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RELATIVE CENTRALISERS OF RELATIVE SUBGROUPS

Abstract. Let R be an associative ring with 1, $G = \mathrm{GL}(n, R)$ be the general linear group of degree $n \geqslant 3$ over R. In this paper we calculate the relative centralisers of the relative elementary subgroups or the principal congruence subgroups, corresponding to an ideal $A \trianglelefteq R$ modulo the relative elementary subgroups or the principal congruence subgroups, corresponding to another ideal $B \trianglelefteq R$. Modulo congruence subgroups the results are essentially easy exercises in linear algebra. But modulo the elementary subgroups they turned out to be quite tricky, and we could get definitive answers only over commutative rings, or, in some cases, only over Dedekind rings/Dedekind rings of arithmetic type. Bibliography: 43 titles.

§1. Introduction

Let F, $H \leqslant G$ be two subgroups of G. We consider the centraliser of F modulo H

$$C_G(F, H) = \big\{g \in G \mid \forall f \in F, [f, g] \in H\big\}.$$

If $H \trianglelefteq G$ is a normal subgroup, and $\pi_H : G \longrightarrow G/H$ is the corresponding projection, then

$$C_G(F, H) = \pi_H^{-1}(C_{G/H}(FH/H))$$

is the preimage of the corresponding absolute centraliser in the factor-group G/H.

In the present paper, we are interested in the case of the general linear group $G = \mathrm{GL}(n, R)$ of degree $n \geqslant 3$ over an associative ring R with 1. In connection with our project on subgroups normalised by unrelative and relative elementary subgroups $E(n, J)$ and $E(n, R, J)$ (see [29] for further references), we had to compute the centralisers of relative subgroups modulo some other relative subgroups.

Key words and phrases: General linear groups, elementary subgroups, congruence subgroups, standard commutator formula, unrelativised commutator formula, elementary generators.

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We could not find the corresponding results in the available literature. In fact, in many cases such similar centralisers were extensively studied, starting with Bass' classical results on the structure of $\mathrm{GL}(n,R)$ in the stable range [5]. However, Hyman Bass himself and his followers only considered the absolute case, where the subgroups F was either perfect itself, or contained a perfect subgroup of the same level.

Zenon Borewicz, the first author, and their schools, have performed diverse calculations in this spirit, in connection with structure theory, and description of various classes of intermediate subgroups, see, for instance, [1, 8, 30–35], etc. However, in most of these calculations the subgroup F was perfect as well.

Here, we calculate some of these centralisers for the case where $G = \mathrm{GL}(n,R)$, whereas *both* F and H are various relative subgroups of G, corresponding to proper ideals, such as $E(n,I)$, $E(n,R,I)$, $\mathrm{GL}(n,R,I)$, or the like. None of these groups is anywhere close to being perfect, so that our calculations here are quite different in spirit from the calculations in the above papers.

For the case of congruence subgroups the corresponding results are mostly exercises in linear algebra, and hold over arbitrary associative rings. But for elementary subgroups the answers crucially depend on difficult results of commutator calculus developed in our joint works with Roozbeh Hazrat and Alexei Stepanov, and then recently by ourselves, and only hold in modified forms, or under miscellaneous assumptions.

The paper is organised as follows. In §2 we recall some ideal arithmetic, and in §3 we collect the definitions of various relative subgroups. After that in §4 we prove our first main result, Theorem 1, which calculates $C_{GL(n,R)}(E(n,A),\mathrm{GL}(n,R,B))$, over arbitrary associative rings. In §5 we recall the requisite facts on generation of relative elementary subgroups and their commutators. After that, in §6 we explore what can be done in this spirit for $C_{GL(n,R)}(E(n,A),E(n,R,B))$, over commutative rings, and prove our second main result, Theorem 2. In particular, it gives the definitive answer for Dedekind rings of arithmetic type, Theorem 3. Finally, in §7 we state some further unsolved problems.

§2. Ideal quotient

Let, as above, R be an associative but not necessarily commutative ring with 1. The results of this paper heavily depend on the operations on ideals of R. For two-sided ideals $A, B \trianglelefteq R$ their sum $A+B$, their intersection

$A \cap B$, their products AB and BA, their symmetrised product $A \circ B = AB + BA$, and their commutator $[A, B]$ are again two sided ideals, and their properties are classically known. However, in the non-commutative case we could not find an authoritative source on ideal quotient $(B : A)$, so in this section we collect the some basic facts used in the sequel.

For two *left* ideals A and B in R we consider their *right ideal quotient*

$$BA^{-1} = \{x \in R \mid xA \subseteq B\}.$$

Obviously, it is a two sided ideal of R. Indeed, for any $x \in BA^{-1}$, $y, z \in R$ one has $(xy)A = x(yA) \leqslant xA \leqslant B$ and $(zx)A = z(xA) \leqslant zB \leqslant B$.

Similarly, for two *right* ideals A and B their *left ideal quotient*

$$A^{-1}B = \{x \in R \mid Ax \subseteq B\},$$

is a two sided ideal of R.

Warning. In many texts the *right* ideal quotient BA^{-1} of two left ideals A and B is called *left* ideal quotient, and is denoted by $(B : A)_L$ or $(B :_L A)$, see [25], for instance. The most amazing notational convention is adopted in [9]. There, the *right* ideal quotient BA^{-1} is denoted by ${}_R(B : A)$ – and is still called *left* ideal quotient. Our notation follows that of [20, 24].

Now, for two sided ideals A and B of R their *ideal quotient* is defined as

$$(B : A) = BA^{-1} \cap A^{-1}B = \{x \in R \mid xA, Ax \subseteq B\},$$

Clearly, $(B : A)$ is a two sided ideal such that $(B : A) \geqslant B$, and $A \leqslant B$ implies that $(B : A) = R$. In particular $(A : A) = R$ and $(A : R) = A$. For commutative rings $(B : A) = BA^{-1} = A^{-1}B$ coincides with the usual ideal quotient in commutative algebra.

Let us list some obvious properties of the ideal quotient.

- Clearly,

$$(B : A) \circ A = A(B : A) + (B : A)A \leqslant A(A^{-1}B) + (BA^{-1})A \leqslant B,$$

thus, $(B : A)$ can be defined as the largest two sided ideal $C \trianglelefteq R$ such that $C \circ A \leqslant B$. However, only very rarely this inclusion becomes an equality.

Warning. The ideal quotient is not a fractional ideal, and even when $A \geqslant B$ the ideal quotient $(B : A)$ should not be interpreted as the *fraction* of B by A. In fact, among commutative domains the equality $A(B : A) = B$ characterises Dedekind domains, in this case $(B : A)$ is indeed BA^{-1} in the group of fractional ideals of R. The same condition imposed on finitely generated ideals characterises Prüfer domains.

- $(A : (B + C)) = (A : B) \cap (A : C)$.

Clearly, this equality implies that $(A : (A + B)) = (A : A) \cap (A : B) = (A : B)$. In other words, every ideal quotient $(A : B)$ coincides with such an ideal quotient that $A \leqslant B$.

- $((A \cap B) : C) = (A : C) \cap (B : C)$.
- Intersection of any two of the ideals

$$((A : B) : C), \qquad (A : (B \circ C)), \qquad ((A : C) : B)$$

is contained in the third one. In particular, when R is commutative, one has

$$((A : B) : C) = (A : (BC)) = ((A : C) : B).$$

- $((A + B) : C) \geqslant (A : C) + (B : C)$.
- $(A : (B \cap C)) \geqslant (A : B) + (A : C)$.

Warning. Hardly ever these last inequalities become equalities. Again, among commutative domains any of the equalities

$$((A + B) : C) = (A : C) + (B : C) \quad \text{or} \quad (A : (B \cap C)) = (A : B) + (A : C)$$

characterises Dedekind domains. Any of these conditions imposed on finitely generated ideals characterises Prüfer domains.

Now, let $Z \subseteq R$ be a subset. Its centraliser

$$\mathrm{Cent}_R(Z) = \{x \in R \mid \forall z \in Z, xz - zx = 0\}$$

is a unital subring of R containing the centre $\mathrm{Cent}(R) = \mathrm{Cent}_R(R)$. In the next section we also encounter the *relative* centraliser of a subset Z modulo an ideal B:

$$\mathrm{Cent}_R(Z, B) = \{x \in R \mid \forall z \in Z, xz - zx \in B\}.$$

Clearly,

$$\mathrm{Cent}_R(Z, B) = \rho_B^{-1}\big(\,\mathrm{Cent}_{R/B}\left(\rho_B(Z)\right)\big),$$

is a unital subring of R containing both the absolute centraliser $\mathrm{Cent}_R(Z)$, and the ideal B itself, $\mathrm{Cent}_R(Z) + B \leqslant \mathrm{Cent}_R(Z, B)$.

Mostly, we consider relative centralisers of ideals. Let $Z = A \trianglelefteq R$. Then $\rho_B(A) = (A + B)/B$ and for $x \in (B : A)$ and $a \in A$ one has $xa, ax \in B$, so that in fact even

$$\mathrm{Cent}_R(A) + (B : A) \leqslant \mathrm{Cent}_R(A, B).$$

§3. Relative subgroups

For two subgroups $F, H \leqslant G$, we denote by $[F, H]$ their mutual commutator subgroup spanned by all commutators $[f, h]$, where $f \in F$, $h \in H$. Observe that our commutators are always left-normed, $[x, y] = xyx^{-1}y^{-1}$. The double commutator $[[x, y], z]$ will be denoted simply by $[x, y, z]$. As usual, $C(G)$ denotes the centre of a group G, whereas $C_{\mathrm{GL}(n,R)}(H)$ denotes the centraliser of a subgroup $H \leqslant G$ in G.

As usual, e denotes the identity matrix and e_{ij} is a standard matrix unit. For $\xi \in R$ and $1 \leqslant i \neq j \leqslant n$, we denote by $t_{ij}(\xi) = e + \xi e_{ij}$, we denote the corresponding [elementary] transvection. To any ideal $I \trianglelefteq R$ one associates the elementary subgroup

$$E(n, I) = \big\langle t_{ij}(\xi),\ \xi \in I,\ 1 \leqslant i \neq j \leqslant n \big\rangle,$$

generated by all elementary transvections of level I, and the *relative* elementary subgroup $E_I = E(n, R, I)$ of level I is defined as the normal closure of $E(n, I)$ in the *absolute* elementary subgroup $E = E(n, R)$.

Further, consider the reduction homomorphism

$$\rho_I : \mathrm{GL}(n, R) \longrightarrow \mathrm{GL}(n, R/I)$$

modulo I.

• By definition, the *principal* congruence subgroup $\mathrm{GL}(n, R, I)$ is the kernel of ρ_I. In other words, $\mathrm{GL}(n, R, I)$ consists of all matrices g congruent to e modulo I.

$$\mathrm{GL}(n, R, I) = \big\{ g = (g_{ij}) \in \mathrm{GL}(n, R) \mid g_{ij} \equiv \delta_{ij} \pmod I \big\}.$$

• In turn, the *full* congruence subgroup $C(n, R, I)$ is the full preimage of the center of $\mathrm{GL}(n, R/I)$ with respect to ρ_I. In other words, $C(n, R, I)$ consists of matrices, which become scalar modulo I, i.e. have the form λe, where λ is central modulo I, $\lambda \in \mathrm{Cent}(R/I)^*$.

We need also some of the less familiar congruence subgroups.

• The *brimming* congruence subgroup $G(n, R, I)$, which is the full preimage of the diagonal subgroup $D(n, R/I) \leqslant \mathrm{GL}(n, R/I)$. In the terminology of Zenon Borewicz, $G(n, R, I) = G(\sigma)$ is the net subgroup corresponding to the D-net $\sigma = (\sigma_{ij})$, $1 \leqslant i \neq j \leqslant n$, such that $\sigma_{ij} = I$ for all $i \neq j$, while $\sigma_{ii} = R$ as they should be, for D-nets, see [7, 8].

$$G(n, R, I) = \big\{ g = (g_{ij}) \in \mathrm{GL}(n, R) \mid g_{ij} \equiv 0 \pmod I, i \neq j \big\}.$$

• For a subgroup $\Omega \leqslant (R/I)^*$ we can define $C_\Omega(n,R,I)$ consists of matrices which modulo I, have the form λe, for some $\lambda \in \Omega$. The largest one of those is the group consisting of all matrices that become (non-central!) homotheties modulo I, it corresponds to $\Omega = (R/I)^*$:

$$C^*(n,R,I) = \big\{g = (g_{ij}) \in G(n,R,I) \mid g_{ii} \equiv g_{jj} \pmod I \big\}.$$

• But actually, we will be most interested in the following special case. Let $A, B \trianglelefteq R$ be two ideals of R. We consider the subgroup

$$\Omega = \Omega(A,B) = \rho_{(B:A)/B}\Big(\operatorname{Cent}_{R/B}\big((A+B)/B\big)\Big) \cap \big(R/(B:A)\big)^*$$

Let $G = \mathrm{GL}(n,R)$ and

$$C_{\Omega(A,B)}\big(n,R,(B:A)\big) = \big\{g \in \mathrm{GL}(n,R) \mid g_{ij}, g_{ii} - g_{jj} \in (B:A) \\ \text{for } i \neq j, \text{ and } g_{ii} \in \operatorname{Cent}_R(A,B)\big\}.$$

In other words, this group is defined in exactly the same way as the full congruence subgroup $C\big(n,R,(B:A)\big)$, only that now instead of requiring that the diagonal entries of matrices become central modulo $(B:A)$, we impose a weaker condition that modulo B they commute with elements of A. Of course, since $(B:A) \circ A \leqslant B$, this condition depends not on the entry itself, but only on its congruence class modulo $(B:A)$, which secures correctness of this definition.

In particular, when R is commutative,

$$C_{\Omega(A,B)}\big(n,R,(B:A)\big) = C\big(n,R,(B:A)\big)$$

is the usual full congruence subgroup of level $(B:A)$.

§4. Centralisers of $E(n,R,A)$ and $\mathrm{GL}(n,R,A)$, modulo $\mathrm{GL}(n,R,B)$

Now we are all set to prove the first main result of the present paper. Here we consider relative centralisers modulo the principal congruence subgroups $\mathrm{GL}(n,R,B)$, which are *always* normal in $\mathrm{GL}(n,R)$, which makes the analysis considerable

Theorem 1. *Let R be an arbitrary associative ring with 1, $A, B \trianglelefteq R$, and $n \geqslant 3$. Further, let $H \leqslant \mathrm{GL}(n,R)$ be any subgroup such that $E(n,A) \leqslant H \leqslant \mathrm{GL}(n,R,A)$. Then*

$$C_{\mathrm{GL}(n,R)}\big(H, \mathrm{GL}(n,R,B)\big) = C_{\Omega(A,B)}\big(n,R,(B:A)\big).$$

Proof. First, we calculate $C_{\mathrm{GL}(n,R)}\big(E(n,A),\mathrm{GL}(n,R,B)\big)$. Assume that $g\in\mathrm{GL}(n,R)$ commutes with $E(n,A)$ modulo $\mathrm{GL}(n,R,B)$. In particular this means that for all $1\leqslant r\neq s\leqslant n$ and all $a\in A$ one has

$$[g,t_{rs}(a)]\in\mathrm{GL}(n,R,B),$$

so that

$$gt_{rs}(a)\equiv t_{rs}(a)g \pmod{B}.$$

The left hand side only differs from g in the s-th column, while the right hand side only differs from g in the r-th row. Comparing the entries of these matrices in positions $(i,s),(r,j)\neq(r,s)$, we see that

- $g_{ir}a\in B$, for all $i\neq r$,
- $ag_{sj}\in B$, for all $j\neq s$,

in particular, we can conclude that all non-diagonal entries of g belong to $(B:A)$.

It remains to compare the entries in the position (r,s).

- $g_{rr}a\equiv ag_{ss}\pmod{B}$, or, what is the same, $g_{rr}a-ag_{ss}\in B$, for all $r\neq s$.

Now, since $n\geqslant 3$, we can choose an index $t\neq r,s$, and conclude that $g_{rr}a-ag_{tt}\in B$. Comparing the above inclusions, we see that $a(g_{ss}-g_{tt})\in B$, so that $g_{ss}-g_{tt}\in A^{-1}B$, for all $s\neq t$. By the same token, $ag_{ss}-g_{tt}a\in B$, and again comparing the above inclusions we see that $(g_{rr}-g_{tt})a\in B$, so that $g_{rr}-g_{tt}\in BA^{-1}$, for all $r\neq t$. This means that pairwise differences of the diagonal entries of g belong to $(B:A)$. But then the congruence in the last item implies that $g_{rr}a-ag_{rr}\in B$, for all r, $1\leqslant r\leqslant n$, and all $a\in A$. Summarising the above, we see that

$$C_{\mathrm{GL}(n,R)}\big(E(n,A),\mathrm{GL}(n,R,B)\big)\leqslant C_{\Omega(A,B)}\big(n,R,(B:A)\big).$$

On the other hand,

$$\begin{aligned}C_{\mathrm{GL}(n,R)}\big(\mathrm{GL}(n,R,A),\mathrm{GL}(n,R,B)\big)\leqslant C_{\mathrm{GL}(n,R)}\big(H,\mathrm{GL}(n,R,B)\big)\leqslant\\ C_{\mathrm{GL}(n,R)}\big(E(n,A),\mathrm{GL}(n,R,B)\big),\end{aligned}$$

and to prove the theorem it only remains to verify that

$$C_{\Omega(A,B)}\big(n,R,(B:A)\big)\leqslant C_{\mathrm{GL}(n,R)}\big(\mathrm{GL}(n,R,A),\mathrm{GL}(n,R,B)\big).$$

Indeed, let $g=(g_{ij})\in C_{\Omega(A,B)}\big(n,R,(B:A)\big)$ and $h\in\mathrm{GL}(n,R,A)$. We claim that then $gh\equiv hg\pmod{B}$. Indeed,

$$(gh)_{rs}=\sum g_{rt}h_{ts},\qquad (hg)_{rs}=\sum h_{rt}g_{ts},$$

where both sums are taken over $1 \leqslant t \leqslant n$.

• First, let $r \neq s$. Then the summands corresponding to $t \neq r, s$ belong to $(B : A)A \leqslant B$ on the left hand side, and to $A(B : A) \leqslant B$ on the right hand side, and can be discarded.

• This means that for the case $r \neq s$ it only remains to take care of the summands corresponding to $t = r, s$. But since $r \neq s$ one has $h_{rs} \in A$, so that $g_{rr}h_{rs} \equiv h_{rs}g_{ss} \pmod B$ by the very definition of $C_{\Omega(A,B)}\big(n, R, (B : A)\big)$. On the other hand, $h_{rr} \equiv h_{ss} \equiv 1 \pmod A$ and since $g_{rs} \in (B : A)$, also $g_{rs}h_{ss} \equiv h_{rr}g_{rs} \pmod B$.

• By the same token, for the remaining case $r = s$ all summands $g_{rt}h_{tr}$ and $h_{rt}g_{tr}$ belong to B and can be discarded. On the other hand, $h_{rr} \equiv 1 \pmod A$, and since 1 commutes with g_{rr}, while elements of A commute with g_{rr} modulo B, one has $g_{rr}h_{rr} \equiv h_{rr}g_{rr} \pmod B$, as claimed.

This proves the desired inclusion, and thus the theorem. □

§5. Generation of relative subgroups and commutator formulas

In the present section we collect the requisite results on relative elementary subgroups that will be used in the rest of this paper.

The following lemma on generation of relative elementary subgroups $E(n, R, A)$ is a classical result discovered in various contexts by Stein, Tits and Vaserstein, see, for instance, [26] (or [16], Lemma 3, for a complete elementary proof). It is stated in terms of the *Stein–Tits–Vaserstein generators*:

$$z_{ij}(a, c) = t_{ij}(c)t_{ji}(a)t_{ij}(-c), \qquad 1 \leqslant i \neq j \leqslant n, \quad a \in A, \quad c \in R.$$

Lemma 1. *Let R be an associative ring with 1, $n \geqslant 3$, and let A be a two-sided ideal of R. Then as a subgroup $E(n, R, A)$ is generated by $z_{ij}(a, c)$, for all $1 \leqslant i \neq j \leqslant n$, $a \in A$, $c \in R$.*

In the following theorem a further type of generators occur, the *elementary commutators*:

$$y_{ij}(a, b) = [t_{ij}(a), t_{ji}(b)], \qquad 1 \leqslant i \neq j \leqslant n, \quad a \in A, \quad b \in B.$$

The following analogue of Lemma 1 for commutators

$$[E(n, R, A), E(n, R, B)]$$

was discovered (in slightly less precise forms) by Roozbeh Hazrat and the second author, see [18], Lemma 12 and then in our joint paper with

Hazrat [16], Theorem 3A. The strong form reproduced below is established only in our paper [39], Theorem 1 (see also [40]), as a spin-off of our papers [27, 38].

Lemma 2. *Let R be any associative ring with 1, let $n \geqslant 3$, and let A, B be two-sided ideals of R. Then the mixed commutator subgroup*

$$\big[E(n, R, A), E(n, R, B)\big]$$

is generated as a group by the elements of the form

- $z_{ij}(ab, c)$ *and* $z_{ij}(ba, c)$,
- $y_{ij}(a, b)$,

where $1 \leqslant i \neq j \leqslant n$, $a \in A$, $b \in B$, $c \in R$. Moreover, for the second type of generators, it suffices to fix one pair of indices (i, j).

In the proofs below we use not just Lemma 2 itself, but also some of the results used in its proof. The first of them is standard, see, for instance, [16, 36, 37] and references there.

Lemma 3. *R be an associative ring with 1, $n \geqslant 3$, and let A and B be two-sided ideals of R. Then*

$$E(n, R, A \circ B) \leqslant \big[E(n, A), E(n, B)\big] \leqslant \big[E(n, R, A), E(n, R, B)\big] \leqslant \mathrm{GL}(n, R, A \circ B).$$

The first of the following lemmas is [39], Lemma 3, or [40], Lemma 9. The second is [40], Lemma 10. And the third one is [39], Lemma 5, or [40], Lemma 11.

Lemma 4. *Let R be an associative ring with 1, $n \geqslant 3$, and let A, B be two-sided ideals of R. Then for any $1 \leqslant i \neq j \leqslant n$, $a \in A$, $b \in B$, and any $x \in E(n, R)$ one has*

$${}^{x}y_{ij}(a, b) \equiv y_{ij}(a, b) \pmod{E(n, R, A \circ B)}\,.$$

Lemma 5. *Let R be an associative ring with 1, $n \geqslant 3$, and let A, B be two-sided ideals of R. Then for any $1 \leqslant i \neq j \leqslant n$, $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$ one has*

$$\begin{aligned}
&y_{ij}(a_1 + a_2, b) \equiv y_{ij}(a_1, b) \cdot y_{ij}(a_2, b) \pmod{E(n, R, A \circ B)}\,,\\
&y_{ij}(a, b_1 + b_2) \equiv y_{ij}(a, b_1) \cdot y_{ij}(a, b_2) \pmod{E(n, R, A \circ B)}\,,\\
&y_{ij}(a, b)^{-1} \equiv y_{ij}(-a, b) \equiv y_{ij}(a, -b) \pmod{E(n, R, A \circ B)}\,,\\
&y_{ij}(ab_1, b_2) \equiv y_{ij}(a_1, a_2 b) \equiv e \pmod{E(n, R, A \circ B)}\,.
\end{aligned}$$

Lemma 6. *Let R be an associative ring with 1, $n \geqslant 3$, and let A, B be two-sided ideals of R. Then for any $1 \leqslant i \neq j \leqslant n$, any $1 \leqslant k \neq l \leqslant n$, and all $a \in A$, $b \in B$, $c \in R$, one has*

$$y_{ij}(ac, b) \equiv y_{kl}(a, cb) \pmod{E(n, R, A \circ B)}.$$

For quasi-finite rings the following result is [37], Theorem 5 and [16], Theorem 2A, but for arbitrary associative rings it was only established in [40], Theorem 2.

Lemma 7. *Let R be any associative ring with 1, let $n \geqslant 3$, and let A and B be two-sided ideals of R. If A and B are comaximal, $A + B = R$, then*

$$[E(n, A), E(n, B)] = E(n, R, A \circ B).$$

The following result is [37], Theorem 4.

Lemma 8. *Let A and B be two ideals of a commutative ring R and $n \geqslant 3$. Then*

$$\big[E(n, R, A), C(n, R, B)\big] = \big[E(n, R, A), E(n, R, B)\big].$$

Finally, the following lemma is [28], Theorem 2.

Lemma 9. *Let A and B be two ideals of a Dedekind ring of arithmetic type $R = \mathcal{O}_S$. Assume that the multiplicative group R^* is infinite and that $n \geqslant 3$. Then*

$$\big[\operatorname{GL}(n, R, A), \operatorname{GL}(n, R, B)\big] = E(n, R, AB).$$

§6. Centralisers of $E(n, R, A)$ and $\operatorname{GL}(n, R, A)$, modulo $E(n, R, B)$

The group $Z(n, R, I)$ is defined as the centraliser of $\operatorname{GL}(n, R)$ modulo $E(n, R, I)$:

$$Z(n, R, I) = \big\{g \in \operatorname{GL}(n, R) \mid [g, \operatorname{GL}(n, R)] \leqslant E(n, R, I)\big\}.$$

When $E(n, R, I)$ is normal in $\operatorname{GL}(n, R)$, the quotient $Z(n, R, I)/E(n, R, I)$ is the centre of

$$\operatorname{GL}(n, R)/E(n, R, I).$$

Let us make some obvious observations concerning this group.

- By definition

$$C(n, R, I) = \big\{g \in \operatorname{GL}(n, R) \mid \big[g, \operatorname{GL}(n, R)\big] \leqslant \operatorname{GL}(n, R, I)\big\}.$$

In other words, $C(n,R,I)/\operatorname{GL}(n,R,I)$ is the centre of
$$\operatorname{GL}(n,R)/\operatorname{GL}(n,R,I).$$
Since $E(n,R,I) \leqslant \operatorname{GL}(n,R,I)$, one has $Z(n,R,I) \leqslant C(n,R,I)$.

• Since $\big[\operatorname{GL}(n,R,I), \operatorname{GL}(n,R)\big] \leqslant E(n,R,I)$ for $n \geqslant \max(\operatorname{sr}(R)+1,3)$, in this case $E(n,R,I)$ is normal in $\operatorname{GL}(n,R)$ and $\operatorname{GL}(n,R,I)/E(n,R,I)$ is contained in the centre of $\operatorname{GL}(n,R)/E(n,R,I)$. Thus, in the stable range
$$\operatorname{GL}(n,R,I) \leqslant Z(n,R,I) \leqslant C(n,R,I).$$
However even in the stable range, it may happen that $Z(n,R,I)$ is strictly smallser than $C(n,R,I)$.

• Below the stable range funny things may happen. In particular, below the stable range even for commutative rings and $n \geqslant 3$ the group
$$K_1(n,R,I) = \operatorname{GL}(n,R,I)/E(n,R,I)$$
does not have to be abelian. The first such counter-examples were constructed by Wilberd van der Kallen [19] and Anthony Bak [2]. For finite dimensional rings this group is indeed nilpotent by abelian, but the nilpotent part may have arbitrarily large nilpotency class.

This means that $Z(n,R,I)$ may sit at the very bottom of $\operatorname{GL}(n,R,I)$. Both Alec Mason [22] and Anthony Bak [2] used the fact that $Z(n,R,I) < C(n,R,I)$ to construct subgroups of level I that are normalised by $E(n,R)$, but not normal in $\operatorname{GL}(n,R)$.

This means that in general such relative centralisers as
$$C_{\operatorname{GL}(n,R)}\big(\operatorname{GL}(n,R,A), E(n,R,B)\big) \text{ and } C_{\operatorname{GL}(n,R)}\big(E(n,R,A), E(n,R,B)\big)$$
do not have an obvious description in the style of the previous section. But for *commutative* rings it is very easy to slightly modify $E(n,R,B)$, to get exactly the same answer, as above. In this case all commutativity assumptions are automatically satisfied, so that $C(n,R,I) = C^*(n,R,I)$ for all ideals.

Theorem 2. *Let R be a commutative ring and $A, B \trianglelefteq R$, $n \geqslant 3$. Then*
$$C_{\operatorname{GL}(n,R)}\Big(E(n,R,A), \big[E\big(n,R,(B:A)\big), E(n,R,A)\big]\Big) = C\big(n,R,(B:A)\big).$$

Proof. By Lemma 8 one has
$$\big[C\big(n,R,(B:A)\big), E(n,R,A)\big] = \big[E\big(n,R,(B:A)\big), E(n,R,A)\big].$$
In other words, the right hand side of the equality in the statement of theorem is contained in the left hand side.

On the other hand,

$$\big[E\big(n,R,(B:A)\big),E(n,R,A)\big] \leqslant \big[\,\mathrm{GL}\,\big(n,R,(B:A)\big),\mathrm{GL}(n,R,A)\big] \leqslant \mathrm{GL}\,\big(n,R,(B:A)A\big),$$

and thus, by Theorem 1

$$C_{\mathrm{GL}(n,R)}\Big(E(n,R,A),\big[E\big(n,R,(B:A)\big),E(n,R,A)\big]\Big) \leqslant C_{\mathrm{GL}(n,R)}\Big(E(n,R,A),\mathrm{GL}\,\big(n,R,(B:A)A\big)\Big) = C\big(n,R,\big((B:A)A:A\big)\big).$$

It remains only to observe that the level of this last subgroup is precisely the what it should be, $((B:A)A:A)=(B:A)$. Indeed, $x\in((B:A)A:A)$ means that $xA\leqslant(B:A)A\leqslant B$, so that $((B:A)A:A)\leqslant(B:A)$. On the other hand, if $x\in(B:A)$, then $xA\leqslant(B:A)A$, so that $(B:A)\leqslant((B:A)A:A)$. This means that $C\big(n,R,\big((B:A)A:A\big)\big)=C\big(n,R,(B:A)\big)$, which proves the theorem. ☐

Of course, the level of the commutator subgroup

$$\big[E\big(n,R,(B:A)\big),E(n,R,A)\big]$$

is $(B:A)A$, which in general is *smaller* than B. However, for Dedekind rings always $(B:A)A=B$.

On the other hand, when the levels coincide, the mixed commutator subgroup

$$\big[E\big(n,R,(B:A)\big),E(n,R,A)\big]$$

is in general *larger* than $E(n,R,B)$. It suffices to take the known examples, where $A=(B:A)=I$, while $B=I^2$, see [40]. The simplest such example was constructed by Alec Mason and Wilson Stothers [21, 23] already for the ring $\mathbb{Z}[i]$ of Gaussian integers. However, in [28] the first author noticed that this cannot possibly occur for Dedekind rings of arithmetic type with *infinite* multiplicative group.

Theorem 3. *Let R be a Dedekind ring of arithmetic type with infinite multiplicative group and $A,B\trianglelefteq R$, $n\geqslant 3$. Then*

$$C_{\mathrm{GL}(n,R)}\big(E(n,R,A),E(n,R,B)\big)=C\big(n,R,(B:A)\big).$$

Proof. Indeed, by Lemma 9, one has

$$\big[E\big(n,R,(B:A)\big),E(n,R,A)\big]=E\big(n,R,(B:A)A\big)=E(n,R,B),$$

and it remains to apply the previous theorem. □

§7. Final remarks

It would be natural to generalise results of the present paper to more general contexts.

Problem 1. *Generalise Theorems* 1 *and* 2 *to Chevalley groups.*

Problem 2. *Generalise Theorems* 1 *and* 2 *to Bak's unitary groups.*

It seems, that in both cases the strategy is clear, but there are a lot of technical details to take care of. For Chevalley groups the ground ring R is commutative anyway, which makes many technical details considerably less burdensome. On the other hand, calculations in representations themselves necessary to establish analogues of Theorem 1, will be somewhat more delicate, especially for exceptional groups. But the pattern of such calculations should be mostly known from [30–33]. Most tools necessary to derive from there an analogue of Theorem 2, are in our recent papers [38, 42]. On the other hand, so far we are still missing several key components necessary to generalise to this case the results of [28], and before we do that, there is no hope to generalise Theorem 3.

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