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HARMONIC MEASURE OF ARCS OF FIXED LENGTH

Abstract. Jordan domains Ω with piece-wise smooth boundaries are treated such that all arcs $\alpha \subset \partial\Omega$ having fixed length l, $0 < l <$ length$(\partial\Omega)$, have equal harmonic measures $\omega(z_0, \alpha, \Omega)$ evaluated at some point $z_0 \in \Omega$. It is proved that Ω is a disk centered at z_0 if the ratio l/length$(\partial\Omega)$ is irrational and that Ω possesses rotational symmetry by some angle $2\pi/n$, $n \geqslant 2$, around the point z_0, if this ratio is rational.

Let Ω be a Jordan domain on $\mathbb{C}$ with rectifiable boundary $\partial\Omega$ of length L, $0 < L < \infty$. We recall that the harmonic measure with respect to Ω is a function $\omega(z, E, \Omega)$ such that: (1) for each $z \in \Omega$, $\omega(z, \cdot, \Omega)$ is a Borel probability measure on $\partial\Omega$, (2) for each continuous function $\varphi : \partial\Omega \to \mathbb{R}$ the generalized Poisson integral

$$u(z) = \int\limits_{\partial\Omega} \varphi(\zeta)\, d\omega(z, \cdot, \Omega)$$

solves the Dirichlet problem on Ω with boundary values $\varphi(\zeta)$; see, for instance, [8, Chapter 4.3].

Suppose that for a point $z_0 \in \Omega$ there is positive l, $0 < l < L$, such that all open boundary arcs $\alpha \subset \partial\Omega$ of length l have the same harmonic measure with respect to Ω evaluated at z_0; i.e., such that

$$\omega(z_0, \alpha_1, \Omega) = \omega(z_0, \alpha_2, \Omega), \tag{1}$$

whenever α_1 and α_2 are open arcs on $\partial\Omega$ with

$$\text{length}(\alpha_1) = \text{length}(\alpha_2) = l. \tag{2}$$

In this note we discuss the following question: When do conditions (1) and (2) determine the shape of Ω?

To address this question, we first introduce necessary definitions and recall few known results.

Key words and phrases: Harmonic measure, conformal mapping, Smirnov domain.

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Let $f : \mathbb{D} \to \Omega$ be a conformal mapping from the unit disk $\mathbb{D} = \{z : |z| < 1\}$ onto Ω. Since $\partial\Omega$ is Jordan and rectifiable, the derivative $f'(\zeta)$, defined by

$$f'(\zeta) = \lim_{z\to\zeta, z\in\overline{\mathbb{D}}} \frac{f(z) - f(\zeta)}{z - \zeta} \neq 0, \infty, \tag{3}$$

exists for almost all $\zeta \in \mathbb{T} = \partial\mathbb{D}$. Furthermore, the derivative f' belongs to the Hardy space H^1 and

$$\text{length}(f(E)) = \int_E |f'(e^{i\theta})|\, d\theta \tag{4}$$

for every measurable set $E \subset \mathbb{T}$. We recall here that the Hardy space H^1 consists of all functions g analytic in $\mathbb{D}$ such that

$$\sup_{0<r<1} \int_0^{2\pi} |g(re^{i\theta})|\, d\theta < \infty. \tag{5}$$

For all properties mentioned in equations (3)–(5), see, for instance, Theorem 6.8 in [7].

As equation (4) shows, the length of every boundary set $f(E)$ with a measurable $E \subset \mathbb{T}$ can be found via the values of the modulus of the derivative $f'(e^{i\theta})$. On the other hand, surprisingly enough, the derivative $f'(z)$, $z \in \mathbb{D}$, itself cannot be recovered from the boundary values of its modulus $|f'(e^{i\theta})|$, in general. Domains Ω for which such a recovery is possible are called Smirnov domains. More precisely, Ω is a Smirnov domain if $f'(z)$ can be found via the following formula (see Section 7.1 in [7]):

$$\log|f'(z)| = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \log|f'(e^{i\theta})|\, d\theta \quad \text{for } z \in \mathbb{D}. \tag{6}$$

The first example of a non-Smirnov domain was constructed by M. W. Keldysh and M. A. Lavrentiev in 1937; see [5]. The following theorem of K. Øyma [3] implies that the non-Smirnov domains are dense in the set of all simply connected domains in the sense of Carathéodory.

Theorem 1 (K. Øyma, [3]). *Let $g(z)$ be univalent in $\mathbb{D}$ and let $0 < r_1 < r_2 < 1$. Then there exists a non-Smirnov domain Ω such that $g(|z| < r_1) \subset \Omega \subset g(|z| < r_2)$ and there exists a conformal mapping $f : \mathbb{D} \to \Omega$ such that $|f'(e^{i\theta})| = \text{constant}$ almost everywhere on $\mathbb{T}$.*

This theorem by Øyma implies that the shape of Ω cannot be recovered from conditions (1) and (2) if Ω is non-Smirnov. More precisely, the following holds.

Corollary 1. *Let Γ be an analytic Jordan curve and let $\Gamma_\varepsilon = \{z : dist(z,\Gamma) < \varepsilon\}$ denote the neighborhood of Γ of radius $\varepsilon > 0$. Then there is a Jordan rectifiable curve $\gamma \subset \Gamma_\varepsilon$ homotopic to Γ in Γ_ε and a point z_0 in the bounded component (call it Ω) of $\mathbb{C} \setminus \gamma$ such that*

$$\omega(z_0, \alpha, \Omega) = \text{length}(\alpha)/\text{length}(\gamma) \tag{7}$$

for every arc $\alpha \subset \gamma$.

Actually, (7) is true for all measurable sets $\alpha \subset \gamma$, not only for arcs. Our emphasis on arcs here is solely in relation with our question stated above.

Thus, Øyma's theorem stated above shows that the shape of a Jordan rectifiable domain cannot be recovered in general from conditions (1) and (2), not even from the proportionality condition (7) applied to all measurable sets $\gamma \subset \Gamma$!

Therefore, to get more information from conditions (1) and (2), we have to restrict ourselves to the subclass of rectifiable Smirnov domains. In this direction, the following important result was proved independently by P. Ebenfelt, D. Khavinson, and H. S. Shapiro [1] and by S. J. Gardiner [2].

Theorem 2 ([1, 2]). *Suppose Γ is a rectifiable Jordan curve such that the equilibrium measure, which coincides with the harmonic measure*

$$\omega(\infty, \cdot, \Omega_\infty)$$

of the outer domain Ω_∞ of Γ with respect to the point $z = \infty$, is a multiple of arclength measure on Γ.

If the outer domain Ω_∞ is Smirnov, then Γ is a circle.

It was mentioned in [2] that a version of Theorem 2 for a smaller class of bounded Lipshitz domains had first been proved by O. Mendez and W. Reichel [6] in response to a conjecture by P. Gruber.

Our main result, stated in Theorem 3 below, shows that, when restricted to piecewise smooth Jordan curves Γ, the conclusion in the spirit of Theorem 2 can be derived under a much weaker assumption, namely, when proportionality condition between harmonic measure and arclength is assumed only for boundary arcs of specific length.

Theorem 3. *Let Ω be a domain bounded by a piecewise smooth Jordan curve Γ of length L. Suppose that for a point $z_0 \in \Omega$ there is positive l, $0 < l < L$, such that conditions* (1), (2) *are satisfied for all arcs $\alpha_1, \alpha_2 \subset \Gamma$. Then the following holds true.*

1) *If l/L is irrational, then Ω is a disk centered at z_0.*
2) *If l/L is rational of the form m/n in lowest terms, then Ω is invariant under rotations by an angle of $2\pi/n$ around z_0.*

Proof. Let an arc $\alpha \subset \partial\Omega$ of length l have its endpoints at ζ_1 and ζ_2. We assume here that a walk from ζ_1 to ζ_2 along α corresponds to the positive orientation with respect to Ω. Let $f : \mathbb{D} \to \Omega$ be a conformal mapping from $\mathbb{D}$ to Ω such that $f(0) = z_0$. Let $e^{i\theta_k} = f^{-1}(\zeta_k)$, $k = 1, 2$. Since Γ is piecewise smooth, the derivatives $f'(e^{i\theta_k})$, $k = 1, 2$, exist for all arcs α of length l except, possibly, a finite number of such arcs. Suppose that, for our choice of α, $f'(e^{i\theta_1})$ and $f'(e^{i\theta_2})$ exist.

We claim that

$$|f'(e^{i\theta_1})| = |f'(e^{i\theta_2})|. \tag{8}$$

To prove this, we assume that $\varepsilon > 0$ is sufficiently small and consider two arcs $\beta_1 = \beta_1(\varepsilon)$ and $\beta_2 = \beta_2(\varepsilon)$. Here $\beta_1 \subset \alpha$ is a subarc of α of length ε with the initial point at ζ_1 and $\beta_2 \subset \overline{\Gamma \setminus \alpha}$ is a subset of $\overline{\Gamma \setminus \alpha}$ with the initial point at ζ_2.

Let $\zeta_k' = \zeta_k'(\varepsilon)$ be the end-point of the arc β_k, $k = 1, 2$, and let $\alpha' = \alpha'(\varepsilon)$ be the arc on Γ with end-points ζ_1' and ζ_2', such that $\beta_2 \subset \alpha'$. Then $\text{length}(\alpha') = l$ and therefore

$$\omega(z_0, \alpha', \Omega) = \omega(z_0, \alpha, \Omega) \tag{9}$$

by the assumptions of the theorem. Let $\widetilde{\alpha} \subset \alpha$ be a sub-arc of α with end-points ζ_1' and ζ_2. Since the harmonic measure is an additive function on boundary sets, using (9), we obtain

$$\begin{aligned}\omega(z_0, \beta_2, \Omega) &= \omega(z_0, \alpha', \Omega) - \omega(z_0, \widetilde{\alpha}, \Omega) \\ &= \omega(z_0, \alpha, \Omega) - \omega(z_0, \widetilde{\alpha}, \Omega) = \omega(z_0, \beta_1, \Omega).\end{aligned}$$

Thus, the arcs β_1 and β_2 have the same harmonic measures at z_0. Since $f^{-1}(z_0) = 0$, the last statement implies that if, for some $\delta = \delta(\varepsilon) > 0$, the point $e^{i(\theta_1+\delta)} = f^{-1}(\zeta_1')$ is the preimage of ζ_1' under the mapping $f(z)$, then the point $e^{i(\theta_2+\delta)}$ is the preimage of ζ_2' under this mapping. Now, formula (4) gives the following expressions for the length of the arcs β_1 and β_2:

$$\varepsilon = \text{length}(\beta_1) = \int\limits_{\theta_1}^{\theta_1+\delta} |f'(e^{it})|\, dt = |f'(e^{it_1})|\, \delta \tag{10}$$

with some $t_1 = t_1(\delta)$, $\theta_1 < t_1 < \theta_1 + \delta$, and

$$\varepsilon = \text{length}(\beta_2) = \int\limits_{\theta_2}^{\theta_2+\delta} |f'(e^{it})|\, dt = |f'(e^{it_2})|\, \delta \tag{11}$$

with some $t_2 = t_2(\delta)$, $\theta_2 < t_2 < \theta_2 + \delta$. The third identity in equations (10) and (11) follows from the mean value theorem for integrals.

Since Γ is piecewise smooth, the function $|f'(e^{i\theta})|$ is continuous for $0 \leqslant \theta < 2\pi$ except, possibly, a finite number of points. Therefore, using equations (10) and (11), we obtain:

$$|f'(e^{i\theta_1})| = \lim_{\varepsilon\to 0} |f'(e^{i(t_1(\delta(\varepsilon)))})| = \lim_{\varepsilon\to 0} |f'(e^{i(t_2(\delta(\varepsilon)))})| = |f'(e^{i\theta_2})|. \tag{12}$$

Thus, we have proved that for any two points ζ_1 and ζ_2 on Γ, one lying from the other at the distance l along Γ, the moduli of the derivatives $f'(e^{i\theta_1})$ and $f'(e^{i\theta_2})$ at the corresponding preimages must be equal if these derivatives exist. Now we are ready to prove parts 1) and 2) of the theorem.

1) Suppose that l/L is irrational and consider any two distinct points ζ_1 and ζ_2 on Γ. Let $\{\zeta^k\}_{k=1}^{\infty}$ be a sequence of points on Γ such that $\zeta^1 = \zeta_1$ and such that, for all $k = 1, 2, \ldots$, the distance along Γ in positive direction between the points ζ^k and ζ^{k+1} equals l. Let $\{e^{i\theta^k} = f^{-1}(\zeta^k)\}_{k=1}^{\infty}$ be the sequence of preimages of the points ζ^k under the mapping $f(z)$. Since l/L is irrational, from the well-known results of the theory of irrational rotational dynamics on a circle, see, for instance Proposition 1.3.3 in [4], it follows that all points ζ^k are distinct and the sequence $\{\zeta^k\}_{k=1}^{\infty}$ is dense on Γ. Since $f(e^{i\theta})$ is continuous and one-to-one on $\mathbb{T}$, the last fact implies that all points $e^{i\theta^k}$ are also distinct and the sequence $\{e^{i\theta^k}\}_{k=1}^{\infty}$ is dense on $\mathbb{T}$. Therefore, either $e^{i\theta_2} = e^{i\theta^k}$ for some k or there is a subsequence $\{e^{i\theta^{k_s}}\}_{s=1}^{\infty}$ that converges to $e^{i\theta_2}$. From (12), it follows that $|f'(e^{i\theta^{k_s}})| = |f'(e^{i\theta_1})|$ for all $s = 1, 2, \ldots$ Since Γ is smooth at ζ_2, the function $|f'(e^{i\theta})|$ is continuous at $e^{i\theta_2}$. Therefore,

$$|f'(e^{i\theta_2})| = \lim_{s\to\infty} |f'(e^{i\theta^{k_s}})| = |f'(e^{i\theta_1})|.$$

Thus, we have proved that there is a constant $c > 0$ such that $|f'(e^{i\theta})| = c$ for all $e^{i\theta} \in \mathbb{T}$ except, possibly, a finite number of points. Since $\Gamma = \partial\Omega$ is piecewise smooth, and thus Ω is a Smirnov domain, the last claim and equation (6) imply that

$$\log|f'(z)| = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\log c\, d\theta = \log c \quad \text{for } z \in \mathbb{D}.$$

This equation and the assumption $f(0) = z_0$ imply that $f(z) = az + z_0$ with some $a \in \mathbb{C}$ such that $|a| = c$ and therefore Ω is a disk centered at z_0. Furthermore, since $L = \text{length}(\Gamma) = \int_0^{2\pi} |f'(e^{i\theta})|\, d\theta = 2\pi c$, the radius of the disk is $c = L/(2\pi)$.

2) Now, we assume that l/L is rational of the form $r = m/n$, $0 < m < n$, in the lowest terms. Once again, we will use sequences $\{\zeta^k\}_{k=1}^{\infty}$ and $\{e^{i\theta^k}\}_{k=1}^{\infty}$ defined in part 1) of this proof. In this case we deal with the rational rotational dynamics and the behavior of the sequence $\{\zeta^k\}_{k=1}^{\infty}$ is different compared to case 1). In particular, this sequence is periodic in the sense that $\zeta_{k+n} = \zeta_k$ for all $k \geqslant 1$. Thus, the terms ζ_k of this sequence visit only finitely many points on Γ. These points are the points $\nu_s \in \Gamma$, $s = 0, \dots, n-1$, oriented in the positive direction on Γ with $\nu_0 = \zeta_1$ such that these points divide Γ into n arcs of equal length. Let γ_s, $s = 0, \dots, n-1$, denote the arc joining the points ν_s and ν_{s+1}. Here, $\nu_n = \zeta_1$. Then $\text{length}(\gamma_s) = L/n$ for $s = 0, \dots, n-1$. We assume further that the arcs γ_s are parameterized by length. Then $\nu_s(\tau)$, $0 \leqslant \tau \leqslant L/n$, will denote the point on γ_s such that the length of the subarc $\gamma_s(\tau)$ of γ_s between ν_s and $\nu_s(\tau)$ equals τ.

Let $g(w) = f^{-1}(w)$ be the inverse function of f which is defined on $\overline{\Omega}$. For $s = 0, \dots, n-1$, let $e^{i\theta_s(\tau)} = g(\nu_s(\tau))$. The same argument as we used to prove (8), shows that

$$|f'(e^{i\theta_s(\tau)})| = |f'(e^{i\theta_0(\tau)})| \tag{13}$$

for all $s = 0, \dots, n-1$ and all $0 \leqslant \tau \leqslant L/n$ if the derivatives exist. Therefore, similar conclusion holds for the inverse function $g(w)$:

$$|g'(\nu_s(\tau))| = |g'(\nu_0(\tau))| \tag{14}$$

for all $s = 0, \dots, n-1$ and all $0 \leqslant \tau \leqslant L/n$ if the derivatives exist.

Integrating (14) over the arcs $\gamma_s(\tau)$ and $\gamma_0(\tau)$ with respect to arclength, we obtain the following equation:

$$\begin{aligned}\theta_s(\tau)-\theta_s(0) &= \mathrm{length}(f^{-1}(\gamma_s(\tau))) = \int\limits_{\gamma_s(\tau)} |g'(\nu_s(t))|\,dt \\ &= \int\limits_{\gamma_0(\tau)} |g'(\nu_0(t))|\,dt = \mathrm{length}(f^{-1}(\gamma_0(\tau))) = \theta_0(\tau)-\theta_0(0)\end{aligned}$$

for all $s=0,\dots,n-1$ and $0\leqslant\tau\leqslant L/n$.

This equation implies that

$$\theta_s(\tau)-\theta_0(\tau)=2\pi s/n \quad \text{for } s=0,\dots,n-1 \text{ and } 0\leqslant\tau\leqslant n-1.$$

In particular,

$$\theta_s(L/n)-\theta_s(0)=2\pi/n \quad \text{for } s=0,\dots,n-1.$$

This equation together with (13) implies

$$|f'(e^{i\theta_s(\tau)})| = |f'(e^{i(\theta_s(\tau)+2\pi s/n)})|,$$

which means that the function $|f'(e^{i\theta})|$ is periodic with the period $2\pi/n$, except possibly a finite number of points.

Since, as we mentioned above, Ω is a Smirnov domain, we have

$$\begin{aligned}\log|f'(e^{2\pi i/n}z)| &= \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-e^{2\pi i/n}z|^2}\,|f'(e^{i\theta})|\,d\theta \\ &= \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{1-|z|^2}{|e^{i(\theta-2\pi/n)}-z|^2}\,|f'(e^{i(\theta-2\pi/n)})|\,d(\theta-2\pi/n) \\ &= \log|f'(z)|.\end{aligned}$$

Since $f'(z)\neq 0$ for $z\in\mathbb{D}$, the functions $\log f'(e^{2\pi i/n}z)$ and $\log f'(z)$ are analytic on $\mathbb{D}$. Since these functions have equal real parts, we must have

$$\log f'(e^{2\pi i/n}z) = \log f'(z) + i\mu \quad \text{with some } \mu\in\mathbb{R}.$$

Hence,

$$f'(e^{2\pi i/n}z) = e^{i\mu} f'(z) \quad \text{for all } z\in\mathbb{D}. \tag{15}$$

Since $f'(0) = e^{i\mu} f'(0)$, we obtain $e^{i\mu} = 1$ and therefore equation (15) becomes

$$f'(e^{2\pi i/n} z) = f'(z) \quad \text{for all } z \in \mathbb{D}. \tag{16}$$

Integrating equation (16) from 0 to z and then multiplying by $e^{2\pi i/n}$, we obtain

$$f(e^{2\pi i/n} z) - f(0) = e^{2\pi i/n}(f(z) - f(0)).$$

This implies that the image domain $\Omega = f(\mathbb{D})$ is invariant under rotation by the angle $2\pi/n$ around the point z_0, as required. $\square$

Remark 1. Let D_n denote a regular polygon with $n \geqslant 3$ sides centered at the origin $z = 0$, with perimeter 1. One can easily find that all boundary arcs $\alpha \subset \partial D_n$ of length $\frac{1}{n}$ have equal harmonic measures $\omega(0, \alpha, D_n) = \frac{1}{n}$. Furthermore, the length $\frac{1}{n}$ is the smallest length with this property; i.e., for every l, $0 < l < \frac{1}{n}$, there exist two arcs $\alpha_1 \subset \partial D_n$ and $\alpha_2 \subset \partial D_n$ such that length(α_1) = length$(\alpha_2) = l$ but $\omega(0, \alpha_1, D_n) \neq \omega(0, \alpha_2, D_n)$.

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