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A PURITY THEOREM FOR QUADRATIC SPACES

Abstract. It is proved a purity theorem for quadratic spaces over semi-local regular integral domain containing a field of odd characteristic. This theorem extends to the semi-local case the corresponding results proven previously by the author and by the author jointly with K. Pimenov. To get this result we extend the purity theorem of Ojanguren–Panin to this more general setting.

§1. Introduction

Let A be a commutative ring and P be a finitely generated projective A-module. An element $v \in P$ is called unimodular if the A-submodule vA of P splits off as a direct summand. If $P = A^n$ and $v = (a_1, a_2, \dots, a_n)$ then v is unimodular if and only if $a_1A + a_2A + \cdots + a_nA = A$.

Let $\frac{1}{2} \in A$. A quadratic space over A is a pair (P, α) consisting of a finitely generated projective A-module P and an A-isomorphism $\alpha : P \to P^*$ satisfying $\alpha = \alpha^*$, where $P^* = \mathrm{Hom}_R(P, R)$. Two spaces (P, α) and (Q, β) are *isomorphic* if there exists an A-isomorphism $\varphi : P \to Q$ such that $\alpha = \varphi^* \circ \beta \circ \varphi$.

Let (P, φ) be a quadratic space over A. One says that it is *isotropic* over A, if there exists a unimodular $v \in P$ with $\varphi(v) = 0$.

Recall the notion of unramified spaces. Let R be a Noetherian domain and K be its quotient field. Recall that a quadratic space (W, ψ) over K is *unramified over R* if for every height one prime ideal $\mathfrak{p}$ of R there exists a quadratic space $(V_{\mathfrak{p}}, \varphi_{\mathfrak{p}})$ over $R_{\mathfrak{p}}$ such that the spaces $(V_{\mathfrak{p}}, \varphi_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} K$ and (W, ψ) are isomorphic.

The main aim of the present paper is to prove the following *purity theorem for quadratic spaces* over semi-local regular integral domain. This theorem extends to the semi-local case the corresponding results [7, Corollary 1] and [8, Corollary 3.1].

Key words and phrases: quadratic form, regular local ring, isotropic vector, Grothendieck–Serre conjecture.

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Theorem 1 (Main)**.** *Let R be a semi-local regular integral domain containing a field of odd characteristic. Let K be the field of fractions of R. Let (W, ψ) be a quadratic space over K which is unramified over R. Then there exists a quadratic space (V, φ) over R extending the space (W, ψ), that is the spaces $(V, \varphi) \otimes_R K$ and (W, ψ) are isomorphic.*

As indicated in [7, Remark 4] the main difficulty in proving Theorem 1 is in an extension of the purity theorem [4, Theorem A] to that semi-local case. Let W be the Witt functor of quadratic spaces on the category of commutative rings. Here is the desired extension of the purity theorem [4, Theorem A].

Theorem 2 (A purity theorem)**.** *Let R be a semi-local regular integral domain containing a field k of odd characteristic. Let K be the field of fractions of R. Then the map $W(R) \to W(K)$ is injective and the sequence*

$$\{0\} \to W(R) \to W(K) \xrightarrow{\sum r_{\mathfrak{p}}} \bigoplus_{\mathfrak{p}} W(K)/W(R_{\mathfrak{p}}) \tag{1}$$

is exact. Here $\mathfrak{p}$ runs over all height one prime ideals of R and each $r_{\mathfrak{p}}$ is the natural map (the projection to the factor group).

Remark 3. We will use often below the following well-known terminology. Let R and K be as in the latter Theorem. Let $\mathfrak{p}$ be a height one prime ideal in R. Recall that an element $a \in W(K)$ is *unramified at* $\mathfrak{p}$ if it can be lifted up to an element of $W(R_{\mathfrak{p}})$.

An element $a \in W(K)$ is *R-unramified* if it is unramified at every height one prime ideal $\mathfrak{p}$ of R. So, a part of the latter Theorem can be restated as follows: each R-unramified element of $W(K)$ can be lifted up to an element in $W(R)$.

Derivation Theorem 1 from Theorem 2. By Theorem 2 there exist a quadratic space (V, φ) over R and an integer $n \geqslant 0$ such that $(V, \varphi) \otimes_R K \cong (W, \psi) \perp \mathbb{H}_K^n$, where $\mathbb{H}_K$ is a hyperbolic plane. If $n > 0$ then the space $(V, \varphi) \otimes_R K$ is isotropic. By [12, Theorem 5.1] the space (V, φ) is isotropic too. Thus $(V, \varphi) \cong (V', \varphi') \perp \mathbb{H}_R$ for a quadratic space (V', φ') over R. Now Witt's Cancellation theorem over a field [3, Chap. I, Theorem 4.2] shows that $(V', \varphi') \otimes_R K \cong (W, \psi) \perp \mathbb{H}_K^{n-1}$. Repeating this procedure several times we may assume that $n = 0$, which means that $(V, \varphi) \otimes_R K \cong (W, \psi)$. □

Proof of Theorem 2. The so called geometric case (see below) is an output of general machineries developed in [5, 1] and [2]. We start with some recollections. Let k be a field. Recall that objects of the category $SmOp/k$ are pairs (X,U), where X is a smooth k-variety and U is its Zarisky open subset. A morphism between (X,U) and (X',U') is a morphism $f: X \to X'$ which takes U to U'. A cohomology theory on $SmOp/k$ in the sense of Panin–Smirnov [10, Sec. 1] is a contra-variant functor

$$A: SmOp/k \to Gr - Ab$$

to the category of graded abelian groups together with functor transformations $\partial_{X,U}: A^n(U) \to A^{n+1}(X,U)$, which satisfies homotopy invariance, étale excision and for any pair $(X,U) \in SmOp/k$ the sequence of abelian groups

$$\dots \to A^n(X) \to A^n(U) \xrightarrow{\partial_{X,U}} A^{n+1}(X,U) \to A^{n+1}(X) \to A^{n+1}(U) \to \dots$$

is long exact. The Balmer–Witt theory $(X, X-Z) \mapsto \oplus W^n_Z(X)$ together with functor transformations $\partial_{X,X-Z}: W^n(U) \to W^{n+1}_Z(X)$ is a cohomology theory in the sense above (see [2] for details). On the category of affine Noetherian regular schemes the functor W^0 coincides with the classical functor of Witt groups W. For each discrete valuation ring A, the scheme $V = \text{Spec}(A)$, its closed point v and its general point ν one has an isomorphism $W(\nu)/W(V) = W^0(\nu)/W^0(V) \cong W^1_v(V)$ induced by the boundary map $\partial_{V,V-v}: W^0(\nu) \to W^1_v(V)$.

The geometric case of Theorem 2. Let R be the semi-local ring of finitely many **closed points** on a k-smooth irreducible affine k-variety X. Let $U = \text{Spec}(R)$ and let η be the generic point of U. Write $U^{(1)}$ for the set of codimension one points of U. By [5, Theorem 9.1] the comlex

$$0 \to W^0(U) \xrightarrow{\eta^*} W^0(\eta) \xrightarrow{\partial} \oplus_{x \in U^{(1)}} W^1_x(U) \tag{2}$$

is exact, where for each point $x \in U^{(1)}$ and its local ring R_x the group $W^1_x(U)$ is defined as the group $W^1_x(\text{Spec}(R_x))$. Let K be the fraction field of R. As mentioned just above one has equalities $W^0(U) = W(R)$, $W^0(\eta) = W(K)$ and $W^1_x(U) = W(K)/W(R_x)$, where x is an arbitrary codimension one point in U. Combining these arguments all together we see that the sequence (1) is exact if the ring R is as in this paragraph. Particularly, the map $W(R) \to W(K)$ is injective in this case.

The quasi-geometric case of Theorem 2. Let X be a k-smooth irreducible affine k-variety and $\xi_1, \dots, \xi_n$ be points of the scheme $\text{Spec}(k[X])$ such that for each pair r, s the point ξ_r is not in the closure $\overline{\{\xi_s\}}$ of ξ_s. Let R be the semi-local ring $\mathcal{O}_{X,\xi_1,\dots,\xi_n}$ of scheme points $\xi_1, \dots, \xi_n$ of $\text{Spec}(k[X])$.

First, prove the injectivity of the map $W(R) \to W(K)$. Our assumption on points ξ_r's yield the following: one can choose closed points $x_s \in \overline{\{\xi_s\}}$ such that for each $r \neq s$ the point x_r is not in $\overline{\{\xi_s\}}$. Particularly, for each $r \neq s$ one has $x_r \neq x_s$. Set $\tilde{R} = \mathcal{O}_{X,x_1,\dots,x_n}$ Moreover, for a given element $\alpha \in W(R)$ one can choose the points x_r's such that additionally there exists an $\tilde{\alpha} \in W(\tilde{R})$ which is a lift of α. Let $\alpha \in W(R)$ be an element vanishing in $W(K)$. Find certain points $x_s \in \overline{\{\xi_s\}}$ as just above and an element $\tilde{\alpha} \in W(\tilde{R})$ which is a lift of α. The element $\tilde{\alpha} \in W(\tilde{R})$ vanishes in $W(K)$ and the map $W(\tilde{R}) \to W(K)$ is injective by the geometric case of Theorem 2. Thus, $\tilde{\alpha} = 0$ and hence $\alpha = 0$.

Prove now the exactness of the complex (1) at the term $W(K)$ for the semi-local ring $R = \mathcal{O}_{X,\xi_1,\dots,\xi_n}$. To do this take an element $\alpha \in W(k(X))$ which is unramified at each irreducible divisor D containing at least one of the points ξ_r. We have to prove that the element α is in the image of $W(R)$.

Clearly, there is a non-zero $f \in k[X]$ and an element $\tilde{\alpha} \in W(k[X_f])$ which is a lift of α. Write down the divisor $\mathrm{div}(f) \in \mathrm{Div}(X)$ in the form $\mathrm{div}(f) = \Sigma m_i D_i + \Sigma n_j D'_j$ such that for each index i there is an index r with $\xi_r \in D_i$ and for any index j and any index r the point ξ_r does not belong to D'_j. There is an element $g \in k[X]$ such that for any index j the D'_j is contained in the closed subset $\{g = 0\}$ and g does not belong to any of ξ_r'r. Replacing X with X_g we may and will assume that $\tilde{\alpha} \in W(k[X_f])$, $\mathrm{div}(f) = \Sigma m_i D_i$ and α is unramified at each irreducible divisor D_i. Hence α is unramified at each height one prime ideal of $k[X]$. Our assumption on points ξ_r's yield the following: one can choose closed points $x_s \in \overline{\{\xi_s\}}$ such that for each $r \neq s$ the point x_r is not in $\overline{\{\xi_s\}}$. Particularly, for each $r \neq s$ one has $x_r \neq x_s$. The element α is unramified at each height one prime ideal of $k[X]$. Thus, by the geometric case of Theorem 2 the element α is in the image of $W(\mathcal{O}_{X,x_1,\dots,x_n})$. So, the element α is in the image of $W(\mathcal{O}_{X,\xi_1,\dots,\xi_n}) = W(R)$. The proof of the quasi-geometric case of Theorem 2 is completed.

The general case of Theorem 2. Clearly, we may assume that k is a prime field and hence k is perfect. It follows from Popescu's theorem [11, 13] that R is a filtered inductive limit of smooth k-algebras R_α. Modifying the inductive system R_α if necessary, we can assume that each R_α is integral. For each maximal ideal $\mathfrak{m}_i$ in R $(i = 1, \dots, n)$ set $\mathfrak{p}_i = \phi_\alpha^{-1}(\mathfrak{m}_i)$. The homomorphism $\phi_\alpha : R_\alpha \to R$ induces a homomorphism of semi-local rings

$\varphi_\alpha : (R_\alpha)_{\mathfrak{p}_1,\dots,\mathfrak{p}_n} \to R$. Since this moment we will write A_α for $(R_\alpha)_{\mathfrak{p}_1,\dots,\mathfrak{p}_n}$. Thus, R is a filtered inductive limit of regular semi-local k-algebras A_α. And for each index α the k-algebra A_α is quasi-geometric in the sense above. These observations yield the following intermediate result
$(**)$ the sequence (2) is exact for each ring A_α. Particularly,
$(***)$ if K_α is the fraction field of A_α then the map $W(A_\alpha) \to W(K_\alpha)$ is injective.

Let now K be the field of fractions of R and, for each index α, let K_α be the field of fractions of A_α. For each index α let $\mathfrak{a}_\alpha$ be the kernel of the map $\varphi_\alpha : A_\alpha \to R$ and $B_\alpha = (A_\alpha)_{\mathfrak{a}_\alpha}$. Clearly, for each index α, K_α is the field of fractions of B_α. The composition map $A_\alpha \to R \to K$ factors through B_α. Since R is a filtering direct limit of the A_α's we see that K is a filtering direct limit of the B_α's. We will write ψ_α for the canonical morphism $B_\alpha \to K$. The intermediate result $(***)$ yields now the injectivity of the maps $W(A_\alpha) \to W(B_\alpha)$. *Hence the map* $W(R) \to W(K)$ *is injective.*

It remains to prove the exactness of the sequence (2) at the term $W(K)$. We need in the following two lemmas.

Lemma 4. *For each index* α *the group map* $W(B_\alpha) \to W(K_\alpha)$ *is injective.*

Proof. Just apply the general case of Theorem 2 to the k-algebra B_α. □

Lemma 5. *Let* $a \in W(K)$ *be an* R*-unramified element. Then there exists an index* α *and an element* $b_\alpha \in W(B_\alpha)$ *such that* $\psi_\alpha(b_\alpha) = b$ *and the class* $b_\alpha \in W(K_\alpha)$ *is* A_α*-unramified.*

Proof. Repeat literally respecting arguments from the proof of [4, Proof of Theorem A]. They work for the semi-local case as well. □

We complete the proof of the general case of Theorem 2 as follows. Let $a \in W(K)$ be an R-unramified element. We have to check that it comes from $W(R)$. By Lemma 5 there exists an index α and an element $b_\alpha \in W(B_\alpha)$ such that $\psi_\alpha(b_\alpha) = b$ and the class $b_\alpha \in W(K_\alpha)$ is A_α-unramified. For this index α consider a commutative diagram of k-algebras

$$\begin{array}{ccc} A_\alpha & \xrightarrow{\varphi_\alpha} & R \\ \downarrow & & \downarrow \\ B_\alpha & \xrightarrow{\psi_\alpha} & K \\ \downarrow & & \\ K_\alpha & & \end{array}$$

The class $b_\alpha \in W(K_\alpha)$ is A_α-unramified. Hence by the statement $(**)$ there exists an element $a_\alpha \in W(A_\alpha)$ such that $b_\alpha = a_\alpha$ in $W(K_\alpha)$. By Lemma 4 one has an equality $b_\alpha = a_\alpha$ in $W(B_\alpha)$. Hence $b \in W(K)$ coincides with the image of the element $\varphi_\alpha(a_\alpha)$ in $W(K)$. Thus, the sequence (1) is exact at the term $W(K)$. The Theorem 2 is proved. □

References

1. P. Balmer, *Derived Witt groups of a scheme.* — J. Pure Appl. Algebra **141** (1999), 101–129.
2. P. Balmer, *Witt cohomology, Mayer-Vietoris, homotopy invariance and the Gersten conjecture.* — K-Theory **23**, No. 1 (2001), 15–30.
3. T. Y. Lam, *Introduction to quadratic forms over fields. Graduate Studies in Mathematics*, Vol. 67, American Mathematical Society, Providence, RI (2005).
4. M. Ojanguren, I. Panin, *A purity theorem for the Witt group.* — Ann. Sci. Ecole Norm. Sup. (4) **32**, No. 1 (1999), 71–86.
5. I. A. Panin, *Nice triples and moving lemmas for motivic spaces.* — Izv. RAN. Ser. Mat. **83**, No. 4 (2019), 158–193; Izv. Math. **83**, No. 4 (2019), 796–829.
6. I. Panin, U. Rehmann, *A variant of a theorem by Springer.* — Algebra Analyz **19** (2007), 117–125.
7. I. Panin, *Rationally isotropic quadratic spaces are locally isotropic.* — Invent. Math. **176** (2009), 397–403.
8. I. Panin, K. Pimenov, *Rationally isotropic quadratic spaces are locally isotropic: II.* — Documenta Mathematica, Extra Volume: Andrei A. Suslin's Sixtieth Birthday, 515–523, (2010).
9. I. Panin, K. Pimenov, *Rationally isotropic quadratic spaces are locally isotropic.* III. — Algebra Analiz **27**, No. 6 (2015), 234–241.
10. I. Panin, *Oriented cohomology theories of algebraic varieties.* — K-Theory **30** (2003), 265–314.
11. D. Popescu, *General Néron desingularization and approximation.* — Nagoya Math. J. **104** (1986), 85–115.
12. S. Scully, *The Artin–Springer theorem for quadratic forms over semi-local rings with finite residue fields.* — Proc. Amer. Math. Soc. **146** (2018), 1–13.
13. R. G. Swan, *Néron–Popescu desingularization.* — In Algebra and Geometry (Taipei, 1995), Lect. Algebra Geom., Vol. 2, Internat. Press, Cambridge, MA (1998), 135–192.

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