N}=\det\big((e^{t\boldsymbol{\Delta}})_{j_r,m_s}\big)_{r,s=1,2,\dots,N}\,,\end{aligned}\tag{44}$$

where $G(j,m|t)$ is given by (11). Expanding the determinant in right-hand side of (44) in powers of $\boldsymbol{\Delta}$, we obtain:

$$\begin{aligned}\det\big((e^{t\boldsymbol{\Delta}})_{j_r,m_s}\big)_{r,s=1,2,\dots,N}\\ =\sum_{\sigma(s_1,\dots,s_N)}(-1)^{\epsilon(\sigma)}(e^{t\boldsymbol{\Delta}})_{j_1,m_{s_1}}(e^{t\boldsymbol{\Delta}})_{j_2,m_{s_2}}\cdots(e^{t\boldsymbol{\Delta}})_{j_N,m_{s_N}}\,,\end{aligned}\tag{45}$$

where the sum is over all permutations of the set $\{s_1,\dots,s_N\}$, and $\epsilon(\sigma)$ is the parity of permutation. Differentiating this expression by t and taking into account the definition (3), we find that (44) is the solution of Eq. (43). The determinantal form of the solution guarantees the non-intersection of the paths.

To find the number of the lattice paths made by N random turns walkers in K steps defined by the average (38), we have to expand (45) in powers of t:

$$\det\big((e^{t\boldsymbol{\Delta}})_{j_r,m_s}\big)_{r,s=1,2,\dots,N} = \sum_{k_1,k_2,\dots,k_N=0}^{\infty} \frac{t^{k_1+k_2+\ldots+k_N}}{k_1!k_2!\ldots k_N!} \sum_{\sigma(s_1,s_2,\dots,s_N)} (-1)^{\epsilon(\sigma)} (\boldsymbol{\Delta}^{k_1})_{j_1,m_{s_1}} \times (\boldsymbol{\Delta}^{k_2})_{j_2,m_{s_2}} \cdots (\boldsymbol{\Delta}^{k_N})_{j_N,m_{s_N}}\,. \quad (46)$$

Gathering the terms at $t^K/K!$, we obtain:

$$\mathfrak{G}(j_1,j_2,\dots,j_N;\, m_1,m_2,\dots,m_N|K) = \sum_{|\mathbf{k}|=K} P(\mathbf{k}) \det\big((\boldsymbol{\Delta}^{k_r})_{j_r,m_s}\big)_{r,s=1,2,\dots,N}\,, \quad (47)$$

where $|\mathbf{k}| \equiv k_1 + k_2 + \ldots + k_N$, and $P(\mathbf{k})$ is the multinomial coefficient:

$$P(\mathbf{k}) \equiv \frac{(k_1+k_2+\ldots+k_N)!}{k_1!k_2!\ldots k_N!}\,.$$

The other representation of the exponential generating function may be obtained by substitution of expression (36) into the solution (44):

$$G(j_1,j_2,\dots,j_N;\, m_1,m_2,\dots,m_N|t) = \frac{1}{M^N N!} \sum_{k_1,k_2,\dots,k_N=0}^{M-1} e^{2t\sum\limits_{n=1}^{N}\cos\frac{2\pi k_n}{M}} \det\big(e^{i\frac{2\pi k_r}{M}(j_r-m_s)}\big)_{r,s=1,2,\dots,N}\,. \quad (48)$$

The antisymmetry of the summand with respect to permutations enables to transform $\det\big(e^{i\frac{2\pi k_r}{M}(j_r-m_k)}\big)$ in (48) into the product of the determinants $\det\big(e^{i\frac{2\pi k_r}{M}j_r}\big)$ and $\det\big(e^{i\frac{2\pi k_r}{M}m_s}\big)$, so that right-hand side of (48) can be expressed in terms of the Schur functions:

$$G(j_1,j_2,\dots,j_N;\, m_1,m_2,\dots,m_N|t) = \sum_{M-1\geqslant k_1>k_2>\ldots>k_N\geqslant 0} \frac{e^{2t\sum_{n=1}^{N}\cos\phi_{k_n}}}{\mathcal{N}^2(\boldsymbol{\phi}_N)} S_{\boldsymbol{\lambda}^{\mathbf{m}}}(e^{i\boldsymbol{\phi}_N}) S_{\boldsymbol{\lambda}^{\mathbf{j}}}(e^{-i\boldsymbol{\phi}_N})\,, \quad (49)$$

where

$$\mathcal{N}^{-2}(\boldsymbol{\phi}_N) \equiv M^{-N} \prod_{1\leqslant s<t\leqslant N} 2\bigl(1-\cos(\phi_{k_t}-\phi_{k_s})\bigr)\,. \tag{50}$$

The notations in (49) and (50) are:

$$\begin{aligned} e^{i\boldsymbol{\phi}_N} &\equiv (e^{i\phi_{k_1}}, e^{i\phi_{k_2}}, \ldots, e^{i\phi_{k_N}})\,, \\ \boldsymbol{\phi}_N &\equiv (\phi_{k_1}, \phi_{k_2}, \ldots, \phi_{k_N})\,, \quad \phi_{k_l} \equiv \frac{2\pi k_l}{M}\,. \end{aligned} \tag{51}$$

The *Schur function* $S_{\boldsymbol{\lambda}}$ in (49) is defined by the Jacobi–Trudi relation, [5]:

$$S_{\boldsymbol{\lambda}}(\mathbf{x}_N) \;\equiv\; S_{\boldsymbol{\lambda}}(x_1, x_2, \ldots, x_N) \;\equiv\; \frac{\det(x_j^{\lambda_k+N-k})_{1\leqslant j,k\leqslant N}}{\mathcal{V}(\mathbf{x}_N)}\,, \tag{52}$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ is the partition $M-N \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_N \geqslant 0$, and $\mathcal{V}(\mathbf{x}_N)$ is the Vandermonde determinant

$$\mathcal{V}(\mathbf{x}_N) \equiv \det(x_j^{N-k})_{1\leqslant j,k\leqslant N} \;=\; \prod_{1\leqslant m<l\leqslant N} (x_l - x_m)\,. \tag{53}$$

The partitions $\boldsymbol{\lambda}^{\mathbf{m}}$ and $\boldsymbol{\lambda}^{\mathbf{j}}$ in (49) are related to the starting and final positions of the walkers $m_1 > m_2 > \ldots > m_N$ and $j_1 > j_2 > \ldots > j_N$: $(\lambda^{\mathbf{m}})_k \equiv m_k - N + k$, $(\lambda^{\mathbf{j}})_k \equiv j_k - N + k$.

The Schur functions are equal to unity, $S_{\boldsymbol{\lambda}^{\mathbf{m}}}(e^{i\boldsymbol{\phi}_N}) = S_{\boldsymbol{\lambda}^{\mathbf{j}}}(e^{-i\boldsymbol{\phi}_N}) = 1$, provided that $\mathbf{m} = \mathbf{j} = \boldsymbol{\delta}_N$, where $\boldsymbol{\delta}_N$ is the "staircase" partition

$$\boldsymbol{\delta}_N \equiv (N-1, N-2, \ldots, 1, 0)\,. \tag{54}$$

Then, Eq. (49) provides the representation for the determinant:

$$\begin{aligned} \det\bigl((e^{t\boldsymbol{\Delta}})_{j_r,m_s}\bigr)_{r,s=1,2,\ldots,N} = M^{-N} &\sum_{M-1\geqslant k_1>k_2>\ldots>k_N\geqslant 0} e^{2t\sum_{n=1}^{N}\cos\frac{2\pi k_n}{M}} \\ &\times \prod_{1\leqslant s<t\leqslant N} 2\bigl(1-\cos(\phi_{k_t}-\phi_{k_s})\bigr)\,. \end{aligned} \tag{55}$$

The use of (47) and (55) leads to the following.

Proposition. *The Ramus's identity* (37) *admits the determinantal generalization*:

$$\sum_{|\mathbf{k}|=K} P(\mathbf{k}) \det\big((\boldsymbol{\Delta}^{k_r})_{r-1,s-1}\big)_{r,s=1,2,\ldots,N} = \frac{2^K}{M^N} \sum_{M-1\geqslant k_1>k_2>\ldots>k_N\geqslant 0} \times \Big(\sum_{n=1}^{N} \cos\frac{2\pi k_n}{M}\Big)^K \prod_{1\leqslant s<t\leqslant N} 2\big(1-\cos(\phi_{k_t}-\phi_{k_s})\big)\,, \quad (56)$$

where left-hand side is expressed by means of the entries of powers of the circulant matrix $\boldsymbol{\Delta}$ (3)*, which are given by Eq.* (33).

Let us consider the *centroid* operator

$$\hat{Q} = \frac{1}{2N} \sum_{j=1}^{M-1} j(1-\sigma_j^z)\,, \quad (57)$$

which acts on N-particle state (7) as

$$\hat{Q}|\boldsymbol{\mu}\rangle \equiv \hat{Q}|\mu_1, \mu_2, \ldots, \mu_N\rangle = \frac{|\boldsymbol{\mu}|}{N}|\boldsymbol{\mu}\rangle\,, \quad (58)$$

where $\frac{|\boldsymbol{\mu}|}{N} = \frac{1}{N}\sum_{j=1}^{N} \mu_j$ may be considered as the arithmetic mean of positions of N random turns walkers.

In order to study the time evolution of the exponential $e^{-h\hat{Q}}$ of the centroid operator $\hat{Q}$, we shall obtain the "two-time" correlation function:

$$\begin{aligned} &F(j_1, j_2, \ldots, j_N;\, m_1, m_2, \ldots, m_N|\, t_1, t_2, h) \\ &\quad\equiv \langle\Uparrow|\, \sigma_{j_1}^+ \sigma_{j_2}^+ \cdots \sigma_{j_N}^+ e^{t_1\hat{H}} e^{-h\hat{Q}} e^{t_2\hat{H}} \sigma_{m_1}^- \sigma_{m_2}^- \cdots \sigma_{m_N}^- \,|\Uparrow\rangle \\ &\quad= \langle j_1, j_2, \ldots, j_N \mid e^{t_1\hat{H}} e^{-h\hat{Q}} e^{t_2\hat{H}} \mid m_1, m_2, \ldots, m_N\rangle\,, \end{aligned} \quad (59)$$

or using (7), (8)

$$F_{\mathbf{j};\mathbf{m}}(t_1, t_2, h) \equiv \langle\mathbf{j}| e^{t_1\hat{H}} e^{-h\hat{Q}} e^{t_2\hat{H}} |\mathbf{m}\rangle\,. \quad (60)$$

Our main result is the following.

Theorem. *Explicit expression for* $F_{\mathbf{j};\mathbf{m}}(t_1, t_2, h)$ (59), (60) *is given by the representation*:

$$F_{\mathbf{j};\mathbf{m}}(t_1, t_2, h) = e^{-h\frac{N-1}{2}}(1 - e^{-\frac{hM}{N}})^N \sum_{M-1 \geqslant k_1 > k_2 > \ldots > k_N \geqslant 0}$$
$$\times \sum_{M-1 \geqslant p_1 > p_2 > \ldots > p_N \geqslant 0} \frac{e^{2t_1 \sum_{n=1}^{N} \cos\phi_{k_n} + 2t_2 \sum_{n=1}^{N} \cos\chi_{p_n}}}{\mathcal{N}^2(\boldsymbol{\phi}_N)\,\mathcal{N}^2(\boldsymbol{\chi}_N)}$$
$$\times S_{\boldsymbol{\lambda}^{\mathbf{m}}}(e^{-i\boldsymbol{\chi}_N})\, S_{\boldsymbol{\lambda}^{\mathbf{j}}}(e^{i\boldsymbol{\phi}_N}) \prod_{1 \leqslant m \leqslant j \leqslant N} \frac{1}{1 - e^{-h/N} e^{i(\chi_{p_j} - \phi_{k_m})}}\,, \quad (61)$$

where the notations given by (51) *are used together with the analogous notations*

$$e^{i\boldsymbol{\chi}_N} \equiv (e^{i\chi_{p_1}}, e^{i\chi_{p_2}}, \ldots, e^{i\chi_{p_N}})\,,$$
$$\boldsymbol{\chi}_N \equiv (\chi_{p_1}, \chi_{p_2}, \ldots, \chi_{p_N})\,, \quad \chi_{p_l} \equiv \frac{2\pi p_l}{M}\,. \quad (62)$$

Besides, $\mathcal{N}^2(\boldsymbol{\phi}_N)$ *and* $\mathcal{N}^2(\boldsymbol{\chi}_N)$ *in* (61) *are given by* (50). *Further,*

$$\mathcal{V}^{-1}(e^{-i\boldsymbol{\phi}_N}) = \prod_{1 \leqslant m < l \leqslant N} (e^{-i\phi_{k_l}} - e^{-i\phi_{k_m}})^{-1}\,,$$
$$\mathcal{V}^{-1}(e^{-h/N} e^{i\boldsymbol{\chi}_N}) = \prod_{1 \leqslant r < s \leqslant N} e^{\frac{h}{N}} (e^{i\chi_{p_s}} - e^{i\chi_{p_r}})^{-1}\,,$$

and $|\boldsymbol{\mu}| \equiv \sum_{j=1}^{N} \mu_j = |\boldsymbol{\lambda}| + \frac{N(N-1)}{2}$ *are taken into account.*

Proof. First of all, let us insert the resolution of the identity operator (10) into (59), (60) and obtain:

$$F(j_1, j_2, \ldots, j_N; m_1, m_2, \ldots, m_N | t_1, t_2, h)$$
$$= \sum_{M-1 \geqslant \mu_1 > \mu_2 > \cdots > \mu_N \geqslant 0} e^{-h\frac{|\boldsymbol{\mu}|}{N}} G(j_1, j_2, \ldots; \mu_1, \mu_2, \ldots, \mu_N | t_1)$$
$$\times\, G(\mu_1, \mu_2, \ldots, \mu_N; m_1, m_2, \ldots, m_N | t_2)\,, \quad (63)$$

or, using (7), (8),

$$F_{\mathbf{j};\mathbf{m}}(t_1, t_2, h) = \sum_{\boldsymbol{\mu}_N \subseteq \{(M-1)^N\}} e^{-h\frac{|\boldsymbol{\mu}|}{N}} G_{\mathbf{j};\boldsymbol{\mu}}(t_1)\, G_{\boldsymbol{\mu};\mathbf{m}}(t_2)\,, \quad (64)$$

where the summation is defined by (10), and (58) is taken into account. The correlation functions $G_{\mathbf{j};\boldsymbol{\mu}}(t_1)$ and $G_{\boldsymbol{\mu};\mathbf{m}}(t_2)$ are defined by (42).

To calculate (63) and (64) we need the Binet–Cauchy formula expressed in terms of the Schur functions, [5]:

$$\mathcal{P}_M(x_1, x_2, \dots, x_N;\, y_1, y_2, \dots, y_N) \equiv \sum_{\boldsymbol{\lambda} \subseteq \{(M-N)^N\}} S_{\boldsymbol{\lambda}}(x_1, x_2, \dots, x_N)\, S_{\boldsymbol{\lambda}}(y_1, y_2, \dots, y_N)$$

$$= \prod_{1 \leqslant m < l \leqslant N} (x_l - x_m)^{-1} \prod_{1 \leqslant p < s \leqslant N} (y_s - y_p)^{-1} \det \left(\frac{1 - (x_k y_j)^M}{1 - x_k y_j} \right)_{j,k=1,2,\dots,N}, \tag{65}$$

where the summation is over all partitions $\boldsymbol{\lambda}$ satisfying: $M - N \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_N \geqslant 0$. The Schur function is a homogeneous function:

$$\alpha^{|\boldsymbol{\lambda}|} S_{\boldsymbol{\lambda}}(x_1, x_2, \dots, x_N) = S_{\boldsymbol{\lambda}}(\alpha x_1, \alpha x_2, \dots, \alpha x_N)\,, \tag{66}$$

where $|\boldsymbol{\lambda}| = \lambda_1 + \lambda_2 + \ldots + \lambda_N$ is the weight of the partition $\boldsymbol{\lambda}$.

After substitution of (49) into (63) and (64) we use (65) and (66) and obtain:

$$F_{\mathbf{j};\mathbf{m}}(t_1, t_2, h) = \sum_{M-1 \geqslant k_1 > k_2 > \ldots > k_N \geqslant 0} \; \sum_{M-1 \geqslant p_1 > p_2 > \ldots > p_N \geqslant 0} \frac{e^{2t_1 \sum_{n=1}^N \cos \phi_{k_n} + 2t_2 \sum_{n=1}^N \cos \chi_{p_n}}}{\mathcal{N}^2(\boldsymbol{\phi}_N) \mathcal{N}^2(\boldsymbol{\chi}_N)}$$
$$\times\; e^{-h \frac{N-1}{2}} \frac{S_{\boldsymbol{\lambda}^{\mathbf{m}}}(e^{-i\boldsymbol{\chi}_N})\, S_{\boldsymbol{\lambda}^{\mathbf{j}}}(e^{i\boldsymbol{\phi}_N})}{\mathcal{V}(e^{-i\boldsymbol{\phi}_N}) \mathcal{V}(e^{-h/N} e^{i\boldsymbol{\chi}_N})}$$
$$\times\; \det \left(\frac{1 - e^{-hM/N}}{1 - e^{-h/N} e^{i(\chi_{p_j} - \phi_{k_m})}} \right)_{j,m=1,2,\dots,N}, \tag{67}$$

where the notations (50), (51), and (62) are used. The Cauchy determinant in (67) may be calculated, and the final result (61) arises. □

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