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ON THE MAXIMAL L_p-L_q REGULARITY THEOREM FOR THE LINEARIZED ELECTRO-MAGNETIC FIELD EQUATIONS WITH INTERFACE CONDITIONS

Abstract. This paper deals with the maximal L_p-L_q regularity theorem for the linearized electro-magnetic field equations with interface conditions and perfect wall condition. This problem is motivated by linearization of the coupled magnetohydrodynamics system, which generates two separate problems. The first problem is associated with the well studied Stokes system. Another problem related to the magnetic field is studied in this paper. The maximal L_p-L_q regularity theorem for the Stokes equations with interface and non-slip boundary conditions has been proved by Pruess and Simonett [15], Maryani and Saito [12]. Combination of these results and the result obtained in this paper yields local well-posedness for MHD problem in the case of two incompressible liquids separated by a closed interface. We plan to prove it in a forthcoming paper.

The main part of the paper is devoted to proving the existence of $\mathcal{R}$ bounded solution operators associated with the generalized resolvent problem. The maximal L_p-L_q regularity is established by applying the Weis operator valued Fourier multiplier theorem.

§1. Introduction

First of all, we formulate the magneto-hydro-dynamic (MHD) equations in the two liquids case. Let Ω be a bounded domain in the N-dimensional Euclidean space $\mathbb{R}^N$ and let Ω_+ be a subdomain of Ω, $\Omega_- = \Omega \setminus \overline{\Omega_+}$. The boundary of Ω we denote by S, let it be a smooth closed surface. The boundary of Ω_+ is a closed surface Γ. We assume that $\operatorname{dist}(\Gamma, S) = \inf\{|x-y| \mid x \in \Gamma, y \in S\} \geqslant d > 0$. Let Ω_{t+} and Γ_t be the evolution of Ω_+ and Γ for $t > 0$, $\Omega_{t-} = \Omega \setminus (\Omega_{t+} \cup \Gamma_t)$. Let $\mathbf{n}_0$, $\mathbf{n}_t$, and $\mathbf{n}$ be unit outer normals to Γ, Γ_t and S, respectively ($\mathbf{n}_t$ are oriented from Ω_{t+} to Ω_{t-}).

Key words and phrases: L_p-L_q maximal regularity, linearized electro-magneto field equations, interface condition.

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For any given functions $h_\pm(x)$ defined in $\Omega_{t\pm}$, we denote by h the function $h(x) = h_\pm(x)$ for $x \in \Omega_{t\pm}$, $t \geqslant 0$ ($\Omega_{0\pm} = \Omega_\pm$). The jump of h across Γ is defined by

$$[[h]](x_0) = \lim_{\substack{x\to x_0\\ x\in\Omega_{t+}}} h_+(x) - \lim_{\substack{x\to x_0\\ x\in\Omega_{t-}}} h_-(x)$$

for every point $x_0 \in \Gamma_t$. We also use the notations $\dot\Omega_t = \Omega_{t+} \cup \Omega_{t-}$, $\dot Q_T = \{(x,t)\big| t \in (0,T), \quad x \in \dot\Omega_t\}$, $G_T = \{(x,t)\big| t \in (0,T), \quad x \in \Gamma_t\}$. The MHD equations in the case of two liquids are as follows:

$$\begin{aligned}
&\rho(\partial_t\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{v})-\mathrm{Div}\,(\mathbf{T}(\mathbf{v},\mathfrak{p})+\mathbf{T}_M(\mathbf{H}))=0,\ \mathrm{div}\,\mathbf{v}=0\ \text{ in } \dot Q_T\ ,\\
&[[(\mathbf{T}(\mathbf{v},\mathfrak{p})+\mathbf{T}_M(\mathbf{H}))\mathbf{n}_t]]=\sigma\mathcal{H}(\Gamma_t)\mathbf{n}_t,\ [[\mathbf{v}]]=0,\ V_{\Gamma_t}=\mathbf{v}\cdot\mathbf{n}_t\ \text{ on } G_T\ ,\\
&\mu\partial_t\mathbf{H}+\mathrm{Div}\,\{\alpha^{-1}\mathrm{curl}\,\mathbf{H}-\mu(\mathbf{v}\otimes\mathbf{H}-\mathbf{H}\otimes\mathbf{v})\}=0,\ \mathrm{div}\,\mathbf{H}=0\ \text{ in } \dot Q_T\ ,\\
&[[\{\alpha^{-1}\mathrm{curl}\,\mathbf{H}-\mu(\mathbf{v}\otimes\mathbf{H}-\mathbf{H}\otimes\mathbf{v})\}\mathbf{n}_t]]=0\ \text{ on } G_T\ ,\\
&[[\mu\mathbf{H}\cdot\mathbf{n}_t]]=0,\ [[\mathbf{H}-<\mathbf{H},\mathbf{n}_t>\mathbf{n}_t]]=0\ \text{ on } G_T\ ,\\
&\mathbf{v}=0,\ \mathbf{n}\cdot\mathbf{H}_-=0,\ (\mathrm{curl}\,\mathbf{H}_-)\mathbf{n}=0\ \text{ on } S\times(0,T),\\
&(\mathbf{v},\mathbf{H})|_{t=0}=(\mathbf{v}_0,\mathbf{H}_0)\quad\text{in } \dot\Omega.
\end{aligned}\tag{1.1}$$

Here, $\mathbf{v} = \mathbf{v}_\pm = (v_{\pm 1}(x,t),\dots,v_{\pm N}(x,t))^\top$ is the velocity vector field, $M^\top$ stands for the transposed M, $\mathfrak{p} = \mathfrak{p}_\pm(x,t)$ is the pressure field, $\mathbf{H} = \mathbf{H}_\pm = (H_{\pm 1}(x,t),\dots,H_{\pm N}(x,t))^\top$ is the magnetic field, while $\mathbf{v}_0$ and $\mathbf{H}_0$ are prescribed initial data for $\mathbf{v}$ and $\mathbf{H}$, respectively. Furthermore, $\mathbf{T} = \nu_\pm\mathbf{D}(\mathbf{v}_\pm) - \mathfrak{p}_\pm\mathbf{I}$ is the viscous stress tensor, $\mathbf{D}(\mathbf{v}_\pm) = \nabla\mathbf{v}_\pm + (\nabla\mathbf{v}_\pm)^\top$ is the doubled deformation tensor whose (i,j) component is $\partial_j v_{\pm i} + \partial_i v_{\pm j}$ with $\partial_i = \partial/\partial x_i$, $\mathbf{I}$ is the $N\times N$ unit matrix, $\mathbf{T}_M(\mathbf{H}) = \mathbf{T}_M(\mathbf{H}_\pm) = \mu_\pm(\mathbf{H}_\pm\otimes\mathbf{H}_\pm - \frac{1}{2}|\mathbf{H}_\pm|^2\mathbf{I})$ is the magnetic stress tensor, $\mathrm{curl}\,\mathbf{v} = \mathrm{curl}\,\mathbf{v}_\pm = (\nabla\mathbf{v}_\pm)^\top - (\nabla\mathbf{v}_\pm)$ is the doubled rotation tensor whose (i,j) component is $\partial_i v_{\pm j} - \partial_j v_{\pm i}$ (see for example [6]), V_{Γ_t} is the velocity of the evolution of Γ_t in the direction of $\mathbf{n}_t$, and $\mathcal{H}(\Gamma_t)$ is the doubled mean curvature of Γ_t that is given by the relation $\mathcal{H}(\Gamma_t)\mathbf{n}_t = \Delta_{\Gamma_t}x$, where Δ_{Γ_t} is the Laplace Beltrami operator on Γ_t. The positive constants $\rho = \rho_\pm$, $\mu = \mu_\pm$, $\nu = \nu_\pm$, and $\alpha = \alpha_\pm$ correspond to the mass density, the magnetic permeability, the kinematic viscosity, and conductivity, respectively. By σ we denote the coefficient of the surface tension, it is also assumed to be a positive constant. For any matrix field $\mathbf{K}$ with (i,j) component K_{ij}, the quantity $\mathrm{Div}\,K$ is an N-vector -functions with the ith component $\sum\limits_{j=1}^N \partial_j K_{ij}$. For any vector-functions $\mathbf{u} = (u_1,\dots,u_N)^\top$ and $\mathbf{w} = (w_1,\dots,w_N)^\top$,

$\operatorname{div} \mathbf{u} = \sum_{j=1}^N \partial_j u_j$, $\mathbf{u}\cdot\nabla\mathbf{w}$ is a N-vector with the ith component $\sum_{j=1}^N u_j\partial_j w_i$, and $\mathbf{u}\otimes\mathbf{w}$ is a $N\times N$ matrix with the (i,j)th components u_iw_j.

Note that

$$\begin{aligned}\Delta\mathbf{v} = -\operatorname{Div}\operatorname{curl}\mathbf{v} + \nabla\operatorname{div}\mathbf{v}, \quad \operatorname{Div}(\mathbf{v}\otimes\mathbf{H} - \mathbf{H}\otimes\mathbf{v})\\ = \mathbf{v}\operatorname{div}\mathbf{H} - \mathbf{H}\operatorname{div}\mathbf{v} + \mathbf{H}\cdot\nabla\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{H}.\end{aligned} \tag{1.2}$$

In the three dimensional case, we have

$$\operatorname{rot}\operatorname{rot}\mathbf{H} = \operatorname{Div}\operatorname{curl}\mathbf{H}, \quad \operatorname{rot}(\mathbf{v}\times\mathbf{H}) = \operatorname{Div}(\mathbf{v}\otimes\mathbf{H} - \mathbf{H}\otimes\mathbf{v}),$$

$(\operatorname{curl}\mathbf{H})\mathbf{n}_t$ corresponds to the tangential components of $\operatorname{curl}\mathbf{H}$ on Γ_t. In the three dimensional case, when the domain Ω_{t-} is a vacuum region MHD problem has been studied by Solonnikov [21], [22], Padula and Solonnikov [14], Frolova and Solonnikov [20]. In particular, L_p estimates to the corresponding linear problem has been obtained in [22]. Corresponding to (1.1) linear problem for magnetic field in the three dimensional case has been studied by Frolova in [9], where the unique solvability in Sobolev–Slobodetskii spaces $W_2^{2+l,1+l/2}$ was proved.

System (1.1) is overdetermined, because we have too many equations for the magnetic field $\mathbf{H}$. In this paper, we consider the equivalent system of MHD equations:

$$\begin{aligned}\rho(\partial_t\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) - \operatorname{Div}(\mathbf{T}(\mathbf{v},\mathfrak{p}) + \mathbf{T}_M(\mathbf{H})) = 0,\ \operatorname{div}\mathbf{v} = 0 \quad &\text{in } \dot{Q}_T\ ,\\ [[(\mathbf{T}(\mathbf{v},\mathfrak{p}) + \mathbf{T}_M(\mathbf{H}))\mathbf{n}_t]] = \sigma H(\Gamma_t)\mathbf{n}_t,\ [[\mathbf{v}]] = 0,\ V_{\Gamma_t} = \mathbf{v}\cdot\mathbf{n} \quad &\text{on } G_T\ ,\\ \mu\partial_t\mathbf{H} - \alpha^{-1}\Delta\mathbf{H} - \operatorname{Div}\mu(\mathbf{v}\otimes\mathbf{H} - \mathbf{H}\otimes\mathbf{v}) = 0 \quad &\text{in } \dot{Q}_T\ ,\\ [[\{\alpha^{-1}\operatorname{curl}\mathbf{H} - \mu(\mathbf{v}\otimes\mathbf{H} - \mathbf{H}\otimes\mathbf{v})\}\mathbf{n}_t]] = 0,\ [[\mu\operatorname{div}\mathbf{H}]] = 0 \quad &\text{on } G_T\ ,\\ [[\mu\mathbf{H}\cdot\mathbf{n}_t]] = 0,\ [[\mathbf{H} - <\mathbf{H},\mathbf{n}_t>\mathbf{n}_t]] = 0 \quad &\text{on } G_T\ ,\\ \mathbf{v} = 0,\ \mathbf{n}\cdot\mathbf{H}_- = 0,\ (\operatorname{curl}\mathbf{H}_-)\mathbf{n} = 0 \quad &\text{on } S\times(0,T),\\ (\mathbf{v},\mathbf{H})|_{t=0} = (\mathbf{v}_0,\mathbf{H}_0) \quad &\text{in } \dot{\Omega}.\end{aligned} \tag{1.3}$$

Namely, instead of the conditions $\operatorname{div}\mathbf{H}_\pm = 0$ in $\Omega_\pm$, we set the condition: $[[\mu\operatorname{div}\mathbf{H}]] = 0$ on Γ. In Appendix, we prove that if the solution to (1.3) satisfies the condition $\operatorname{div}\mathbf{H} = 0$ at the initial moment of time, then $\operatorname{div}\mathbf{H} = 0$ in $\dot{\Omega}_t$ for any $t > 0$ as long as the solution exists. It yields equivalence of the problems (1.1) and (1.3).

To prove local well-posedness of Eq. (1.3), the key step is to show the maximal L_p-L_q regularity for the linearized equations. Since the coupling of $\mathbf{v}$ and $\mathbf{H}$ in Eq. (1.3) is of lower order, it is sufficient to consider the Stokes equations with interface condition and non-slip boundary condition and linearized equations for the magnetic field separately.

The Stokes equations with interface and non-slip conditions have been studied by Pruess and Simonett [15], Solonnikov [23], and also by Maryani and Saito [12] (by different approaches). Parabolic systems were studied by Zhitarashu [10] and Zhitarashu and Eidelman [11] in the L_p framework (where conjugation problems also were considered), by Denk, Hieber and Pruss [4] in the L_p-L_q framework. At the present paper, we prove the maximal L_p-L_q regularity for the system of heat equations with interface conditions and perfect wall conditions, which corresponds to the linear equations for the magnetic field and has the form

$$\begin{aligned}
&\mu\partial_t\mathbf{H} - \alpha^{-1}\Delta\mathbf{H} = \mathbf{f} && \text{in } \dot\Omega\times(0,\infty),\\
&[[\alpha^{-1}\mathrm{curl}\,\mathbf{H}]]\mathbf{n}_0 = \mathbf{g}', \quad [[\beta\mathrm{div}\,\mathbf{H}]] = g_N && \text{on } \Gamma\times(0,\infty),\\
&[[\mathbf{H} - \langle\mathbf{H},\mathbf{n}_0\rangle\mathbf{n}_0]] = \mathbf{h}', \quad [[\beta\mathbf{H}\cdot\mathbf{n}_0]] = h_N && \text{on } \Gamma\times(0,\infty),\\
&(\mathrm{curl}\,\mathbf{H}_-)\mathbf{n} = \mathbf{g}_-, \quad \mathbf{n}\cdot\mathbf{H}_- = h_- && \text{on } S\times(0,\infty),\\
&\mathbf{H}|_{t=0} = \mathbf{H}_0 && \text{in } \dot\Omega.
\end{aligned}\tag{1.4}$$

Here $\beta = \beta_\pm$, $\alpha = \alpha_\pm$, $\mu = \mu_\pm$ are positive constants. Henceforth, we use the notation $\mathbf{g} = (\mathbf{g}', g_N)$, $\mathbf{h} = (\mathbf{h}', h_N)$. We assume that the domain Ω satisfies the following conditions.

Definition 1.1. Let $1 < r < \infty$. We say that Ω is a uniform $W_r^{3-1/r}$ domain, if there exist positive constants α_i $(i = 1,2,3)$, and K such that the following two assertions hold:

- For any $x_0 = (x_{01},\dots,x_{0N}) \in \Gamma$, there exist a coordinate number j and a function $h(x_j') \in W_r^{3-1/r}(B'_{\alpha_1}(x'_{0j}))$ such that

$$\begin{aligned}
&\|h\|_{W_r^{3-1/r}(B'_{\alpha_1}(x'_{0j}))} \leqslant K,\\
&\Omega\cap B_{\alpha_2}(x_0) = \{x\in\mathbb{R}^N \mid -\alpha_3 + h(x_j') < x_j < h(x_j') + \alpha_3\\
&\qquad\qquad (x_j' \in B'_{\alpha_1}(x'_{0j}))\}\cap B_{\alpha_2}(x_0),\\
&\Gamma\cap B_{\alpha_2}(x_0) = \{x\in\mathbb{R}^N \mid x_j = h(x_j')\ \ (x_j'\in B'_{\alpha_1}(x'_{0j}))\}\cap B_{\alpha_2}(x_0).
\end{aligned}\tag{1.5}$$

- For any $x_0 = (x_{01}, \dots, x_{0N}) \in S$ there exist a coordinate number j and a function $h(x'_j) \in W_r^{3-1/r}(B'_{\alpha_1}(x'_{0j}))$ such that

$$\begin{aligned}
&\|h\|_{W_r^{3-1/r}(B'_{\alpha_1}(x'_{0j}))} \leqslant K,\\
&\Omega \cap B_{\alpha_2}(x_0) = \{x \in \mathbb{R}^N \mid x_j > h(x'_j) \ \ (x'_j \in B'_{\alpha_1}(x'_{0j}))\} \cap B_{\alpha_2}(x_0),\\
&S \cap B_{\alpha_2}(x_0) = \{x \in \mathbb{R}^N \mid x_j = h(x'_j) \ \ (x'_j \in B'_{\alpha_1}(x'_{0j}))\} \cap B_{\alpha_2}(x_0).
\end{aligned} \tag{1.6}$$

Here,

$$\begin{aligned}
x'_j &= (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_N),\\
x'_{0j} &= (x_{01}, \dots, x_{0j-1}, x_{0j+1}, \dots, x_{0N}),\\
B'_{\alpha_1}(x'_{0j}) &= \{x'_j \in \mathbb{R}^{N-1} \mid |x'_j - x'_{0j}| < \alpha_1\}
\end{aligned}$$

and

$$B_{\alpha_2}(x_0) = \{x \in \mathbb{R}^N \mid |x - x_0| < \alpha_2\}.$$

Theorem below is the main result of the present paper.

Theorem 1.2. *Let $1 < p, q < \infty$, $2/p + 1/q \neq 1$ and $\neq 2$. Let Ω be a uniform $W_r^{3-1/r}$ domain with $N < r < \infty$. Assume that there exists a constant $\gamma > 0$ such that the given functions $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$, $\mathbf{g}_-$, and h_- in* (1.4) *satisfy the following conditions*: $e^{-\gamma t}\mathbf{f} \in L_p(\mathbb{R}, L_q(\Omega)^N)$,

$$\begin{aligned}
e^{-\gamma t}\mathbf{g} &\in L_p(\mathbb{R}, H_q^1(\Omega)^N) \cap H_p^{1/2}(\mathbb{R}, L_q(\Omega)^N),\\
e^{-\gamma t}\mathbf{h} &\in L_p(\mathbb{R}, H_q^2(\Omega)^N) \cap H_p^1(\mathbb{R}, L_q(\Omega)^N),\\
e^{-\gamma t}\mathbf{g}_- &\in L_p(\mathbb{R}, H_q^1(\Omega_-)^{N-1}) \cap H_p^{1/2}(\mathbb{R}, L_q(\Omega_-)^{N-1}),\\
e^{-\gamma t}h_- &\in L_p(\mathbb{R}, H_q^2(\Omega_-)) \cap H_p^1(\mathbb{R}, L_q(\Omega_-)).
\end{aligned}$$

Let at the initial moment of time $\mathbf{H}_0 \in B_{q,p}^{2(1-1/p)}(\dot\Omega)$ and the following compatibility conditions hold:

$$\begin{aligned}
&[[\alpha^{-1}\mathrm{curl}\,\mathbf{H}_0]]\mathbf{n}_0 = \mathbf{g}'|_{t=0}, \quad [[\beta\mathrm{div}\,\mathbf{H}_0]] = g_N|_{t=0} \ \ \textit{on}\ \Gamma,\\
&(\mathrm{curl}\,\mathbf{H}_{0-})\mathbf{n} = \mathbf{g}_-|_{t=0} \quad \textit{on} \quad S
\end{aligned}$$

if $1 > 2/p + 1/q$, and

$$\begin{aligned}
&[[\mathbf{H}_0 - < \mathbf{H}_0, \mathbf{n}_0 > \mathbf{n}_0]] = \mathbf{h}'|_{t=0}, \quad [[\beta\mathbf{H}_0 \cdot \mathbf{n}_0]] = h_N|_{t=0} \quad \textit{on} \quad \Gamma,\\
&\mathbf{n} \cdot \mathbf{H}_0 = h_-|_{t=0} \quad \textit{on} \quad S
\end{aligned} \tag{1.7}$$

if $2 > 2/p + 1/q > 1$. *We do not impose any compatibility conditions if* $2/p + 1/q > 2$. *Then, problem* (1.4) *admits a unique solution*

$$\mathbf{H} \in H^1_p((0,\infty), L_q(\dot\Omega)^N) \cap L_p((0,\infty), H^2_q(\dot\Omega)^N))$$

satisfying the estimate:

$$\begin{aligned}
&\|e^{-\gamma t}\partial_t \mathbf{H}\|_{L_p((0,\infty),L_q(\dot\Omega)^N)} + \|e^{-\gamma t}\mathbf{H}\|_{L_p((0,\infty),H^2_q(\dot\Omega)^N)} \\
&\leqslant C\{\|\mathbf{H}_0\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)} + \|e^{-\gamma t}\mathbf{f}\|_{L_p(\mathbb{R},L_q(\dot\Omega))} \\
&+ \|e^{-\gamma t}\mathbf{g}\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))} + \|e^{-\gamma t}\mathbf{g}\|_{L_p(\mathbb{R},H^1_q(\Omega))} + \|e^{-\gamma t}\partial_t\mathbf{h}\|_{L_p(\mathbb{R},L_q(\Omega))} \\
&+ \|e^{-\gamma t}\mathbf{h}\|_{L_p(\mathbb{R},H^2_q(\Omega))} + \|e^{-\gamma t}\mathbf{g}_-\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega_-))} \\
&+ \|e^{-\gamma t}\mathbf{g}_-\|_{L_p(\mathbb{R},H^1_q(\Omega_-))} + \|e^{-\gamma t}\partial_t h_-\|_{L_p(\mathbb{R},L_q(\Omega_-))} \\
&+ \|e^{-\gamma t}h_-\|_{L_p(\mathbb{R},H^2_q(\Omega_-))}\}.
\end{aligned}$$

To prove Theorem 1.2, we use an $\mathcal{R}$ bounded solution operator associated with the following generalized resolvent equations corresponding to problem (1.4):

$$\begin{array}{rll}
\mu\lambda\mathbf{H} - \alpha^{-1}\Delta\mathbf{H} = \mathbf{f} & & \text{in } \dot\Omega, \\
{[[\alpha^{-1}\mathrm{curl}\,\mathbf{H}]]\mathbf{n}_0 = \mathbf{g}'}, & [[\beta\mathrm{div}\,\mathbf{H}]] = g_N & \text{on } \Gamma, \\
{[[\mathbf{H}- <\mathbf{H},\mathbf{n}_0> \mathbf{n}_0]] = \mathbf{h}'}, & [[\beta\mathbf{H}\cdot\mathbf{n}_0]] = h_N & \text{on } \Gamma, \\
(\mathrm{curl}\,\mathbf{H}_-)\mathbf{n} = \mathbf{g}_-, & \mathbf{n}\cdot\mathbf{H}_- = h_- & \text{on } S.
\end{array} \tag{1.8}$$

Now we give the definition of $\mathcal{R}$ boundedness of an operator family.

Definition 1.3. Let X and Y be two Banach spaces. A family of operators $\mathcal{T} \subset \mathcal{L}(X,Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X,Y)$, if there exist constants $q \in [1,\infty)$ and $C_q > 0$ such that for each $n \in \mathbb{N}$, $\{T_j\}_{j=1}^n \subset \mathcal{T}$, $\{f_j\}_{j=1}^n \subset X$ and for all sequences $\{r_j(u)\}_{j=1}^n$ of independent, symmetric, $\{-1,1\}$-valued random variables on $[0,1]$, there holds the inequality:

$$\int_0^1 \|\sum_{j=1}^n r_j(u)T_jf_j\|_Y^q\,du \leqslant C_q\int_0^1 \|\sum_{j=1}^n r_j(u)f_j\|_X^q\,du. \tag{1.9}$$

The smallest C_q in (1.9) is called $\mathcal{R}$-bound of $\mathcal{T}$ on $\mathcal{L}(X,Y)$, and is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$.

Remark 1.4. The definition of $\mathcal{R}$-boundedness is independent of $q \in [1,\infty)$ (cf. [3, p.26 3.2. Remarks (2)]). Namely, if there exist constants

$q \in [1, \infty)$ and C_q for which (1.9) holds, then for any $q \in [1, \infty)$, there exists a constant C_q for which (1.9) holds.

Theorem 1.5. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$, and $N < r < \infty$. Assume that Ω is a uniform $W_r^{3-1/r}$ domain in $\mathbb{R}^N$. Let*

$$\begin{aligned} Z_q(\Omega) =& \{\mathbf{F} = (\mathbf{f}, \mathbf{g}, \mathbf{h}, \mathbf{g}_-, h_-) \mid \mathbf{f} \in L_q(\Omega)^N,\ \ \mathbf{g} \in H_q^1(\Omega)^N,\ \ \mathbf{h} \in H_q^2(\Omega)^N, \\ & \mathbf{g}_- \in H_q^1(\Omega_-)^{N-1},\ \ h_- \in H_q^2(\Omega_-)\}, \\ \mathcal{Z}_q(\Omega) =& \{(F_0, F_1, \ldots, F_{10}) \mid F_0, F_1, F_3 \in L_q(\Omega)^N,\ \ F_2, F_4 \in H_q^1(\Omega)^N, \\ & F_5 \in H_q^2(\Omega)^N,\ \ F_6 \in L_q(\Omega_-)^{N-1},\ \ F_7 \in H_q^1(\Omega_-)^{N-1}, \\ & F_8 \in L_q(\Omega_-),\ \ F_9 \in H_q^1(\Omega_-),\ \ F_{10} \in H_q^2(\Omega_-)\}. \end{aligned}$$

Let

$$\Sigma_\epsilon = \{\lambda \in \mathbb{C}\setminus\{0\} \mid |\arg\lambda| \leqslant \pi - \epsilon\}, \quad \Sigma_{\epsilon,\lambda_1} = \{\lambda \in \Sigma_\epsilon \mid |\lambda| \geqslant \lambda_1\}. \tag{1.10}$$

Then, there exist a constant $\lambda_1 \geqslant 1$ and an operator family

$$\mathcal{A}(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_1}, \mathcal{L}(\mathcal{Z}_q(\Omega), H_q^2(\dot\Omega)^N))$$

such that for any $\lambda \in \Sigma_{\epsilon,\lambda_1}$, $\mathbf{F} \in Z_q(\Omega)$, the unique solution of Eq. (1.8) is given by $\mathbf{H} = \mathcal{A}(\lambda)F_\lambda\mathbf{F}$, where

$$F_\lambda\mathbf{F} = (\mathbf{f}, \lambda^{1/2}\mathbf{g}, \mathbf{g}, \lambda\mathbf{h}, \lambda^{1/2}\mathbf{h}, \mathbf{h}, \lambda^{1/2}\mathbf{g}_-, \mathbf{g}_-, \lambda h_-, \lambda^{1/2}h_-, h_-) \in \mathcal{Z}_q(\Omega). \tag{1.11}$$

The estimate

$$\mathcal{R}_{\mathcal{L}(\mathcal{Z}_q(\Omega), H_q^{2-k}(\dot\Omega)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{k/2}\mathcal{A}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_1}\}) \leqslant \gamma,\ \tau = Im\lambda$$

is valid for $\ell = 0, 1$ and $k = 0, 1, 2$. Here, γ is a positive constant depending on $\mu_\pm$, $\alpha_\pm$, $\beta_\pm$, ϵ, q, and N.

Remark 1.6. (1) The variables F_0, F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9, and F_{10} correspond to $\mathbf{f}$, $\lambda^{1/2}\mathbf{g}$, $\mathbf{g}$, $\lambda\mathbf{h}$, $\lambda^{1/2}\mathbf{h}$, $\mathbf{h}$, $\lambda^{1/2}\mathbf{g}_-$, $\mathbf{g}_-$, λh_-, $\lambda^{1/2}h_-$, and h_-, respectively.
(2) The norms of $Z_q(\Omega)$ and $\mathcal{Z}_q(\Omega)$ are defined by

$$\begin{aligned} &\|\mathbf{F}\|_{\mathcal{Z}_q} = \|\mathbf{f}\|_{L_q(\Omega)} + \|\mathbf{g}\|_{H_q^1(\Omega)} + \|\mathbf{h}\|_{H_q^2(\Omega)} + \|\mathbf{g}_-\|_{H_q^1(\Omega_-)} + \|h_-\|_{H_q^2(\Omega_-)}, \\ &\|(F_0, F_1, \ldots, F_{10})\|_{\mathcal{Z}_q(\Omega)} = \|(F_0, F_1, F_3)\|_{L_q(\Omega)} + \|(F_2, F_4)\|_{H_q^1(\Omega)} \\ &+ \|F_5\|_{H_q^2(\Omega)} + \|(F_6, F_8)\|_{L_q(\Omega_-)} + \|(F_7, F_9)\|_{H_q^1(\Omega_-)} + \|F_{10}\|_{H_q^2(\Omega_-)}. \end{aligned}$$

Finally, we explain some symbols used throughout the paper. For any multi-index $\kappa = (\kappa_, \ldots, \kappa_N)$, $\kappa_j \in \mathbb{N}_0$, we set $\partial_x^\kappa = \partial_1^{\kappa_1} \cdots \partial_N^{\kappa_N}$, $|\kappa| = \kappa_1 +$

$\cdots+\kappa_N$. For $1\leqslant q\leqslant\infty$, $m\in\mathbb{N}$, $s\in\mathbb{R}$, and any domain $D\subset\mathbb{R}^N$, we denote by $L_q(D)$, $H_q^m(D)$, and $B_{q,p}^s(D)$ the standard Lebesgue, Sobolev, and Besov spaces, respectively ($W_q^s(D)=B_{q,q}^s(D)$), while $\|\cdot\|_{L_q(D)}$, $\|\cdot\|_{H_q^m(D)}$, and $\|\cdot\|_{B_{q,p}^s(D)}$ denote the norms of these spaces. We set $(u,v)_D=\int\limits_D u(x)\overline{v(x)}\,dx$, $(u,v)_{\Gamma_t}=\int\limits_{\Gamma_t} u\overline{v}\,d\sigma$, and $(u,v)_S=\int\limits_S u\overline{v}\,d\sigma$.

For $\mathcal{H}\in\{H_q^m,B_{q,p}^s\}$, the function spaces $\mathcal{H}(\dot D)$ ($\dot D=D_+\cup D_-$) and their norms are defined by setting

$$\mathcal{H}(\dot D)=\{f=f_\pm\mid f_\pm\in\mathcal{H}(D_\pm)\},\quad \|f\|_{\mathcal{H}(\dot D)}=\|f_+\|_{\mathcal{H}(D_+)}+\|f_-\|_{\mathcal{H}(D_-)}.$$

For any Banach space X with the norm $\|\cdot\|_X$, X^d denotes the d product space defined by $\{x=(x_1,\ldots,x_d)\mid x_i\in X\}$, while the norm of X^d is simply written by $\|\cdot\|_X$, that is $\|x\|_X=\sum\limits_{j=1}^d\|x_j\|_X$. For any time interval (a,b), $L_p((a,b),X)$ and $H_p^m((a,b),X)$ denote the standard X-valued Lebesgue space and X-valued Sobolev space, while $\|\cdot\|_{L_p((a,b),X)}$ and $\|\cdot\|_{H_p^m((a,b),X)}$ denote their norms, respectively. Let L and L_λ^{-1} be the Laplace transform and the Laplace inverse transform defined by

$$L[f](\lambda)=\int\limits_{-\infty}^{\infty}e^{-(\gamma+i\tau)t}f(t)\,dt,$$

$$L_\lambda^{-1}[g(\lambda)](t)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}e^{(\gamma+i\tau)t}g(\gamma+i\tau)\,d\tau,$$

where $\lambda=\gamma+i\tau\in\mathbb{C}$. Let $H_p^s(\mathbb{R},X)$, $s>0$, be the Bessel potential space of order s defined by

$$H_p^s(\mathbb{R},X)=\{f\in L_p(\mathbb{R},X)\mid \|e^{-\gamma t}f\|_{H_p^s(\mathbb{R},X)}=\|e^{-\gamma t}\Lambda^s f\|_{L_p(\mathbb{R},X)}<\infty\},$$
$$\Lambda^s f=L_\lambda^{-1}[\lambda^s L[f](\lambda)]$$

for $\gamma>0$. For any two Banach spaces X and Y, $\mathcal{L}(X,Y)$ denotes the set of all linear bounded operators from X into Y, while $\|\cdot\|_{\mathcal{L}(X,Y)}$ denotes the operator norm. We write $\mathcal{L}(X,X)$ simply by $\mathcal{L}(X)$. For a domain U in $\mathbb{C}$, $\mathrm{Hol}\,(U,\mathcal{L}(X,Y))$ denotes the set of all $\mathcal{L}(X,Y)$ valued holomorphic functions defined on U. Given a vector $\mathbf{a}$ and a matrix K, $\mathbf{a}|_i$ and $K|_{(i,j)}$ denote the i-th component of $\mathbf{a}$ and (i,j)th component of K, respectively. For two N-vectors $\mathbf{a}$ and $\mathbf{b}$ with $\mathbf{a}|_i=a_i$ and $\mathbf{b}|_i=b_i$, $<\mathbf{a},\mathbf{b}>=\mathbf{a}\cdot$

$\mathbf{b} = \sum_{i=1}^N a_i b_i$. For two $N \times N$ matrices A and B with $A|_{(i,j)} = a_{ij}$ and $B|_{(i,j)} = b_{ij}$, $A : B = \sum_{i,j=1}^N a_{ij} b_{ij}$. Throughout the paper, the letter C denotes generic constants and $C_{a,b,\cdots}$ the constant which depends on $a, b, \cdots$. Values of C, $C_{a,b,\cdots}$ may be changed from line to line.

The paper is organized as follows. In Sect. 2, the existence of $\mathcal{R}$ bounded solution operators for the model problems are proved. In Sect. 3, the existence of $\mathcal{R}$ bounded solution operators for the bent space is proved. In Sect. 4, Theorem 1.5 is proved. In Sect. 5, Theorem 1.2 is proved with the help of $\mathcal{R}$ bounded solution operators given in Theorem 1.5 and Weis' operator valued Fourier multiplier theorem [26]. In the Appendix, it is proved that any solution $\mathbf{H}$ of Eq. (1.3) with $\operatorname{div} \mathbf{H}|_{t=0} = 0$ satisfies Eq. (1.1).

§2. Model Problems

2.1. Model problem in the whole space. In this subsection, we consider the whole space problem:

$$\mu_k \lambda \mathbf{H} - \alpha_k^{-1} \Delta \mathbf{H} = \mathbf{f} \quad \text{in } \mathbb{R}^N \tag{2.1}$$

with $k \in \{+, -\}$. Let Σ_ϵ be the set defined in (1.10). We know that

$$|\mu_k \lambda + \alpha_k^{-1} |\xi|^2| \geqslant C_k(|\lambda| + |\xi|^2)$$

for any $\lambda \in \Sigma_\epsilon$ and $\xi \in \mathbb{R}^N$ with some constant C_k depending on μ_k and α_k. Let $\mathcal{F}$ and $\mathcal{F}_\xi^{-1}$ be the Fourier transform and the inverse Fourier transform defined by

$$\mathcal{F}[f](\xi) = \int\limits_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)\,dx, \quad \mathcal{F}_\xi^{-1}[g(\xi)](x) = \frac{1}{(2\pi)^N} \int\limits_{\mathbb{R}^N} e^{ix\cdot\xi} g(\xi)\,d\xi,$$

respectively. We define the operators $\mathcal{K}_\pm(\lambda)$ acting on $\mathbf{f} \in L_q(\mathbb{R}^N)^N$ by the formula

$$\mathcal{K}_\pm(\lambda)\mathbf{f} = \mathcal{F}_\xi^{-1}\Big[\frac{\mathcal{F}[\mathbf{f}](\xi)}{\mu_\pm \lambda + \alpha_\pm^{-1} |\xi|^2}\Big], \tag{2.2}$$

then $\mathbf{H} = \mathcal{K}_\pm(\lambda)\mathbf{f} \in H_q^2(\mathbb{R}^N)^N$ is a unique solution of Eq. (2.1) for any $\lambda \in \Sigma_\epsilon$ and $\mathbf{f} \in L_q(\mathbb{R}^N)^N$. Our proof is based on the theory of L_p multipliers in Fourier integrals initiated by Mihlin [13]. To prove the $\mathcal{R}$ boundedness of $\mathcal{K}_\pm(\lambda)$, we use the following lemma due to Denk and Schnaubelt [5, Lemma 2.1] and Enomoto and Shibata [7, Theorem 3.3].

Lemma 2.1. *Let $1 < q < \infty$ and let $\Lambda \subset \mathbb{C}$, $m = m(\lambda, \xi)$ be a function defined on $\Lambda \times (\mathbb{R}^N \setminus \{0\})$ which is infinitely differentiable with respect to $\xi \in \mathbb{R}^N \setminus \{0\}$ for each $\lambda \in \Lambda$. Assume that for any multi-index $\kappa \in \mathbb{N}_0^N$ there exists a constant C_κ depending on κ and Λ such that*

$$|\partial_\xi^\kappa m(\lambda, \xi)| \leqslant C_\kappa |\xi|^{-|\kappa|} \tag{2.3}$$

for any $(\lambda, \xi) \in \Lambda \times (\mathbb{R}^N \setminus \{0\})$. Let K_λ be an operator defined by $K_\lambda f = \mathcal{F}_\xi^{-1}[m(\lambda, \xi)\mathcal{F}f(\xi)]$. Then, the family of operators $\{K_\lambda \mid \lambda \in \Lambda\}$ is $\mathcal{R}$-bounded on $\mathcal{L}(L_q(\mathbb{R}^N))$ and

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N))}(\{K_\lambda \mid \lambda \in \Lambda\}) \leqslant C_{q,N} \max_{|\kappa| \leqslant N+1} C_\kappa \tag{2.4}$$

with some constant $C_{q,N}$ depending only on q and N.

At this point, we introduce some fundamental properties of $\mathcal{R}$-bounded operators and Bourgain's results concerning Fourier multiplier theorems with scalar multiplier.

Proposition 2.2. a) *Let X and Y be Banach spaces, $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X, Y)$. Then, $\mathcal{T} + \mathcal{S} = \{T + S \mid T \in \mathcal{T},\ S \in \mathcal{S}\}$ is also an $\mathcal{R}$-bounded family in $\mathcal{L}(X, Y)$ and*

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T} + \mathcal{S}) \leqslant \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}) + \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S}).$$

b) *Let X, Y and Z be Banach spaces, $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X, Y)$ and $\mathcal{L}(Y, Z)$, respectively. Then, $\mathcal{S}\mathcal{T} = \{ST \mid T \in \mathcal{T},\ S \in \mathcal{S}\}$ is also an $\mathcal{R}$-bounded family in $\mathcal{L}(X, Z)$ and*

$$\mathcal{R}_{\mathcal{L}(X,Z)}(\mathcal{S}\mathcal{T}) \leqslant \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})\mathcal{R}_{\mathcal{L}(Y,Z)}(\mathcal{S}).$$

c) *Let $1 < p, q < \infty$ and let D be a domain in $\mathbb{R}^N$. Let $m = m(\lambda)$ be a bounded function defined on a subset Λ in $\mathbb{C}$ and let $M_m(\lambda)$ be a map defined by $M_m(\lambda)f = m(\lambda)f$ for any $f \in L_q(D)$. Then, $\mathcal{R}_{\mathcal{L}(L_q(D))}(\{M_m(\lambda) \mid \lambda \in \Lambda\}) \leqslant C_{N,q,D}\|m\|_{L_\infty(\Lambda)}$.*

d) *Let $n = n(\tau)$ be a C^1-function defined on $\mathbb{R} \setminus \{0\}$ which satisfies the conditions $|n(\tau)| \leqslant \gamma$ and $|\tau n'(\tau)| \leqslant \gamma$ with some constant $\gamma > 0$ for any $\tau \in \mathbb{R} \setminus \{0\}$. Let T_n be the operator-valued Fourier multiplier defined by $T_n f = \mathcal{F}^{-1}[n\mathcal{F}[f]]$ for any f with $\mathcal{F}[f] \in \mathcal{D}(\mathbb{R}, L_q(D))$. Then, T_n can be extended to a bounded linear operator from $L_p(\mathbb{R}, L_q(D))$ into itself. Moreover, denoting this extension also by T_n, we have*

$$\|T_n\|_{\mathcal{L}(L_p(\mathbb{R}, L_q(D)))} \leqslant C_{D,p,q}\gamma.$$

Here, $\mathcal{D}(\mathbb{R}, L_q(D))$ denotes the set of all $L_q(D)$-valued C^∞-functions on $\mathbb{R}$ with compact support.

Remark 2.3. (1) In view of (a) and (b), we can treat $\mathcal{R}$ norms like usual norms.
(2) Let $\Lambda = \{\lambda \in \mathbb{C} \mid |\lambda| \geqslant \lambda_0\}$ with some $\lambda_0 > 0$. Let $m_a = \lambda^{-a}$ with $0 < a \leqslant 1$. By (c),

$$\mathcal{R}_{\mathcal{L}(L_q(D))}(\{M_{m_a}(\lambda) \mid \lambda \in \Lambda\}) \leqslant C\lambda_0^{-a}. \tag{2.5}$$

(3) Let $\mathcal{K}(\lambda) \in \operatorname{Hol}(\Lambda, \mathcal{L}(L_q(D)))$ such that $\mathcal{R}_{\mathcal{L}(L_q(D))}(\{\mathcal{K}(\lambda) \mid \lambda \in \Lambda\}) \leqslant \gamma$. Then, by (b) we have

$$\begin{aligned}&\mathcal{R}_{\mathcal{L}(L_q(D))}(\{\lambda^{-a}\mathcal{K}(\lambda) \mid \lambda \in \Lambda\})\\ &\leqslant \mathcal{R}_{\mathcal{L}(L_q(D))}(\{M_{m_a}(\lambda) \mid \lambda \in \Lambda\})\mathcal{R}_{\mathcal{L}(L_q(D))}(\{\mathcal{K}(\lambda) \mid \lambda \in \Lambda\}) \leqslant \lambda_0^{-a}\gamma.\end{aligned}$$

Applying Lemma 2.1 and Proposition 2.2, we have the following theorem.

Theorem 2.4. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$, and $\lambda_0 > 0$. Let Σ_ϵ and $\Sigma_{\epsilon,\lambda_0}$ be the sets defined in* (1.10)*, $\mathcal{K}_\pm(\lambda)$ be the operators defined in* (2.2)*. Then, $\mathcal{K}_\pm(\lambda) \in \operatorname{Hol}(\Sigma_\epsilon, \mathcal{L}(L_q(\mathbb{R}^N)^N, H_q^2(\mathbb{R}^N)^N)$ and the estimate*

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N)^N, H_q^{2-j}(\mathbb{R}^N)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{K}_\pm(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leqslant \gamma_{\lambda_0},\ \tau = \operatorname{Im}\lambda$$

holds for $\ell = 0, 1$ and $j = 0, 1, 2$. The constant γ_{λ_0} depends on λ_0 in such a way that $\gamma_{\lambda_0} \to \infty$ as $\lambda_0 \to 0$.

2.2. Model problem in a half-space. Let

$$\begin{aligned}\mathbb{R}^N_+ &= \{x = (x_1, \dots, x_N) \in \mathbb{R}^N \mid x_N > 0\},\\ \mathbb{R}^N_0 &= \{x = (x_1, \dots, x_N) \in \mathbb{R}^N \mid x_N = 0\}.\end{aligned} \tag{2.6}$$

In this subsection, we consider the half space problem:

$$\begin{cases}\mu_-\lambda\mathbf{H} - \alpha_-^{-1}\Delta\mathbf{H} = \mathbf{f}_- & \text{in } \mathbb{R}^N_+,\\ (\operatorname{curl}\mathbf{H})\mathbf{n} = \mathbf{g}_-, \quad \mathbf{H}\cdot\mathbf{n} = h_- & \text{on } \mathbb{R}^N_0,\end{cases} \tag{2.7}$$

where $\mathbf{n} = (0, \dots, 0, -1)$, $\mathbf{g}_- = (g_1, \dots, g_{N-1})$.

Theorem 2.5. *Let* $1 < q < \infty$, $0 < \epsilon < \pi/2$, *and* $\lambda_0 > 0$. *Let*

$$X_q(\mathbb{R}^N_+) = \{(\mathbf{f}_-, \mathbf{g}_-, h_-) \mid \mathbf{f}_- \in L_q(\mathbb{R}^N_+)^N,\ \mathbf{g}_- \in H^1_q(\mathbb{R}^N_+)^{N-1},\ h_- \in H^2_q(\mathbb{R}^N_+)\},$$
$$\mathcal{X}_q(\mathbb{R}^N_+) = \{(F_{0-}, F_6, F_7, F_8, F_9, F_{10}) \mid F_{0-} \in L_q(\mathbb{R}^N_+)^N,$$
$$F_6 \in L_q(\mathbb{R}^N_+)^{N-1},\ \ F_7 \in H^1_q(\mathbb{R}^N_+)^{N-1},\ \ F_8 \in L_q(\mathbb{R}^N_+),$$
$$F_9 \in H^1_q(\mathbb{R}^N_+),\ \ F_{10} \in H^2_q(\mathbb{R}^N_+)\}.$$

Then, there exists an operator family $\mathcal{B}(\lambda) \in \mathrm{Hol}\ (\Sigma_\epsilon,\ \mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+), H^2_q(\mathbb{R}^N_+)^N))$ *such that for any* $\lambda \in \Sigma_{\epsilon,\lambda_0}$ *and* $(\mathbf{f}_-, \mathbf{g}_-, h_-) \in X_q(\mathbb{R}^N_+)$, *the unique solution of* (2.7) *is given by* $\mathbf{H} = \mathcal{B}(\lambda) F^1_\lambda(\mathbf{f}_-, \mathbf{g}_-, h_-)$, *where*

$$F^1_\lambda(\mathbf{f}_-, \mathbf{g}_-, h_-) = (\mathbf{f}_-, \lambda^{1/2}\mathbf{g}_-, \mathbf{g}_-, \lambda h_-, \lambda^{1/2} h_-, h_-) \in \mathcal{X}_q(\mathbb{R}^N_+).$$

The estimate

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+), H^{2-j}_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{B}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leqslant \gamma_{\lambda_0},\ \tau = \mathrm{Im}\,\lambda$$

holds for $\ell = 0, 1$ *and* $j = 0, 1, 2$ *with some constant* γ_{λ_0}, *which depends on* λ_0 *in such a way that* $\gamma_{\lambda_0} \to \infty$ *as* $\lambda_0 \to 0$.

Remark 2.6. (1) The variables F_{0-}, F_6, F_7, F_8, F_9, and F_{10} correspond to $\mathbf{f}_-$, $\lambda^{1/2}\mathbf{g}_-$, $\mathbf{g}_-$, λh_-, $\lambda^{1/2} h_-$, and h_- respectively.
(2) The norm of $\mathcal{X}_q(\mathbb{R}^N_+)$ is defined by

$$\|(F_{0-}, F_6, F_8)\|_{L_q(\mathbb{R}^N_+)} + \|(F_7, F_9)\|_{H^1_q(\mathbb{R}^N_+)} + \|F_{10}\|_{H^2_q(\mathbb{R}^N_+)}.$$

Extending $\mathbf{f}_-$ by 0 into $\mathbb{R}^N_-$, and denoting this extension by $\mathbf{f}$, we can reduce (2.7) to the similar problem with homogeneous equation using the solution $\mathcal{K}_-(\lambda)\mathbf{f}$ constructed in the previous section. Thus, it is sufficient to consider the problem:

$$\begin{cases} \mu_-\lambda\mathbf{H} - \alpha_-^{-1}\Delta\mathbf{H} = 0 & \text{in } \mathbb{R}^N_+, \\ (\mathrm{curl}\,\mathbf{H})\mathbf{n} = \mathbf{g}_-, \quad \mathbf{H}\cdot\mathbf{n} = h_- & \text{on } \mathbb{R}^N_0. \end{cases} \tag{2.8}$$

Let $\mathcal{F}'$ and $\mathcal{F}^{-1}_{\xi'}$ be the partial Fourier transform with respect to $x' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$ and the inverse partial Fourier transform with respect to $\xi' = (\xi_1, \ldots, \xi_{N-1}) \in \mathbb{R}^{N-1}$ defined by

$$\widehat{f} = \mathcal{F}'[f] = \int_{\mathbb{R}^{N-1}} e^{-ix'\cdot\xi'} f(x', x_N)\, dx',$$
$$\mathcal{F}^{-1}_{\xi'}[g(\xi', x_N)] = \frac{1}{(2\pi)^{N-1}} \int_{\mathbb{R}^{N-1}} e^{ix'\cdot\xi'} g(\xi', x_N)\, d\xi'.$$

Applying the partial Fourier transform to the first equation in (2.8), we have

$$\alpha_-\mu_-\lambda\widehat{\mathbf{H}} + |\xi'|^2\widehat{\mathbf{H}} - D_N^2\widehat{\mathbf{H}} = 0.$$

Bounded solutions to this equation in $\mathbb{R}^N_+$ have the form $\widehat{H}_j = \beta_j e^{-\omega_- x_N}$, where $\omega_- = \sqrt{\alpha_-\mu_-\lambda + |\xi'|^2}$, $\beta_j \in \mathbb{R}$, $j = 1, ..., N$.
We rewrite the boundary conditions in (2.8) componentwise:

$$H_N = -h_-,\ \partial_N H_j = g_j + \partial_j H_N = g_j - \partial_j h_-,\ j = 1, \dots, N-1.$$

Applying the partial Fourier transform, we obtain

$$\partial_N \widehat{H}_j|_{x_N=0} = \mathcal{F}'[g_j - \partial_j h_-], \qquad \widehat{H}_N|_{x_N=0} = -\widehat{h}_-. \tag{2.9}$$

Inserting $\widehat{H}_j = \beta_j e^{-\omega_- x_N}$ into (2.9) yields the relation

$$\beta_j = -\frac{1}{\omega_-}\mathcal{F}'[g_j - \partial_j \widehat{h}_-], \quad \beta_N = -\widehat{h}_-.$$

Thus, we have

$$\widehat{H}_j(\xi', x_N) = -\frac{e^{-\omega_- x_N}}{\omega_-}\mathcal{F}'[g_j - \partial_j h_-](\xi', 0),$$
$$\widehat{H}_N(\xi', x_N) = -e^{-\omega_- x_N}\widehat{h}_-(\xi', 0).$$

We use the method suggested by Volevich [25] and obtain

$$\begin{aligned}\widehat{H}_j(\xi', x_N) = &-\int_0^\infty e^{-\omega_-(x_N+y_N)}\mathcal{F}'[g_j - \partial_j h_-](\xi', y_N)\, dy_N \\ &+ \int_0^\infty \frac{e^{-\omega_-(x_N+y_N)}}{\omega_-}\mathcal{F}'[\partial_N g_j - \partial_j\partial_N h_-](\xi', y_N)\, dy_N, \\ \widehat{H}_N(\xi', x_N) = &-\int_0^\infty \omega_- e^{-\omega(x_N+y_N)}\widehat{h}_-(\xi', y_N)\, dy_N \\ &+ \int_0^\infty e^{-\omega_-(x_N+y_N)}\partial_N\widehat{h}_-(\xi', y_N)\, dy_N.\end{aligned}$$

Using the identities:

$$\omega_- = \frac{\alpha_-\mu_-\lambda}{\omega_-} + \frac{|\xi'|^2}{\omega_-}, \quad 1 = \frac{\alpha_-\mu_-\lambda}{\omega_-^2} + \frac{|\xi'|^2}{\omega_-^2}, \tag{2.10}$$

we have

$$\widehat{H}_j(\xi', x_N) = -\int_0^\infty \lambda^{1/2} e^{-\omega_-(x_N+y_N)} \frac{\alpha_-\mu_-}{\omega_-^2} \mathcal{F}'[\lambda^{1/2} g_j - \lambda^{1/2}\partial_j h_-](\xi', y_N)\, dy_N$$

$$- \sum_{k=1}^{N-1} \int_0^\infty |\xi'| e^{-\omega_-(x_N+y_N)} \frac{i\xi_k}{|\xi'|\omega_-^2} \mathcal{F}'[\partial_k g_j - \partial_j \partial_k h_-](\xi', y_N)\, dy_N$$

$$+ \int_0^\infty (\lambda^{1/2} e^{-\omega_-(x_N+y_N)} \frac{\alpha_-\mu_-\lambda^{1/2}}{\omega_-^3} + |\xi'| e^{-\omega_-(x_N+y_N)} \frac{|\xi'|}{\omega_-^3})$$

$$\Big(\mathcal{F}'[\partial_N g_j - \partial_j \partial_N h_-](\xi', y_N)\Big)\, dy_N,$$

$$\widehat{H}_N(\xi', x_N) = \int_0^\infty (\lambda^{1/2} e^{-\omega_-(x_N+y_N)} \frac{\alpha_-\mu_-\lambda^{1/2}}{\omega_-^3} + |\xi'| e^{-\omega_-(x_N+y_N)} \frac{|\xi'|}{\omega_-^3})$$

$$\Big(\alpha_-\mu_- \mathcal{F}'[\lambda h_-](\xi', y_N) - \sum_{k=1}^{N-1} \mathcal{F}'[\partial_k^2 h_-](\xi', y_N)\Big)\, dy_N$$

$$+ \int_0^\infty (\lambda^{1/2} e^{-\omega_-(x_N+y_N)} \frac{\alpha_-\mu_-\lambda^{1/2}}{\omega_-^3} + |\xi'| e^{-\omega_-(x_N+y_N)} \frac{|\xi'|}{\omega_-^3})$$

$$\Big(\frac{\alpha_-\mu_-\lambda^{1/2}}{\omega_-} \mathcal{F}'[\lambda^{1/2}\partial_N h_-](\xi', y_N) - \sum_{k=1}^{N-1} \frac{i\xi_k}{\omega_-} \mathcal{F}[\partial_k \partial_N h_-](\xi', y_N)\Big)\, dy_N.$$

We define the operators

$$\mathcal{B}_j(\lambda)(F_6, \ldots, F_{10})$$

$$= -\int_0^\infty \mathcal{F}_{\xi'}^{-1} \Big[\lambda^{1/2} e^{-\omega_-(x_N+y_N)} \frac{\alpha_-\mu_-}{\omega_-^2} (\mathcal{F}'[F_{6j} - \partial_j F_9](\xi', y_N)\Big](x')\, dy_N$$

$$- \sum_{k=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \Big[|\xi'| e^{-\omega_-(x_N+y_N)} \frac{i\xi_k}{|\xi'|\omega_-^2} \mathcal{F}'[\partial_k F_{7j} - \partial_j \partial_k F_{10}](\xi', y_N)\Big](x')\, dy_N$$

$$+ \int_0^\infty \mathcal{F}_{\xi'}^{-1} \Big[(\lambda^{1/2} e^{-\omega_-(x_N+y_N)} \frac{\alpha_-\mu_-\lambda^{1/2}}{\omega_-^3} + |\xi'| e^{-\omega_-(x_N+y_N)} \frac{|\xi'|}{\omega_-^3})$$

$$\Big(\mathcal{F}'[\partial_N F_{7j} - \partial_j\partial_N F_{10}](\xi', y_N)\Big)\Big](x')\,dy_N,$$

$$\begin{aligned}
&\mathcal{B}_N(\lambda)(F_6,\dots,F_{10})\\
&= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[(\lambda^{1/2}e^{-\omega_-(x_N+y_N)}\frac{\alpha_-\mu_-\lambda^{1/2}}{\omega_-^3} + |\xi'|e^{-\omega_-(x_N+y_N)}\frac{|\xi'|}{\omega_-^3})\\
&\qquad \Big(\alpha_-\mu_-\mathcal{F}'[F_8](\xi', y_N) - \sum_{k=1}^{N-1}\mathcal{F}'[\partial_k^2 F_{10}](\xi', y_N)\Big)\Big](x')\,dy_N\\
&+ \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[(\lambda^{1/2}e^{-\omega_-(x_N+y_N)}\frac{\alpha_-\mu_-\lambda^{1/2}}{\omega_-^3} + |\xi'|e^{-\omega_-(x_N+y_N)}\frac{|\xi'|}{\omega_-^3})\\
&\qquad \Big(\frac{\alpha_-\mu_-\lambda^{1/2}}{\omega_-}\mathcal{F}'[\partial_N F_9](\xi', y_N) - \sum_{k=1}^{N-1}\frac{i\xi_k}{\omega_-}\mathcal{F}'[\partial_k\partial_N F_{10}](\xi', y_N)\Big)\Big](x')\,dy_N.
\end{aligned}$$

Obviously,

$$H_j(x) = \mathcal{F}^{-1}_{\xi'}[\widehat{H}_j(\xi', x_N)](x') = \mathcal{B}_j(\lambda)(\lambda^{1/2}\mathbf{g}_-, \mathbf{g}_-, \lambda h_-, \lambda^{1/2}h_-, h_-).$$

As a preparation for the proof of the $\mathcal{R}$ boundedness of $\mathcal{B}_j(\lambda)$, we introduce some classes of multipliers.

Definition 2.7. Let $\Xi \subset \Lambda \times (\mathbb{R}^{N-1} \setminus \{0\})$, $\Lambda \subset \mathbb{C}$ and let $m\colon \Xi \to \mathbb{C}$, $(\lambda, \xi') \mapsto m(\lambda, \xi')$ be C^1 with respect to τ ($\lambda = \gamma + i\tau$) and C^∞ with respect to ξ'.

(1) $m(\lambda, \xi')$ is called a multiplier on Ξ of type 1 of the order s if there hold the estimates:

$$\begin{aligned}
&|\partial_{\xi'}^{\kappa'} m(\lambda, \xi')| \leqslant C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{s-|\kappa'|},\\
&|\partial_{\xi'}^{\kappa'}(\tau\partial_\tau m(\lambda, \xi'))| \leqslant C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{s-|\kappa'|}
\end{aligned} \tag{2.11}$$

for any multi-index $\kappa' \in \mathbb{N}_0^{N-1}$ and $(\lambda, \xi') \in \Xi$ with some constant $C_{\kappa'}$ depending solely on κ' and Ξ.

(2) $m(\lambda, \xi')$ is called a multiplier on Ξ of type 2 of the order s if there hold the estimates:

$$\begin{aligned}
&|\partial_{\xi'}^{\kappa'} m(\lambda, \xi')| \leqslant C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{s}|\xi'|^{-|\kappa'|},\\
&|\partial_{\xi'}^{\kappa'}(\tau\partial_\tau m(\lambda, \xi'))| \leqslant C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{s}|\xi'|^{-|\kappa'|}
\end{aligned} \tag{2.12}$$

for any multi-index $\kappa' \in \mathbb{N}_0^{N-1}$ and $(\lambda, \xi') \in \Xi$ with some constant $C_{\kappa'}$ depending solely on κ' and Ξ.

Let $\mathbb{M}_{s,i}(\Xi)$ be the set of all multipliers on Ξ of type i $(i = 1, 2)$ of the order s. Especially, below we write $\mathbb{M}_{s,i}(\Sigma_\epsilon \times (\mathbb{R}^{N-1} \setminus \{0\}))$ simply by $\mathbb{M}_{s,i}$.

The following lemma immediately follows from the inequality

$$(|\lambda|^{1/2} + |\xi'|)^{-|\alpha'|} \leqslant |\xi'|^{-|\alpha'|}$$

and the Leibniz rule.

Lemma 2.8. *Let s_1, s_2 be two real numbers. Then, the following three assertions hold.*

a) *Given $m_i \in \mathbb{M}_{s_i,1}(\Xi)$ $(i = 1, 2)$, we have $m_1 m_2 \in \mathbb{M}_{s_1+s_2,1}(\Xi)$.*
b) *Given $\ell_i \in \mathbb{M}_{s_i,i}(\Xi)$ $(i = 1, 2)$, we have $\ell_1 \ell_2 \in \mathbb{M}_{s_1+s_2,2}(\Xi)$.*
c) *Given $n_i \in \mathbb{M}_{s_i,2}(\Xi)$ $(i = 1, 2)$, we have $n_1 n_2 \in \mathbb{M}_{s_1+s_2,2}(\Xi)$.*

For any $s \in \mathbb{R}$, $\omega^s \in \mathbb{M}_{s,1}$. Moreover, $\xi_j \in \mathbb{M}_{1,1}$, $|\xi'|^2 \in \mathbb{M}_{2,1}$. To prove the $\mathcal{R}$-boundedness of the operators $\mathcal{B}_j(\lambda)$, we use the following lemma due to Shibata and Shimizu [19, Lemma 5.6].

Lemma 2.9. *Let $0 < \epsilon < \pi/2$. For given $\ell_0(\lambda, \xi') \in \mathbb{M}_{-2,1}$ and $\ell_1(\lambda, \xi') \in \mathbb{M}_{-2,2}$, we define the operators $K_j(\lambda)$ $(j = 1, 2)$ by*

$$[K_1(\lambda)h](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[\ell_0(\lambda, \xi')\lambda^{1/2} e^{-\omega_-(x_N + y_N)} \mathcal{F}'[h](\xi', y_N)](x')\, dy_N,$$

$$[K_2(\lambda)h](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[\ell_1(\lambda, \xi')|\xi'| e^{-\omega_-(x_N + y_N)} \mathcal{F}'[h](\xi', y_N)](x')\, dy_N.$$

Then,

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+))}\big(\{(\tau\partial_\tau)^s(\lambda^{i/2}\partial_x^\alpha K_j(\lambda)) \mid \lambda \in \Sigma_\epsilon\}\big) \leqslant C_{N,q}$$
$$(s = 0, 1, \ \ i + |\alpha| = 2, \ \ j = 1, 2).$$

Applying Lemma 2.9 to $\mathcal{B}_j(\lambda)$ and using Proposition 2.2, we observe

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+), H_q^{2-i}(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)(\lambda^{i/2}\mathcal{B}_j(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leqslant \gamma_{\lambda_0}$$

with some constant γ_{λ_0} that depends on λ_0 in such a way that $\gamma_{\lambda_0} \to \infty$ as $\lambda_0 \to 0$. Uniqueness follows from the existence of solutions to the dual problem. This completes the proof of Theorem 2.5.

2.3. Model interface problem. Let $\mathbb{R}^N_+$ and $\mathbb{R}^N_0$ be the symbols defined in (2.6) and

$$\mathbb{R}^N_- = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N < 0\}, \quad \dot{\mathbb{R}}^N = \mathbb{R}^N_+ \cup \mathbb{R}^N_-. \tag{2.13}$$

Let $\mathbf{n} = (0, \ldots, 0, -1)$. Consider the problem

$$\begin{cases} \mu\lambda\mathbf{H} - \alpha^{-1}\Delta\mathbf{H} = \mathbf{f} & \text{in } \dot{\mathbb{R}}^N, \\ [[\alpha^{-1}\mathrm{curl}\,\mathbf{H}]]\mathbf{n} = \mathbf{g}', \quad [[\beta\mathrm{div}\,\mathbf{H}]] = g_N & \text{on } \mathbb{R}^N_0, \\ [[\mathbf{H} - (\mathbf{H}\cdot\mathbf{n})\mathbf{n}]] = \mathbf{h}', \quad [[\beta\mathbf{H}\cdot\mathbf{n}]] = h_N & \text{on } \mathbb{R}^N_0. \end{cases} \tag{2.14}$$

Theorem 2.10. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$, and $\lambda_0 > 0$. Let*

$$Y_q(\mathbb{R}^N) = \{(\mathbf{f}, \mathbf{g}, \mathbf{h}) \mid \mathbf{f} \in L_q(\mathbb{R}^N)^N, \quad \mathbf{g} \in H^1_q(\mathbb{R}^N)^N, \quad \mathbf{h} \in H^2_q(\mathbb{R}^N)^N\},$$

$$\mathcal{Y}_q(\mathbb{R}^N) = \{(F_0, F_1, F_2, F_3, F_4, F_5) \mid F_0, F_1, F_3 \in L_q(\mathbb{R}^N)^N, \\ F_2, F_4 \in H^1_q(\mathbb{R}^N)^N, \quad F_5 \in H^2_q(\mathbb{R}^N)^N\}.$$

Then, there exists an operator family

$$\mathcal{B}_I(\lambda) \in \mathrm{Hol}\,(\Sigma_\epsilon, \mathcal{L}(\mathcal{Y}_q(\mathbb{R}^N), H^2_q(\dot{\mathbb{R}}^N)^N))$$

such that for any $\lambda \in \Sigma_\epsilon$, $(\mathbf{f}, \mathbf{g}, \mathbf{h}) \in Y_q(\mathbb{R}^N)$, the unique solution of Eq. (2.14) is given by $\mathbf{H} = \mathcal{B}_I(\lambda)F^0_\lambda(\mathbf{f}, \mathbf{g}, \mathbf{h})$, where

$$F^0_\lambda(\mathbf{f}, \mathbf{g}, \mathbf{h}) = (\mathbf{f}, \lambda^{1/2}\mathbf{g}, \mathbf{g}, \lambda\mathbf{h}, \lambda^{1/2}\mathbf{h}, \mathbf{h}) \in \mathcal{Y}_q(\mathbb{R}^N).$$

The estimate

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\mathbb{R}^N), H^{2-j}_q(\dot{\mathbb{R}}^N)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{B}_I(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leqslant \gamma_{\lambda_0}, \quad \tau = Im\lambda$$

holds for $\ell = 0, 1$ and $j = 0, 1, 2$. The constant γ_{λ_0} depends on λ_0 in such a way that $\gamma_{\lambda_0} \to \infty$ as $\lambda_0 \to 0$.

Remark 2.11. The norm of $\mathcal{Y}_q(\mathbb{R}^N)$ is defined by

$$\|(F_0, F_1, F_3)\|_{L_q(\mathbb{R}^N)} + \|(F_2, F_4)\|_{H^1_q(\mathbb{R}^N)} + \|F_5\|_{H^2_q(\mathbb{R}^N)}.$$

Let $\mathbf{f} = \mathbf{f}_\pm$ and let $\mathbf{f}^0_\pm$ be the zero extensions of $\mathbf{f}_\pm$ to $\mathbb{R}^N_\mp$. By using $\mathcal{K}_\pm(\lambda)\mathbf{f}^0_\pm$, where $\mathcal{K}_\pm(\lambda)$ are the operators defined in (2.2), we can reduce Eq. (2.14) to the case $\mathbf{f} = 0$. Thus, it is enough to consider the following problem:

$$\begin{cases} \mu\lambda\mathbf{H} - \alpha^{-1}\Delta\mathbf{H} = 0 & \text{in } \dot{\mathbb{R}}^N, \\ [[\alpha^{-1}\mathrm{curl}\,\mathbf{H}]]\mathbf{n} = \mathbf{g}', \quad [[\beta\mathrm{div}\,\mathbf{H}]] = g_N & \text{on } \mathbb{R}^N_0, \\ [[\mathbf{H} - (\mathbf{H}\cdot\mathbf{n})\mathbf{n}]] = \mathbf{h}', \quad [[\beta\mathbf{H}\cdot\mathbf{n}]] = h_N & \text{on } \mathbb{R}^N_0. \end{cases} \tag{2.15}$$

We assume that $\mathbf{g} \in H^1_q(\mathbb{R}^N)^N$ and $\mathbf{h} \in H^2_q(\mathbb{R}^N)^N$. The jump conditions in (2.15) have the form

$$
\begin{aligned}
&-\alpha_+^{-1}(\partial_j H_{+N} - \partial_N H_{+j})|_{x_N=0+} + \alpha_-^{-1}(\partial_j H_{-N} - \partial_N H_{-j})|_{x_N=0-} = g_j,\\
&\beta_+ \sum_{j=1}^N \partial_j H_{+j}|_{x_N=0+} - \beta_- \sum_{j=1}^N \partial_j H_{-j}|_{x_N=0-} = g_N, \qquad (2.16)\\
&H_{+j} - H_{-j} = h_j, \quad -\beta_+ H_{+N} + \beta_- H_{-N} = h_N, \quad j = 1, ..., N-1.
\end{aligned}
$$

Let $\widehat{H}_{\pm j} = \mathcal{F}'[H_{\pm j}](\xi', x_N)$. Applying the partial Fourier transform to the first equation in (2.15), we have

$$
\mu_\pm \lambda \widehat{H}_{\pm j} + \alpha_\pm^{-1}|\xi'|^2 \widehat{H}_{\pm j} - \alpha_\pm^{-1} D_N^2 \widehat{H}_{\pm j} = 0 \quad \text{for } \pm x_N > 0. \qquad (2.17)
$$

Let $\omega_\pm = \sqrt{\alpha_\pm \mu_\pm \lambda + |\xi'|^2}$. Bounded solution to (2.17) has the form $\widehat{H}_{\pm j} = A_{\pm j} e^{\mp \omega_\pm x_N}$. To find $A_{\pm j}$, applying the partial Fourier transform to conditions (2.16) and inserting $\widehat{H}_{\pm j}$, we arrive at the following linear system

$$
-\alpha_+^{-1}(i\xi_j A_{+N} + \omega_+ A_{+j}) + \alpha_-^{-1}(i\xi_j A_{-N} - \omega_- A_{-j}) = \widehat{g}_j \qquad (2.18)
$$

$$
\beta_+ \Big(\sum_{j=1}^{N-1} i\xi_j A_{+j} - \omega_+ A_{+N}\Big) - \beta_- \Big(\sum_{j=1}^{N-1} i\xi_j A_{-j} + \omega_- A_{-N}\Big) = \widehat{g}_N, \qquad (2.19)
$$

$$
A_{+j} = A_{-j} + \widehat{h}_j, \quad A_{+N} = \frac{\beta_-}{\beta_+} A_{-N} - \frac{1}{\beta_+}\widehat{h}_N. \qquad (2.20)
$$

Multiplying (2.18) by $i\xi_j$ and taking a sum from $j = 1$ through $j = N-1$, we have

$$
\alpha_+^{-1}|\xi'|^2 A_{+N} - \alpha_+^{-1}\omega_+ i\xi' \cdot A'_+ - \alpha_-^{-1}|\xi'|^2 A_{-N} - \alpha_-^{-1}\omega_- i\xi' \cdot A'_- = i\xi' \cdot \widehat{g}',
$$

which, combined with (2.20), furnishes that

$$
\begin{aligned}
(\alpha_+^{-1}\omega_+ + \alpha_-^{-1}\omega_-) i\xi' \cdot A'_- + |\xi'|^2 (\alpha_-^{-1} - \alpha_+^{-1}\frac{\beta_-}{\beta_+}) A_{-N}&\\
= -i\xi' \cdot \widehat{g}' - \alpha_+^{-1}\omega_+ i\xi' \cdot \widehat{h}' - \frac{|\xi'|^2}{\alpha_+\beta_+}\widehat{h}_N.& \qquad (2.21)
\end{aligned}
$$

Combination of (2.19) and (2.20) yields the following relation

$$
(\beta_+ - \beta_-) i\xi' \cdot A'_- - \beta_-(\omega_+ + \omega_-) A_{-N} = \widehat{g}_N - \beta_+ i\xi' \cdot \widehat{h}' - \omega_+ \widehat{h}_N. \qquad (2.22)
$$

We set

$$\mathcal{A} = \begin{pmatrix} \alpha_+^{-1}\omega_+ + \alpha_-^{-1}\omega_- & \beta_-|\xi'|^2((\alpha_-\beta_-)^{-1} - (\alpha_+\beta_+)^{-1}) \\ \beta_+ - \beta_- & -\beta_-(\omega_+ + \omega_-) \end{pmatrix}.$$

With the help of (2.21) and (2.22), we obtain

$$\mathcal{A}\begin{pmatrix} i\xi' \cdot A'_- \\ A_{-N} \end{pmatrix} = \begin{pmatrix} -i\xi' \cdot \widehat{g}' - \alpha_+^{-1}\omega_+ i\xi' \cdot \widehat{h}' - (\alpha_+\beta_+)^{-1}|\xi'|^2\widehat{h}_N \\ \widehat{g}_N - \beta_+ i\xi' \cdot \widehat{h}' - \omega_+\widehat{h}_N \end{pmatrix}. \tag{2.23}$$

By simple calculations, we observe that

$$\begin{aligned} \det\mathcal{A} &= -\beta_-\{(\omega_+ + \omega_-)(\alpha_+^{-1}\omega_+ + \alpha_-^{-1}\omega_-) \\ &\quad - |\xi'|^2(\beta_+ - \beta_-)((\alpha_+\beta_+)^{-1} - (\alpha_-\beta_-)^{-1})\} \\ &= -\beta_-\{\alpha_+^{-1}(\alpha_+\mu_+\lambda + |\xi'|^2) + \alpha_-^{-1}(\alpha_-\mu_-\lambda + |\xi'|^2) \\ &\quad + (\alpha_+^{-1} + \alpha_-^{-1})\omega_+\omega_- - |\xi'|^2(\beta_+ - \beta_-)((\alpha_+\beta_+)^{-1} - (\alpha_-\beta_-)^{-1})\} \\ &= -\beta_-\{(\mu_+ + \mu_-)\lambda + (\beta_+(\alpha_-\beta_-)^{-1} \\ &\quad + \beta_-(\alpha_+\beta_+)^{-1})|\xi'|^2 + (\alpha_+^{-1} + \alpha_-^{-1})\omega_+\omega_-\}. \end{aligned}$$

Hence

$$|\det\mathcal{A}| \neq 0, \tag{2.24}$$

provided that $(\lambda, \xi') \in \Sigma_\epsilon \times (\mathbb{R}^{N-1} \setminus \{0\})$. In fact, if $0 \leqslant \arg\lambda < \pi$, then

$$0 \leqslant \arg((\mu_+ + \mu_-)\lambda + (\beta_+(\alpha_-\beta_-)^{-1} + \beta_-(\alpha_+\beta_+)^{-1})|\xi'|^2) < \pi$$

and $0 \leqslant \arg((\alpha_+^{-1} + \alpha_-^{-1})\omega_+\omega_-) < \pi$. And, if $-\pi < \arg\lambda \leqslant 0$, then

$$-\pi < \arg((\mu_+ + \mu_-)\lambda + (\beta_+(\alpha_-\beta_-)^{-1} + \beta_-(\alpha_+\beta_+)^{-1})|\xi'|^2) \leqslant 0$$

and $-\pi < \arg((\alpha_+^{-1} + \alpha_-^{-1})\omega_+\omega_-) \leqslant 0$. Thus, we have (2.24).

Now, we prove that there exists a constant $c_0 > 0$ such that

$$|\det\mathcal{A}| \geqslant c_0(|\lambda| + |\xi'|^2) \tag{2.25}$$

for any $(\lambda, \xi') \in \Sigma_\epsilon \times (\mathbb{R}^{N-1} \setminus \{0\})$. Really, in the case $|\xi'| \geqslant R_1|\lambda|^{1/2}$ with large $R_1 \geqslant 1$, we have

$$|\det\mathcal{A}| \geqslant \beta_-((\beta_+(\alpha_-\beta_-)^{-1} + \beta_-(\alpha_+\beta_+)^{-1}) + \alpha_+^{-1} + \alpha_-^{-1} + O(R_1^{-2}))|\xi'|^2.$$

Choosing $R_1 \geqslant 1$ so large that

$$(\beta_+(\alpha_-\beta_-)^{-1} + \beta_-(\alpha_+\beta_+)^{-1}) + \alpha_+^{-1} + \alpha_-^{-1} + O(R_1^{-2}) \geqslant (\alpha_+^{-1} + \alpha_-^{-1}),$$

we obtain

$$|\det\mathcal{A}| \geqslant (\beta_-/2)(\alpha_+^{-1} + \alpha_-^{-1})(|\lambda| + |\xi'|^2).$$

In the case $|\lambda|^{1/2} \geqslant R_2|\xi'|$ with a large number $R_2 \geqslant 1$, we have

$$|\det A| \geqslant \beta_-(\mu_+ + \mu_- + (\alpha_+^{-1} + \alpha_-^{-1})(\alpha_+\alpha_-\mu_+\mu_-)^{1/2} + O(R_2^{-2}))|\lambda|.$$

Choosing R_2 so large that

$$\mu_+ + \mu_- + (\alpha_+^{-1} + \alpha_-^{-1})(\alpha_+\alpha_-\mu_+\mu_-)^{1/2} + O(R_2^{-2}) \geqslant \mu_+ + \mu_-,$$

we have

$$|\det \mathcal{A}| \geqslant (\beta_-/2)(\mu_+ + \mu_-)(|\lambda| + |\xi'|^2).$$

Finally, we study the case when $R_1^{-1}|\xi'| \leqslant |\lambda|^{1/2} \leqslant R_2|\xi'|$. We introduce the notations $\widetilde{\lambda} = \frac{\lambda}{|\lambda|+|\xi'|^2}$, $\quad \widetilde{\xi}_j = \frac{\xi_j}{\sqrt{|\lambda|+|\xi'|^2}}$,
then $|\widetilde{\lambda}| + |\widetilde{\xi}'|^2 = 1$ and $\frac{1}{1+R_1^2} \leqslant |\widetilde{\lambda}| \leqslant 1$, $\quad \frac{1}{1+R_2^2} \leqslant |\widetilde{\xi}'|^2 \leqslant 1$.
Let

$$\Lambda_\epsilon = \{(\widetilde{\lambda}, \widetilde{\xi}') \in \Sigma_\epsilon \times (\mathbb{R}^{N-1} \setminus \{0\}) \mid |\widetilde{\lambda}| + |\widetilde{\xi}'|^2 = 1,$$
$$\frac{1}{1+R_1^2} \leqslant |\widetilde{\lambda}| \leqslant 1, \quad \frac{1}{1+R_2^2} \leqslant |\widetilde{\xi}'|^2 \leqslant 1\}.$$

Obviously, Λ_ϵ is a compact set. If $(\lambda, \xi') \in \Sigma_\epsilon \times (\mathbb{R}^{N-1} \setminus \{0\})$ satisfies the condition: $R_1^{-1}|\xi'| \leqslant |\lambda|^{1/2} \leqslant R_2|\xi'|$, then $(\widetilde{\lambda}, \widetilde{\xi}') \in \Lambda_\epsilon$. If we set $\widetilde{\omega}_\pm = \sqrt{\alpha_\pm\mu_\pm\widetilde{\lambda} + |\widetilde{\xi}'|^2}$, then we have $|\det \mathcal{A}| = (|\lambda|+|\xi'|^2)\theta(\widetilde{\lambda}, \widetilde{\xi}')$ with $\theta(\widetilde{\lambda}, \widetilde{\xi}') = \beta_-((\mu_+ + \mu_-)\widetilde{\lambda} + (\beta_+(\alpha_-\beta_-)^{-1} + \beta_-(\alpha_+\beta_+)^{-1})|\widetilde{\xi}'|^2 + (\alpha_+^{-1} + \alpha_-^{-1})\widetilde{\omega}_+\widetilde{\omega}_-)$. By (2.24), $\theta(\widetilde{\lambda}, \widetilde{\xi}') \neq 0$ for $(\widetilde{\lambda}, \widetilde{\xi}') \in \Sigma_\epsilon \times (\mathbb{R}^{N-1} \setminus \{0\})$, it implies

$$\inf_{(\widetilde{\lambda}, \widetilde{\xi}') \in \Lambda_\epsilon} \theta(\widetilde{\lambda}, \widetilde{\xi}') = c_1 > 0.$$

Thus, we have (2.25), provided that $(\lambda, \xi') \in \Sigma_\epsilon \times (\mathbb{R}^{N-1} \setminus \{0\})$ and $R_1^{-1}|\xi'| \leqslant |\lambda|^{1/2} \leqslant R_2|\xi'|$. It completes the proof of (2.25).

Since

$$\mathcal{A}^{-1} = \frac{1}{\det \mathcal{A}} \begin{pmatrix} -\beta_-(\omega_+ + \omega_-) & -\beta_-|\xi'|^2((\alpha_-\beta_-)^{-1} - (\alpha_+\beta_+)^{-1}) \\ -(\beta_+ - \beta_-) & \alpha_+^{-1}\omega_+ + \alpha_-^{-1}\omega_- \end{pmatrix},$$

we have

$$
\begin{aligned}
i\xi'\cdot A'_- = {} & \frac{\beta_-}{\det\mathcal{A}}\{(\omega_+ + \omega_-)i\xi'\cdot\widehat{g}'(\xi',0)\\
&+((\alpha_+\beta_+)^{-1}-(\alpha_-\beta_-)^{-1})|\xi'|^2\widehat{g}_N(\xi',0)\\
&+(\mu_+\lambda+\beta_+(\alpha_-\beta_-)^{-1}|\xi'|^2+\alpha_+^{-1}\omega_+\omega_-)i\xi'\cdot\widehat{h}'(\xi',0)\\
&+((\alpha_-\beta_-)^{-1}\omega_+ + (\alpha_+\beta_+)^{-1}\omega_-)|\xi'|^2\widehat{h}_N(\xi',0)\},
\end{aligned}
\tag{2.26}
$$

$$
\begin{aligned}
A_{-N} = {} & \frac{1}{\det\mathcal{A}}\{(\beta_+-\beta_-)i\xi'\cdot\widehat{g}'(\xi',0)+(\alpha_+^{-1}\omega_+ + \alpha_-^{-1}\omega_-)\widehat{g}_N(\xi',0)\\
&-(\beta_-\alpha_+^{-1}\omega_+ + \beta_+\alpha_-^{-1}\omega_-)i\xi'\cdot\widehat{h}'(\xi',0)\\
&-(\mu_+\lambda+\beta_-(\alpha_+\beta_+)^{-1}|\xi'|^2+\alpha_-^{-1}\omega_+\omega_-)\widehat{h}_N(\xi',0)\}.
\end{aligned}
$$

Since $A_{+N} = \beta_-\beta_+^{-1}A_{-N} - \beta_+^{-1}\widehat{h}_N$, using the formula of $\det\mathcal{A}$, we obtain

$$
\begin{aligned}
A_{+N} = {} & \frac{\beta_-\beta_+^{-1}}{\det\mathcal{A}}\{(\beta_+-\beta_-)i\xi'\cdot\widehat{g}'(\xi',0)+(\alpha_+^{-1}\omega_+ + \alpha_-^{-1}\omega_-)\widehat{g}_N(\xi',0)\\
&-(\beta_-\alpha_+^{-1}\omega_+ + \beta_+\alpha_-^{-1}\omega_-)i\xi'\cdot\widehat{h}'(\xi',0)\\
&+(\mu_-\lambda+\beta_+(\alpha_-\beta_-)^{-1}|\xi'|^2+\alpha_+^{-1}\omega_+\omega_-)\widehat{h}_N(\xi',0)\}.
\end{aligned}
\tag{2.27}
$$

By (2.18), we have

$$
\begin{aligned}
-\widehat{g}_j &= \alpha_+^{-1}i\xi_jA_{+N} - \alpha_-^{-1}i\xi_jA_{-N} + \alpha_+^{-1}\omega_+A_{+j} + \alpha_-^{-1}\omega_-A_{-j}\\
&= \Big(\frac{\beta_-}{\alpha_+\beta_+}A_{-N} - \frac{1}{\alpha_+\beta_+}\widehat{h}_N\Big)i\xi_j - \alpha_-^{-1}i\xi_jA_{-N}\\
&+(\alpha_+^{-1}\omega_+ + \alpha_-^{-1}\omega_-)A_{-j} + \alpha_+^{-1}\omega_+\widehat{h}_j.
\end{aligned}
$$

It implies

$$
\begin{aligned}
A_{-j} = {} & \frac{1}{\alpha_+^{-1}\omega_+ + \alpha_-^{-1}\omega_-}(-\widehat{g}_j - \alpha_+^{-1}\omega_+\widehat{h}_j + (\alpha_+\beta_+)^{-1}i\xi_j\widehat{h}_N)\\
&-\frac{((\alpha_+\beta_+)^{-1}\beta_- - \alpha_-^{-1})i\xi_j}{\alpha_+^{-1}\omega_+ + \alpha_-^{-1}\omega_-}A_{-N}.
\end{aligned}
$$

By (2.20)

$$
\begin{aligned}
A_{+j} = {} & \frac{1}{\alpha_+^{-1}\omega_+ + \alpha_-^{-1}\omega_-}(-\widehat{g}_j + \alpha_-^{-1}\omega_-\widehat{h}_j + (\alpha_+\beta_+)^{-1}i\xi_j\widehat{h}_N)\\
&-\frac{((\alpha_+\beta_+)^{-1}\beta_- - \alpha_-^{-1})i\xi_j}{\alpha_+^{-1}\omega_+ + \alpha_-^{-1}\omega_-}A_{-N}.
\end{aligned}
\tag{2.28}
$$

Using the Volevich method [25], we have

$$\begin{aligned}
\widehat{H}_{+j}(x) = &-\int_0^\infty e^{-\omega_+(x_N+y_N)}\partial_N A_{+j}(\xi', y_N)\,dy_N \\
&+\int_0^\infty \omega_+ e^{-\omega_+(x_N+y_N)} A_{+j}(\xi', y_N)\,dy_N \quad \text{for } x_N > 0, \\
\widehat{H}_{-j}(x) = &\int_{-\infty}^0 e^{\omega_-(x_N+y_N)}\partial_N A_{-j}(\xi', y_N)\,dy_N \\
&+\int_{-\infty}^0 \omega_- e^{\omega_-(x_N+y_N)} A_{-j}(\xi', y_N)\,dy_N \quad \text{for } x_N < 0.
\end{aligned}$$

Using the identities (2.10) for $\omega_\pm$, we obtain

$$\begin{aligned}
\widehat{H}_{\pm j}(x) = &-\int_0^{\pm\infty}\Big\{\lambda^{1/2}e^{\mp\omega_\pm(x_N+y_N)}\frac{\alpha_\pm\mu_\pm\lambda^{1/2}}{\omega_\pm^2} \\
&+|\xi'|e^{\mp\omega_\pm(x_N+y_N)}\frac{|\xi'|}{\omega_\pm^2}\Big\}\partial_N A_{\pm j}(\xi', y_N)\,dy_N \\
&\pm\int_0^{\pm\infty}\Big\{\lambda^{1/2}e^{\mp\omega_\pm(x_N+y_N)}\frac{\alpha_\pm\mu_\pm\lambda^{1/2}}{\omega_\pm} \\
&+|\xi'|e^{\mp\omega_\pm(x_N+y_N)}\frac{|\xi'|}{\omega_\pm}\Big\}A_{\pm j}(\xi', y_N)\,dy_N
\end{aligned}\tag{2.29}$$

for $\pm x_N > 0$. In the sequel, we prove that there exist multipliers $m^1_{i,\pm j}$, $m^2_{ik,\pm j}$, $m^3_{i,\pm j}$, $m^4_{ik,\pm j}$ and $m^5_{ik\ell,\pm j}$ belonging to $\mathbb{M}_{-2,1}$, and multipliers $n^1_{i,\pm j}$, $n^2_{i,\pm j}$ and $n^3_{ik,\pm j}$ belonging to $\mathbb{M}_{-1,1}$ such that $A_{\pm j}$ and $\partial_N A_{\pm j}$ are represented as follows:

$$\begin{aligned}
A_{\pm j} = &\sum_{i=1}^N m^1_{i,\pm j}(\lambda, \xi')\mathcal{F}'[\lambda^{1/2}g_i](\xi', y_N) + \sum_{i,k=1}^N m^2_{ik,\pm j}(\lambda, \xi')\mathcal{F}'[\partial_i g_k](\xi', y_N) \\
&+\sum_{i=1}^N m^3_{i,\pm j}(\lambda, \xi')\mathcal{F}'[\lambda h_i](\xi', y_N) + \sum_{i,k=1}^N m^4_{ik,\pm j}(\lambda, \xi')\mathcal{F}'[\lambda^{\frac12}\partial_i h_k](\xi', y_N)
\end{aligned}$$

$$+\sum_{i,k,\ell=1}^N m^5_{ik\ell,\pm j}(\lambda,\xi')\mathcal{F}'[\partial_i\partial_k h_\ell](\xi',y_N) \quad \text{for } \pm x_N>0; \tag{2.30}$$

$$\partial_N A_{\pm j}=\sum_{i=1}^N n^1_{i,\pm j}(\lambda,\xi')\mathcal{F}'[\partial_N g_i](\xi',y_N)+\sum_{i=1}^N n^2_{i,\pm j}(\lambda,\xi')\mathcal{F}'[\lambda^{\frac12}\partial_N h_i](\xi',y_N)$$
$$+\sum_{i,k=1}^N n^3_{ik,\pm j}(\lambda,\xi')\mathcal{F}'[\partial_i\partial_N h_k](\xi',y_N) \quad \text{for } \pm x_N>0. \tag{2.31}$$

It suffices to prove (2.30) and (2.31) for A_{+j} and $\partial_N A_{+j}$, because A_{-j} and $\partial_N A_{-j}$ can be treated in the same manner. First, we treat A_{+N} given by (2.27). We write

$$i\xi'\cdot\widehat{g}'(\xi',y_N)=\sum_{k=1}^{N-1}\mathcal{F}'[\partial_k g_k](\xi',y_N)$$

and use (2.10) for $\omega_\pm$, we obtain

$$(\alpha_+^{-1}\omega_+ + \alpha_-^{-1}\omega_-)\widehat{g}_N(\xi',y_N)=\Big(\frac{\mu_+\lambda^{1/2}}{\omega_+}+\frac{\mu_-\lambda^{1/2}}{\omega_-}\Big)\mathcal{F}'[\lambda^{1/2}g_N](\xi',y_N)$$
$$-\sum_{k=1}^{N-1}\Big(\frac{\alpha_+^{-1}i\xi_k}{\omega_+}+\frac{\alpha_-^{-1}i\xi_k}{\omega_-}\Big)\mathcal{F}'[\partial_k g_N](\xi',y_N);$$
$$(\beta_-\alpha_+^{-1}\omega_+ + \beta_+\alpha_-^{-1}\omega_-)i\xi'\cdot\widehat{h}'(\xi',y_N)=\sum_{k=1}^{N-1}\Big(\frac{\beta_-\mu_+\lambda^{1/2}}{\omega_+}+\frac{\beta_-\mu_-\lambda^{1/2}}{\omega_-}\Big)$$
$$\times\mathcal{F}'[\lambda^{1/2}\partial_k h_k](\xi',y_N)-\sum_{k,\ell=1}^{N-1}\Big(\frac{\beta_-\alpha_+^{-1}i\xi_\ell}{\omega_+}+\frac{\beta_-\alpha_-^{-1}i\xi_\ell}{\omega_-}\Big)\mathcal{F}'[\partial_\ell\partial_k h_k](\xi',y_N);$$
$$(\mu_-\lambda+\beta_+(\alpha_-\beta_-)^{-1}|\xi'|^2+\alpha_+^{-1}\omega_+\omega_-)\widehat{h}_N(\xi',y_N)$$
$$=\Big(\mu_-+\frac{\mu_+\omega_-}{\omega_+}\Big)\mathcal{F}'[\lambda h_N](\xi',y_N)-\sum_{k=1}^{N-1}\Big(\frac{\beta_+}{\alpha_-\beta_-}+\frac{\alpha_+\omega_-}{\omega_+}\Big)\mathcal{F}'[\partial_k^2 h_N](\xi',y_N).$$

Since

$$(\det\mathcal{A})^{-1}\in\mathbb{M}_{-2,1},\quad \lambda^{1/2}\omega_\pm^{-1}\in\mathbb{M}_{0,1},\quad i\xi_k\omega_\pm^{-1}\in\mathbb{M}_{0,1},\quad \omega_-\omega_+^{-1}\in\mathbb{M}_{0,1},$$

we have (2.30) for A_{+N}.

On the other hand, by (2.27), we see that

$$\begin{aligned}\partial_N A_{+N}(\xi', y_N) = \frac{\beta_-\beta_+^{-1}}{\det A}\{&(\beta_- - \beta_+)i\xi'\cdot\partial_N\widehat{g}'(\xi', y_N)\\ &+ (\alpha_+^{-1}\omega_+ + \alpha_-^{-1}\omega_-)\partial_N\widehat{g}_N(\xi', y_N)\\ &- (\beta_-\alpha_+^{-1}\omega_+ + \beta_+\alpha_-^{-1}\omega_-)i\xi'\cdot\partial_N\widehat{h}'(\xi', y_N)\\ &- (\mu_-\lambda + \beta_+(\alpha_-\beta_-)^{-1}|\xi'|^2 + \alpha_+^{-1}\omega_+\omega_-)\partial_N\widehat{h}_N(\xi', y_N)\}.\end{aligned}$$

Notice that

$$\begin{aligned}&i\xi'\cdot\partial_N\widehat{g}'(\xi', y_N) = \sum_{k=1}^{N-1} i\xi_k\mathcal{F}'[\partial_N g_k](\xi', y_N),\\ &i\xi'\cdot\partial_N\widehat{h}'(\xi', y_N) = \sum_{k=1}^{N-1}\mathcal{F}'[\partial_k\partial_N h_k](\xi', y_N),\\ &(\mu_-\lambda + \beta_+(\alpha_-\beta_-)^{-1}|\xi'|^2 + \alpha_+^{-1}\omega_+\omega_-)\partial_N\widehat{h}_N(\xi', y_N)\\ &= (\mu_- + \mu_+\omega_-\omega_+^{-1})\lambda^{1/2}\mathcal{F}'[\lambda^{1/2}\partial_N h_N](\xi', y_N)\\ &- \sum_{k=1}^{N-1}(\beta_+(\alpha_-\beta_-)^{-1} + \alpha_+^{-1}\omega_-\omega_+^{-1})i\xi_k\mathcal{F}'[\partial_k\partial_N h_N](\xi', y_N).\end{aligned}$$

Since

$$\frac{\lambda^{1/2}}{\det\mathcal{A}} \in \mathbb{M}_{-1,1}, \quad \frac{i\xi_k}{\det\mathcal{A}} \in \mathbb{M}_{-1,1}, \quad \frac{\omega_\pm}{\det\mathcal{A}} \in \mathbb{M}_{-1,1},$$
$$\frac{\omega_-\lambda^{1/2}}{\omega_+\det\mathcal{A}} \in \mathbb{M}_{-1,1}, \quad \frac{\omega_- i\xi_k}{\omega_+\det\mathcal{A}} \in \mathbb{M}_{-1,1},$$

we have (2.31) for $\partial_N A_{+N}$.

Taking into account (2.10), we obtain the following relations:

$$\begin{aligned}&\widehat{g}_j \pm \alpha_\mp^{-1}\omega_\mp\widehat{h}_j + (\alpha_+\beta_+)^{-1}i\xi_j\widehat{h}_N = \frac{\alpha_+\mu_+\lambda^{\frac12}}{\omega_+^2}\mathcal{F}'[\lambda^{\frac12}g_j](\xi', y_N)\\ &-\sum_{k=1}^{N-1}\frac{i\xi_k}{\omega_+^2}\mathcal{F}'[\partial_k g_j](\xi', y_N) \pm \Big(\frac{\mu_\mp}{\omega_\mp}\mathcal{F}'[\lambda h_j](\xi', y_N) - \sum_{k=1}^{N-1}\frac{\alpha_\mp^{-1}}{\omega_\mp}\mathcal{F}'[\partial_k^2 h_j](\xi', y_N)\Big)\\ &+\Big(\frac{\mu_+\beta_+^{-1}\lambda^{\frac12}}{\omega_+^2}\mathcal{F}'[\lambda^{\frac12}\partial_j h_N](\xi', y_N) - \sum_{k=1}^{N-1}\frac{(\alpha_+\beta_+)^{-1}i\xi_k}{\omega_+^2}\mathcal{F}'[\partial_k\partial_j h_N](\xi', y_N)\Big).\end{aligned}$$

Since

$$\frac{\lambda^{1/2}}{(\alpha_+^{-1}\omega_+ + \alpha_-^{-1}\omega_-)\omega_+^2} \in \mathbb{M}_{-2,1}, \quad \frac{i\xi_k}{(\alpha_+^{-1}\omega_+ + \alpha_-^{-1}\omega_-)\omega_+^2} \in \mathbb{M}_{-2,1},$$
$$\frac{1}{(\alpha_+^{-1}\omega_+ + \alpha_-^{-1}\omega_-)\omega_\pm} \in \mathbb{M}_{-2,1},$$

in accordance with (2.28), we have (2.30) for $A_{\pm j}$. On the other hand, we see that

$$\begin{aligned}
&\partial_N(\widehat{g}_j \pm \alpha_\mp^{-1}\omega_\mp \widehat{h}_j + (\alpha_+\beta_+)^{-1} i\xi_j \widehat{h}_N) = \mathcal{F}'[\partial_N g_j](\xi', y_N)\\
&\pm \Big(\frac{\mu_\mp \lambda^{\frac12}}{\omega_\mp}\mathcal{F}'[\lambda^{\frac12}\partial_N h_j](\xi', y_N) - \sum_{k=1}^{N-1}\frac{\alpha_\mp^{-1} i\xi_k}{\omega_\mp}\mathcal{F}'[\partial_k\partial_N h_j](\xi', y_N)\Big)\\
&+\Big(\frac{\mu_+\beta_+^{-1}\lambda^{\frac12}}{\omega_+}\mathcal{F}'[\lambda^{\frac12}\partial_j h_N](\xi', y_N) - \sum_{k=1}^{N-1}\frac{(\alpha_+\beta_+)^{-1} i\xi_k}{\omega_+}\mathcal{F}'[\partial_k\partial_j h_N](\xi', y_N)\Big).
\end{aligned}$$

Since

$$\frac{\lambda^{1/2}}{(\alpha_+^{-1}\omega_+ + \alpha_-^{-1}\omega_-)\omega_\pm} \in \mathbb{M}_{-1,1}, \quad \frac{i\xi_k}{(\alpha_+^{-1}\omega_+ + \alpha_-^{-1}\omega_-)\omega_\pm} \in \mathbb{M}_{-1,1},$$

we have (2.31) for $\partial_N A_{\pm j}$.

Let F_1, F_2, F_3, F_4 and F_5 be corresponding to $\lambda^{1/2}\mathbf{g}$, $\mathbf{g}$, $\lambda\mathbf{h}$, $\lambda^{1/2}\mathbf{h}$, and $\mathbf{h}$, respectively. We define operators $\mathcal{B}_{\pm j}(\lambda)$ acting on $\mathbf{F} = (F_1, \dots, F_5)$ by the following rule:

$$\begin{aligned}
\mathcal{B}_{\pm j}(\lambda)\mathbf{F} = &-\int_0^{\pm\infty} \mathcal{F}_{\xi'}^{-1}\Big[\Big(\lambda^{\frac12} e^{\mp\omega_\pm(x_N+y_N)}\frac{\alpha_\pm\mu_\pm\lambda^{\frac12}}{\omega_\pm^2} + |\xi'| e^{\mp\omega_\pm(x_N+y_N)}\frac{|\xi'|}{\omega_\pm^2}\Big)\\
&\times\Big(\sum_{j=1}^N n^1_{j,\pm i}(\lambda,\xi')\mathcal{F}[\partial_N F_{2j}](\xi', y_N) + \sum_{j=1}^N n^2_{j,\pm i}(\lambda,\xi')\mathcal{F}[\partial_N F_{4j}](\xi', y_N)\\
&+ \sum_{j,k=1}^N n^3_{jk,\pm i}(\lambda,\xi')\mathcal{F}[\partial_j\partial_N F_{5k}](\xi', y_N)\Big)\Big](x')\, dy_N\\
&\pm\int_0^{\pm\infty} \mathcal{F}_{\xi'}^{-1}\Big[\Big(\lambda^{\frac12} e^{\mp\omega_\pm(x_N+y_N)}\frac{\alpha_\pm\mu_\pm\lambda^{\frac12}}{\omega_\pm} + |\xi'| e^{\mp\omega_\pm(x_N+y_N)}\frac{|\xi'|}{\omega_\pm}\Big)
\end{aligned}$$

$$\times \Big(\sum_{j=1}^N m^1_{j,\pm i}(\lambda,\xi')\mathcal{F}[F_{1j}](\xi',y_N) + \sum_{j,k=1}^N m^2_{jk,\pm i}(\lambda,\xi')\mathcal{F}[\partial_j F_{2k}](\xi',y_N)$$
$$+ \sum_{j=1}^N m^3_{j,\pm i}(\lambda,\xi')\mathcal{F}[F_{3j}](\xi',y_N) + \sum_{j,k=1}^N m^4_{jk,\pm i}(\lambda,\xi')\mathcal{F}[\partial_j F_{4k}](\xi',y_N)$$
$$+ \sum_{j,k,\ell=1}^N m^5_{jk\ell,\pm i}(\lambda,\xi')\mathcal{F}[\partial_j\partial_k F_{5\ell}](\xi',y_N)\Big)\Big](x')\,dy_N.$$

We set $(\mathcal{B}_I(\lambda)\mathbf{F})(x) = ((\mathcal{B}_{\pm 1}(\lambda)\mathbf{F})(x),\dots,(\mathcal{B}_{\pm N}(\lambda)\mathbf{F})(x))$ for $x \in \mathbb{R}^N_\pm$, in accordance with (2.29), (2.30) and (2.31),

$$\mathbf{H} = \mathcal{B}_I(\lambda)(\lambda^{1/2}\mathbf{g}, \nabla\mathbf{g}, \lambda\mathbf{h}, \lambda^{1/2}\nabla\mathbf{h}, \nabla^2\mathbf{h})$$

is a solution of problem (2.15). Moreover, by Lemma 2.9 and Proposition 2.2, we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}'_q(\mathbb{R}^N), H^{2-k}_q(\dot{\mathbb{R}}^N)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{k/2}\mathcal{B}_I(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leqslant \gamma_{\lambda_0}$$

for $\ell = 0,1$ and $k = 0,1,2$ with some constant γ_{λ_0} that depends on λ_0 in such a way that $\gamma_{\lambda_0} \to \infty$ as $\lambda_0 \to 0$. Here,

$$\begin{aligned}\mathcal{Y}'_q(\mathbb{R}^N) = \{(F_1,\dots,F_5) \mid\ & F_1, F_3 \in L_q(\mathbb{R}^N)^N,\\ & F_2, F_4 \in H^1_q(\mathbb{R}^N)^N,\ \ F_5 \in H^2_q(\mathbb{R}^N)^N\}.\end{aligned}$$

This completes the proof of the existence part of Theorem 2.10. Our final task is to prove the uniqueness. Let $\mathbf{H} = \mathbf{H}_\pm \in H^2_q(\mathbb{R}^N_\pm)^N$ satisfy the homogeneous equations:

$$\begin{cases} \mu\lambda\mathbf{H} - \alpha^{-1}\Delta\mathbf{H} = 0 & \text{in } \dot{\mathbb{R}}^N,\\ [[\alpha^{-1}\mathrm{curl}\,\mathbf{H}]]\mathbf{n} = 0, \quad [[\beta\mathrm{div}\,\mathbf{H}]] = 0 & \text{on } \mathbb{R}^N_0,\\ [[\mathbf{H} - (\mathbf{H}\cdot\mathbf{n})\mathbf{n}]] = 0, \quad [[\beta\mathbf{H}\cdot\mathbf{n}]] = 0 & \text{on } \mathbb{R}^N_0. \end{cases} \tag{2.32}$$

Let $\mathbf{f}$ be any elements in $L_q(\mathbb{R}^N)^N$ and let $\mathbf{G} = \mathbf{G}_\pm \in H^2_{q'}(\mathbb{R}^N_\pm)^N$ be solutions of the equations:

$$\begin{cases} \mu\overline{\lambda}\mathbf{G} - \alpha^{-1}\Delta\mathbf{G} = \mathbf{f} & \text{in } \dot{\mathbb{R}}^N,\\ [[\alpha^{-1}\mathrm{curl}\,\mathbf{G}]]\mathbf{n} = 0, \quad [[(\alpha\beta)^{-1}\mathrm{div}\,\mathbf{G}]] = 0 & \text{on } \mathbb{R}^N_0,\\ [[\mathbf{G} - (\mathbf{G}\cdot\mathbf{n})\mathbf{n}]] = 0, \quad [[(\alpha\beta)^{-1}\mathbf{G}\cdot\mathbf{n}]] = 0 & \text{on } \mathbb{R}^N_0. \end{cases} \tag{2.33}$$

By the divergence theorem of Gauss,

$$(\mathbf{H},\mathbf{f})_{\dot{\mathbb{R}}^N} = (\mathbf{H}, \mu\overline{\lambda}\mathbf{G})_{\dot{\mathbb{R}}^N} - (\mathbf{H}, \alpha^{-1}\Delta\mathbf{G})_{\dot{\mathbb{R}}^N}$$

$$
\begin{aligned}
&= \lambda(\mu\mathbf{H}, \mathbf{G})_{\dot{\mathbb{R}}^N} - (\mathbf{H}, \alpha^{-1}\mathrm{Div}\,\mathrm{curl}\,\mathbf{G})_{\dot{\mathbb{R}}^N} - (\beta\mathbf{H}, (\alpha\beta)^{-1}\nabla\mathrm{div}\,\mathbf{G})_{\dot{\mathbb{R}}^N} \\
&= \lambda(\mu\mathbf{H}, \mathbf{G})_{\dot{\mathbb{R}}^N} + \frac{1}{2}(\alpha^{-1}\mathrm{curl}\,\mathbf{H}, \mathrm{curl}\,\mathbf{G})_{\dot{\mathbb{R}}^N} + (\alpha^{-1}\mathrm{div}\,\mathbf{H}, \mathrm{div}\,\mathbf{G})_{\dot{\mathbb{R}}^N}.
\end{aligned}
$$

On the other hand, by the divergence theorem of Gauß,

$$
\begin{aligned}
0 &= (\mu\lambda\mathbf{H} - \alpha^{-1}\Delta\mathbf{H}, \mathbf{G})_{\dot{\mathbb{R}}^N} \\
&= \lambda(\mu\mathbf{H}, \mathbf{G})_{\dot{\mathbb{R}}^N} - (\alpha^{-1}\mathrm{Div}\,\mathrm{curl}\,\mathbf{H}, \mathbf{G})_{\dot{\mathbb{R}}^N} - (\beta\nabla\mathrm{div}\,\mathbf{H}, (\alpha\beta)^{-1}\mathbf{G})_{\dot{\mathbb{R}}^N} \\
&= \lambda(\mu\mathbf{H}, \mathbf{G})_{\dot{\mathbb{R}}^N} + \frac{1}{2}(\alpha^{-1}\mathrm{curl}\,\mathbf{H}, \mathrm{curl}\,\mathbf{G})_{\dot{\mathbb{R}}^N} + (\alpha^{-1}\mathrm{div}\,\mathbf{H}, \mathrm{div}\,\mathbf{G})_{\dot{\mathbb{R}}^N}.
\end{aligned}
$$

Thus, we have $(\mathbf{H}, \mathbf{f})_{\dot{\mathbb{R}}^N_\pm} = 0$. The arbitrary choice of $\mathbf{f} \in L_q(\mathbb{R}^N)^N$ yields that $\mathbf{H} = 0$, which shows the uniqueness. This completes the proof of Theorem 2.10.

§3. $\mathcal{R}$-bounded solution operators in a bent space

Let $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ be a bijection of C^1 class and let Φ^{-1} be its inverse map. Let $\nabla\Phi = \mathcal{A} + B(x)$ and $\nabla\Phi^{-1} = \mathcal{A}_{-1} + B_{-1}(x)$. We assume that $\mathcal{A}$ and $\mathcal{A}_{-1}$ are orthogonal matrices with constant coefficients and $B(x)$ and $B_{-1}(x)$ are matrices of functions in $W^2_r(\mathbb{R}^N)$ with $N < r < \infty$ such that

$$
\|B\|_{L_\infty(\mathbb{R}^N)}, \|B_{-1}\|_{L_\infty(\mathbb{R}^N)} \leqslant M_1, \quad \|\nabla B\|_{H^1_r(\mathbb{R}^N)}, \|\nabla B_{-1}\|_{H^1_r(\mathbb{R}^N)} \leqslant M_2. \tag{3.1}
$$

Note that $\mathcal{A}_{-1} = \mathcal{A}^\top$ and $\mathcal{A}\mathcal{A}^\top = \mathcal{A}^\top\mathcal{A} = \mathbf{I}$. We will choose M_1 small enough eventually, so that we may assume that $0 < M_1 \leqslant 1 \leqslant M_2$ in the following. We set $\Omega_\pm = \Phi(\mathbb{R}^N_\pm)$, $\Gamma = \Phi(\mathbb{R}^N_0)$. Note that $\dot{\Omega} \cup \Gamma = \mathbb{R}^N$. Let $\mathbf{n}$ be the unit normal to Γ, outward with respect to Ω_+, $\Phi^{-1} = (\Phi_{-1,1}, \ldots, \Phi_{-1,N})$. In this case Γ is represented by $\Phi_{-1,N}(y) = 0$. It implies

$$
\mathbf{n} = \frac{\nabla\Phi_{-1,N}}{|\nabla\Phi_{-1,N}|} = \frac{(\mathcal{A}_{N1} + B_{N1}, \ldots, \mathcal{A}_{NN} + B_{NN})^\top}{(\sum_{i=1}^N (\mathcal{A}_{Ni} + B_{Ni})^2)^{1/2}}, \tag{3.2}
$$

where $\mathcal{A}_{-1} = (\mathcal{A}_{ij})$ and $B_{-1} = (B_{ij})$ ($\mathbf{n}$ is defined on the whole $\mathbb{R}^N$). Choosing $M_1 > 0$ in (3.1) small enough, we have

$$
\mathbf{n} = (\mathcal{A}_{N1} \ldots, \mathcal{A}_{NN})^\top + \mathbf{b}_n, \tag{3.3}
$$

where $\mathbf{b}_n = (b_{n1}, \ldots, b_{nN})^\top \in H^1_r(\mathbb{R}^N)^N$ and satisfies the estimates:

$$
\|\mathbf{b}_n\|_{L_\infty(\mathbb{R}^N)} \leqslant C_N M_1, \quad \|\nabla\mathbf{b}_n\|_{H^1_r(\mathbb{R}^N)} \leqslant C_{N,r} M_2^2.
$$

For the interface problem

$$\begin{cases} \mu\lambda\mathbf{H} - \alpha^{-1}\Delta\mathbf{H} = \mathbf{f} & \text{in } \dot{\Omega}, \\ [[\alpha^{-1}(\operatorname{curl}\mathbf{H})\mathbf{n}]] = \mathbf{g}', \quad [[\beta\operatorname{div}\mathbf{H}]] = g_N & \text{on } \Gamma, \\ [[\mathbf{H}- <\mathbf{H},\mathbf{n}>\mathbf{n}]] = \mathbf{h}', \quad [[\beta\mathbf{H}\cdot\mathbf{n}]] = h_N & \text{on } \Gamma, \end{cases} \tag{3.4}$$

we have the following result.

Theorem 3.1. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. There exist constants $M_1 \in (0,1)$, $\lambda_0 \geqslant 1$, and an operator family*

$$\mathcal{B}_b(\lambda) \in \operatorname{Hol}(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{Y}_q(\mathbb{R}^N), H^2_q(\dot{\Omega})^N))$$

such that for any $(\mathbf{f},\mathbf{g},\mathbf{h}) \in Y_q(\mathbb{R}^N)$ and $\lambda \in \Sigma_{\epsilon,\lambda_0}$, the unique solution to problem (3.4) *is given by $\mathbf{H} = \mathcal{B}_b(\lambda)F_\lambda(\mathbf{f},\mathbf{g},\mathbf{h})$, where $F_\lambda(\mathbf{f},\mathbf{g},\mathbf{h}) = (\mathbf{f}, \lambda^{1/2}\mathbf{g}, \mathbf{g}, \lambda\mathbf{h}, \lambda^{1/2}\mathbf{h}, \mathbf{h})$. Moreover, $\mathcal{B}_b(\lambda)$ possesses the estimate:*

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\mathbb{R}^N), H^{2-j}_q(\dot{\Omega})^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{B}_b(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leqslant \gamma_b, \quad \tau = \operatorname{Im}\lambda.$$

Here, M_1 depends solely on ϵ, $\mu_\pm$, $\alpha_\pm$, $\beta_\pm$, q and N; λ_0 and γ_b depend solely on ϵ, $\mu_\pm$, $\alpha_\pm$, $\beta_\pm$, M_2, q and N. $Y_q(\mathbb{R}^N)$, $\mathcal{Y}_q(\mathbb{R}^N)$ are defined in Theorem 2.10.

We give a sketch of the proof. We transfer (3.4) into a problem in $\dot{\mathbb{R}}^N$ by the change of the variable: $x = \Phi^{-1}(y)$ with $y \in \Omega_\pm$ and $x \in \mathbb{R}^N_\pm$. In this case,

$$\frac{\partial}{\partial y_j} = \sum_{\ell=1}^N (\mathcal{A}_{\ell j} + B_{\ell j})\frac{\partial}{\partial x_\ell}, \tag{3.5}$$

where $\mathcal{A}_{-1} = (A_{ij})$, $B_{-1} \circ \Phi = (B_{ij})$. As by (3.1) $B_{\ell j}$ are small enough, Theorem 3.1 can be deduced from Theorem 2.10 by the help of the following lemma which is a consequence of Sobolev's imbedding theorem (cf. Shibata [16, Lemma 2.4]).

Lemma 3.2. *Let $1 < q \leqslant r < \infty$, $r > N$. There exists a constant $C_{N,r,q}$ such that for any $\sigma > 0$, $a \in L_r(\mathbb{R}^N_+)$ and $b \in W^1_q(\mathbb{R}^N_+)$ the following estimate*

$$\|ab\|_{L_q(\mathbb{R}^N_+)} \leqslant \sigma\|\nabla b\|_{L_q(\mathbb{R}^N_+)} + C_{N,r,q}\sigma^{-\frac{N}{r-N}}\|a\|^{\frac{r}{r-N}}_{L_r(\mathbb{R}^N_+)}\|b\|_{L_q(\mathbb{R}^N_+)}$$

holds.

The detailed proof of the similar result for the Stokes equations with free boundary conditions is given in Shibata [17, 18]. As the proof of Theorem 3.1 is almost parallel to the proof in [17, 18], we may omit the details.

Let $\Omega_+ = \Phi(\mathbb{R}^N_+)$, $\Gamma = \Phi(\mathbb{R}^N_0)$, and $\mathbf{n}$ is the unit normal to Γ, outward with respect to Ω_+. Consider the problem:

$$\begin{cases} \mu_-\lambda\mathbf{H} - \alpha_-^{-1}\Delta\mathbf{H} = \mathbf{f} & \text{in } \Omega_+, \\ (\operatorname{curl}\mathbf{H})\mathbf{n} = \mathbf{g}_-, \quad \mathbf{H}\cdot\mathbf{n} = h_- & \text{on } \Gamma. \end{cases} \tag{3.6}$$

Employing the similar arguments, we deduce from Theorem 2.5 the following result.

Theorem 3.3. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Let*

$$X_q(\Omega_+) = \{(\mathbf{f}, \mathbf{g}_-, h_-) \mid \mathbf{f} \in L_q(\Omega_+)^N,\ \mathbf{g}_- \in H^1_q(\Omega_+)^{N-1},\ h_- \in H^2_q(\Omega_+)\},$$

$$\mathcal{X}_q(\Omega_+) = \{F = (F_0, F_6, F_7, F_8, F_9, F_{10}) \mid F_0 \in L_q(\Omega_+)^N,\ F_6 \in L_q(\Omega_+)^{N-1},$$
$$F_7 \in H^1_q(\Omega_+)^{N-1},\ F_8 \in L_q(\Omega_+),\ F_9 \in H^1_q(\Omega_+),\ F_{10} \in H^2_q(\Omega_+)\}.$$

Then, there exist constants $M_1 \in (0,1)$, $\lambda_0 \geqslant 1$, and an operator family

$$\mathcal{B}_b(\lambda) \in \operatorname{Hol}(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(\Omega_+), H^2_q(\Omega_+)^N))$$

such that for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$ and $(\mathbf{f}, \mathbf{g}_-, h_-) \in X_q(\Omega_+)$,
$\mathbf{H} = \mathcal{B}_b(\lambda)F^1_\lambda(\mathbf{f}, \mathbf{g}_-, h_-)$, where

$$F^1_\lambda(\mathbf{f}, \mathbf{g}_-, h_-) = (\mathbf{f}, \lambda^{1/2}\mathbf{g}_-, \mathbf{g}_-, \lambda h_-, \lambda^{1/2}h_-, h_-),$$

is a unique solution of problem (3.6), and

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega_+), H^{2-j}_q(\Omega_+)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{B}_b(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leqslant \gamma_b, \quad \tau = \operatorname{Im}\lambda$$

for $\ell = 0, 1$ and $j = 0, 1, 2$. The constant M_1 depends solely on ϵ, μ_-, α_-, q, and N. The constants λ_0 and γ_b depend solely on ϵ, μ_-, α_-, M_2, q, and N.

§4. Proof of Theorem 1.5

4.1. Some preparations for the proof of Theorem 1.5. First, we state some properties of a uniform $W^{3-1/r}_r$ domain that we use to prove Theorem 1.5 below. Employing the same argumentation as that in the proof of Proposition 6.1 in Enomoto and Shibata [7], we can prove the following result.

Proposition 4.1. *Let $N < r < \infty$ and let Ω be a uniform $W_r^{3-1/r}$ domain in $\mathbb{R}^N$. Let M_1 be the number given in Sect. 3. Then, there exist constants $M_2 > 0$, $0 < d^0, d^1, d^2 < 1$, at most countably many N-vector of functions $\Phi_j^i \in H^3_{r,\mathrm{loc}}(\mathbb{R}^N)^N$ $(i = 0,1)$ and points $x_j^0 \in \Gamma_0 = \Gamma$, $x_j^1 \in \Gamma_1 = S$ and $x^2_{\pm j} \in \dot\Omega_\pm$ such that the following assertions hold:*

(i) *The maps: $\mathbb{R}^N \ni x \mapsto \Phi_j^i(x) \in \mathbb{R}^N$ $(i = 0,1, j \in \mathbb{N})$ are bijective.*

(ii) $\Omega = \Big(\bigcup_{j=1}^\infty (\Phi_j^0(\mathbb{R}^N) \cap B_{d^0}(x_j^0))\Big) \cup \Big(\bigcup_{j=1}^\infty (\Phi_j^1(\mathbb{R}^N_+) \cap B_{d^1}(x_j^1))\Big) \cup \bigcup_{j=1}^\infty (B_{d^2}(x^2_{+j}) \cup B_{d^2}(x^2_{-j}))$,
$B_{d^2}(x^2_{\pm j}) \subset \Omega_\pm$, $\quad \Phi_j^0(\mathbb{R}^N) \cap B_{d^0}(x_j^0) = \Omega \cap B_{d^0}(x_j^0)$, $\quad \Phi_j^0(\mathbb{R}^N_0) \cap B_{d^0}(x_j^i) = \Gamma \cap B_{d^0}(x_j^0)$,
$\Phi_j^1(\mathbb{R}^N_+) \cap B_{d^1}(x_j^1) = \Omega \cap B_{d^1}(x_j^1)$, $\quad \Phi_j^1(\mathbb{R}^N_0) \cap B_{d^1}(x_j^1) = S \cap B_{d^1}(x_j^1)$.

(iii) *There exist C^∞ functions ζ_j^i, $\zeta^2_{\pm j}$, $\widetilde\zeta_j^i$ $(i = 0, 1)$, and $\widetilde\zeta^2_{\pm j}$, $(j \in \mathbb{N})$, such that*

$$
\begin{aligned}
&0 \leqslant \zeta_j^i,\ \zeta^2_{\pm j},\ \widetilde\zeta_j^i,\ \widetilde\zeta^2_{\pm j} \leqslant 1, \quad \operatorname{supp}\zeta_j^i, \operatorname{supp}\widetilde\zeta_j^i \subset B_{d^i}(x_j^i),\\
&\quad \operatorname{supp}\zeta^2_{\pm j}, \operatorname{supp}\widetilde\zeta^2_{\pm j} \subset B_{d^2}(x^2_{\pm j})\\
&\|(\zeta_j^i, \zeta^2_{\pm j}, \widetilde\zeta_j^i, \widetilde\zeta^2_{\pm j})\|_{H^3_\infty(\mathbb{R}^N)} \leqslant c_0,\ \ \widetilde\zeta_j^i = 1 \ \text{ on } \operatorname{supp}\zeta_j^i,\ \ \widetilde\zeta^2_{\pm j} = 1 \ \text{ on } \operatorname{supp}\zeta^2_{\pm j},\\
&\sum_{i=0}^1\sum_{j=1}^\infty \zeta_j^i + \sum_\pm \sum_{j=1}^\infty \zeta^2_{\pm j} = 1 \ \text{ on } \overline\Omega, \quad \sum_{j=1}^\infty \zeta_j^0 = 1 \ \text{ on } \Gamma,\ \ \sum_{j=1}^\infty \zeta_j^1 = 1 \ \text{ on } S
\end{aligned}
$$

Here, c_0 is a constant which depends on M_2, N, q and r, but independent of $j \in \mathbb{N}$.

(iv) *$\nabla\Phi_j^i = \mathcal{A}_j^i + B_j^i$, $\nabla(\Phi_j^i)^{-1} = \mathcal{A}^i_{j,-1} + B^i_{j,-1}$, where $\mathcal{A}_j^i$ and $\mathcal{A}^i_{j,-1}$ are $N\times N$ constant orthogonal matrices with constant coefficients, and B_j^i and $B^i_{j,-1}$ are $N \times N$ matrices of $H^2_{r,\mathrm{loc}}(\mathbb{R}^N)$ functions defined on $\mathbb{R}^N$ which satisfy the conditions: $\|(B_j^i, B^i_{j,-1})\|_{L_\infty(\mathbb{R}^N)} \leqslant M_1$ and $\|\nabla(B_j^i, B^i_{j,-1})\|_{H^1_r(\mathbb{R}^N)} \leqslant M_2$ for $i = 0, 1$ and $j \in \mathbb{N}$.*

(v) *There exists a natural number $L \geqslant 2$ such that any $L + 1$ distinct sets of $\{B_{d^i}(x_j^i) \mid i = 0,1,\ j \in \mathbb{N}\} \cup \{B_{d^2}(x^2_{\pm j}) \mid j \in \mathbb{N}\}$ have an empty intersection.*

In the sequel, we write $B_{d^i}(x_j^i)$, $B_{d^2}(x^2_{\pm j})$ simply by B_j^i, $B^2_{\pm j}$, respectively. By the finite intersection property stated in Proposition 4.1 (v), for any $r \in (1, \infty)$ and $k \in \mathbb{N}_0$, there exists a constant $C_{k,r,L}$ such that

$$\Big[\sum_{j=1}^{\infty} \|f\|_{H^k_r(\Omega\cap A_j)}^r\Big]^{1/r} \leqslant C_{k,r,L}\|f\|_{H^k_r(\Omega)} \tag{4.1}$$
$$\text{for any } f \in H^k_r(\Omega) \text{ and } A_j \in \{B^0_j, B^1_j, B^2_{\pm j}\}.$$

By the help of (4.1), in the similar way as in the proof of Lemma 4.3 in Shibata [16], we can prove the following proposition:

Proposition 4.2. *Let $1 < q < \infty$, $q' = q/(q-1)$ and $i = 0, 1, 2$. Let $A_j \in \{B^0_j, B^1_j, B^2_{\pm j}\}$. Then, the following assertions hold.*

(i) *Let m be a non-negative integer. Let $\{f_j\}_{j=1}^\infty$ be a sequence in $H^m_q(\Omega)$ and let $\{g^{(\ell)}_j\}_{j=1}^\infty$ $(\ell = 0, 1, \ldots, m)$ be sequences of positive real numbers. Assume that*

$$\sum_{j=1}^{\infty} (g^{(\ell)}_j)^q < \infty, \quad |(\nabla^\ell f_j, \varphi)_\Omega| \leqslant M_3 g^{(\ell)}_j \|\varphi\|_{L_{q'}(\Omega\cap A_j)}$$
$$\text{for any } \varphi \in L_q(\Omega) \text{ and } \ell = 0, 1, \ldots, m$$

with some constant M_3 independent of $j \in \mathbb{N}$. Then, $f = \sum\limits_{j=1}^{\infty} f_j$ exists in the strong topology of $H^m_q(\Omega)$ and

$$\|\nabla^\ell f\|_{L_q(\Omega)} \leqslant C_{q',L} M_3 \Big(\sum_{j=1}^{\infty} (g^{(\ell)}_j)^q\Big)^{\frac{1}{q}}.$$

(ii) *Let n be a natural number. Let $\{f^{(i)}_j\}_{j=1}^\infty$ $(i = 1, \ldots, n)$ be sequences in $L_q(\Omega)$ and let $\{g^{(i)}_j\}_{j=1}^\infty$ $(i = 1, \ldots, n)$ be sequences of positive numbers. Let a_i $(i = 1, \ldots, n)$ be any complex numbers. Assume that*

$$\sum_{j=1}^{\infty} (g^{(i)}_j)^q < \infty, \quad |(f^{(i)}_j, \varphi)_\Omega| \leqslant M_3 g^{(i)}_j \|\varphi\|_{L_{q'}(\Omega\cap A_j)}$$
$$\text{for any } \varphi \in L_q(\Omega) \text{ and } i = 1, \ldots, n$$

with some constant M_3 independent of $j = 1, 2, 3, \ldots$. In addition, we assume that there exists a sequence of positive numbers $\{h_j\}_{j=1}^\infty$

such that

$$\sum_{j=1}^{\infty}(h_j)^q < \infty, \quad \Big|\Big(\sum_{i=1}^{n} a_i f_j^{(i)}, \varphi\Big)_\Omega\Big| \leqslant M_3 h_j \, \|\varphi\|_{L_{q'}(\Omega\cap A_j)}.$$

Then, $f^{(i)} = \sum_{j=1}^{\infty} f_j^{(i)}$ *exist in the strong topology of* $L_q(\Omega)$,

$i = 1, \dots, n$ *and* $\Big\| \sum_{i=1}^{n} a_i f^{(i)} \Big\|_{L_q(\Omega)} \leqslant C_{q',L} M_3 \Big(\sum_{j=1}^{\infty}(h_j)^q\Big)^{\frac{1}{q}}$.

4.2. Local solutions. In what follows, we use the notations $\Phi_j^0(\mathbb{R}^N_\pm) = \mathcal{H}^0_{\pm j}$, $\Phi_j^0(\mathbb{R}^N_0) = \Gamma_j$, $\Phi_j^1(\mathbb{R}^N_+) = \mathcal{H}^1_{-j}$, and $\Phi_j^1(\mathbb{R}^N_0) = S_j^1$, and set $\mathcal{H}^2_{j\pm} = \mathbb{R}^N$. Let $\mathbf{n}_j^0$ and $\mathbf{n}_j^1$ be the unit normal to Γ_j oriented from $\mathcal{H}^0_{+j}(\mathbb{R}^N+)$ into $\mathcal{H}^0_{-j}$ and the unit outer normal to S_j^1, respectively. Let $\dot{\mathcal{H}}_j^0 = \mathcal{H}^0_{+j} \cup \mathcal{H}^0_{-j}$ and $\mathcal{H}_j^0 = \dot{\mathcal{H}}_j^0 \cup \Gamma_j = \Phi_j^0(\mathbb{R}^N) = \mathbb{R}^N$. Let $\mathbf{F} = (\mathbf{f}, \mathbf{g}, \mathbf{h}, \mathbf{g}_-, h_-) \in Z_q(\Omega)$. We consider the following problems:

$$\begin{aligned} \mu\lambda \mathbf{H}_j^0 - \alpha^{-1}\Delta \mathbf{H}_j^0 &= \widetilde{\zeta}_j^0 \mathbf{f} && \text{in } \dot{\mathcal{H}}_j^0, \\ [[\alpha^{-1}(\mathrm{curl}\, \mathbf{H}_j^0)\mathbf{n}_j^0]] = \widetilde{\zeta}_j^0 \mathbf{g}', \quad [[\beta \mathrm{div}\, \mathbf{H}_j^0]] &= \widetilde{\zeta}_j^0 g_N && \text{on } \Gamma_j, \\ [[\mathbf{H}_j^0 - < \mathbf{H}_j^0, \mathbf{n}_j^0 > \mathbf{n}_j^0]] = \widetilde{\zeta}_j^0 \mathbf{h}', \quad [[\mu \mathbf{n}_j^0 \cdot \mathbf{H}_j^0]] &= \widetilde{\zeta}_j^0 h_N && \text{on } \Gamma_j, \end{aligned} \tag{4.2}$$

$$\begin{aligned} \mu_-\lambda \mathbf{H}_j^1 - \alpha_-^{-1}\Delta \mathbf{H}_j^1 &= \widetilde{\zeta}_j^1 \mathbf{f} && \text{in } \mathcal{H}^1_{-j}, \\ \mathbf{n}_j^1 \cdot \mathbf{H}_j^1 = \widetilde{\zeta}_j^1 h_-, \quad (\mathrm{curl}\, \mathbf{H}_j^1)\mathbf{n}_j^1 &= \widetilde{\zeta}_j^1 \mathbf{g}_- && \text{on } S_j^1, \end{aligned} \tag{4.3}$$

$$\mu_\pm\lambda \mathbf{H}^2_{\pm j} - \alpha_\pm^{-1}\Delta \mathbf{H}^2_{\pm j} = \widetilde{\zeta}^2_{\pm j} \mathbf{f} \quad \text{in } \mathcal{H}^2_{\pm j}. \tag{4.4}$$

Note that $\mathbf{n}_j^i$ $(i = 0, 1)$ are defined on the whole $\mathbb{R}^N$ and $\|\mathbf{n}_j^i\|_{H^3_r(B^i_j)} \leqslant C_N M_2$ $(i = 0, 1)$. In addition, $\mathbf{n}_0 = \mathbf{n}_j^0$ on $\Gamma \cap B_j^0$ and $\mathbf{n} = \mathbf{n}_j^1$ on $S \cap B_j^1$. By Theorem 3.1, Theorem 3.3 and Theorem 2.4 there exist a constant $\lambda_0 \geqslant 1$, and operator families

$$\begin{aligned} &\mathcal{T}_j^0(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{Y}_q^0(\mathcal{H}_j^0), H_q^2(\dot{\mathcal{H}}_j^0)^N)), \\ &\mathcal{T}_j^1(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(\mathcal{H}^1_{-j}), H_q^2(\mathcal{H}^1_{-j})^N)), \\ &\mathcal{T}^2_{\pm j}(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(L_q(\mathcal{H}^2_{\pm j}), H_q^2(\mathcal{H}^2_{\pm j})^N) \end{aligned} \tag{4.5}$$

such that

$$\begin{aligned} &\mathbf{H}_j^0 = \mathcal{T}_j^0(\lambda) F_\lambda^0(\widetilde{\zeta}_j^0 \mathbf{f}, \widetilde{\zeta}_j^0 \mathbf{g}, \widetilde{\zeta}_j^0 \mathbf{h}), \;\; \mathbf{H}_j^1 = \mathcal{T}_j^1(\lambda) F_\lambda^1(\widetilde{\zeta}_j^1 \mathbf{f}, \widetilde{\zeta}_j^1 \mathbf{g}_-, \widetilde{\zeta}_j^1 h_-), \\ &\mathbf{H}^2_{\pm j} = \mathcal{T}^2_{\pm j}(\lambda) \widetilde{\zeta}^2_\pm \mathbf{f}, \end{aligned} \tag{4.6}$$

where

$$F^0_\lambda(\widetilde{\zeta}^0_j\mathbf{f}, \widetilde{\zeta}^0_j\mathbf{g}, \widetilde{\zeta}^0_j\mathbf{h}) = (\widetilde{\zeta}^0_j\mathbf{f}, \lambda^{1/2}\widetilde{\zeta}^0_j\mathbf{g}, \widetilde{\zeta}^0_j\mathbf{g}, \lambda\widetilde{\zeta}^0_j\mathbf{h}, \lambda^{1/2}\widetilde{\zeta}^0_j\mathbf{h}, \widetilde{\zeta}^0_j\mathbf{h}),$$
$$F^1_\lambda(\widetilde{\zeta}^1_j\mathbf{f}, \widetilde{\zeta}^1_j\mathbf{g}_-, \widetilde{\zeta}^1_j h_-) = (\widetilde{\zeta}^1_j\mathbf{f}, \lambda^{1/2}\widetilde{\zeta}^1_j\mathbf{g}_-, \widetilde{\zeta}^1_j\mathbf{g}_-, \lambda\widetilde{\zeta}^1_j h_-, \lambda^{1/2}\widetilde{\zeta}^1_j h_-, \widetilde{\zeta}^1_j h_-)$$

are unique solutions to the problems (4.2), (4.3) and (4.4), respectively. Moreover, we have

$$\begin{aligned}
&\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\mathcal{H}^0_j), H^{2-k}_q(\dot{\mathcal{H}}^0_j)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{k/2}\mathcal{T}^0_j(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leqslant \kappa,\\
&\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathcal{H}^1_{-j}), H^{2-k}_q(\mathcal{H}^1_{-j})^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{k/2}\mathcal{T}^1_j(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leqslant \kappa,\\
&\mathcal{R}_{\mathcal{L}(L_q(\mathcal{H}^i_{\pm j})^N, H^{2-k}_q(\mathcal{H}^i_{\pm j})^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{k/2}\mathcal{T}^2_{\pm j}(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leqslant \kappa
\end{aligned} \tag{4.7}$$

for $\ell = 0, 1$ and $k + |\alpha| = 2$ $(k = 0, 1, 2)$ with some constant κ independent of $j \in \mathbb{N}$. By (4.7), we obtain

$$\begin{aligned}
&\sum_{k=0}^{2} |\lambda|^{k/2}\|\mathbf{H}^0_j\|_{H^{2-k}_q(\dot{\mathcal{H}}^0_j)} \leqslant \kappa\|F^0_\lambda(\widetilde{\zeta}^0_j\mathbf{f}, \widetilde{\zeta}^0_j\mathbf{g}, \widetilde{\zeta}^0_j\mathbf{h})\|_{\mathcal{Y}_q(\mathcal{H}^0_j)},\\
&\sum_{k=0}^{2} |\lambda|^{k/2}\|\mathbf{H}^1_j\|_{H^{2-k}_q(\mathcal{H}^1_j)} \leqslant \kappa\|F^1_\lambda(\widetilde{\zeta}^1_j\mathbf{f}, \widetilde{\zeta}^1_j\mathbf{g}_-, \widetilde{\zeta}^1_j h_-)\|_{\mathcal{X}_q(\mathcal{H}^1_j)},\\
&\sum_{k=0}^{2} |\lambda|^{k/2}\|\mathbf{H}^2_{\pm j}\|_{H^{2-k}_q(\mathcal{H}^2_{\pm j})} \leqslant \kappa\|\widetilde{\zeta}^2_{j\pm}\mathbf{f}\|_{L_q(\mathcal{H}^2_{\pm j})},
\end{aligned} \tag{4.8}$$

for any $j \in \mathbb{N}$.

4.3. Construction of a parametrix. We define the parametrix $\mathbf{U}(\lambda)\mathbf{F}$ by the following formula

$$\begin{aligned}
\mathbf{U}(\lambda)\mathbf{F} &= \sum_{i=0}^{1}\sum_{j=1}^{\infty} \zeta^i_j\mathbf{H}^i_j + \sum_{\pm}\sum_{j=1}^{\infty} \zeta^2_{\pm j}\mathbf{H}^2_{\pm j} \qquad (4.9)\\
&= \sum_{j=1}^{\infty} \zeta^0_j\mathcal{T}^0_j(\lambda)F^0_\lambda(\widetilde{\zeta}^0_j\mathbf{f}, \widetilde{\zeta}^0_j\mathbf{g}, \widetilde{\zeta}^0_j\mathbf{h}) + \sum_{j=1}^{\infty} \zeta^1_j\mathcal{T}^1_j(\lambda)F^1_\lambda(\widetilde{\zeta}^1_j\mathbf{f}, \widetilde{\zeta}^1_j\mathbf{g}_-, \widetilde{\zeta}^1_j h_-)\\
&+ \sum_{j=1}^{\infty} \zeta^2_{+j}\mathcal{T}^2_{+j}(\lambda)\widetilde{\zeta}^2_{+j}\mathbf{f} + \sum_{j=1}^{\infty} \zeta^2_{-j}\mathcal{T}^2_{-j}(\lambda)\widetilde{\zeta}^2_{-j}\mathbf{f}, \quad \mathbf{F} = (\mathbf{f}, \mathbf{g}, \mathbf{h}, \mathbf{g}_-, h_-) \in Z_q(\Omega).
\end{aligned}$$

To represent the jump quantity, we make the following preparation: Let $\mathcal{T}_j^0(\lambda) = (\mathcal{T}_{j1}^0(\lambda), \dots, \mathcal{T}_{jN}^0(\lambda))$, and let $E_\mp[\mathcal{T}_{jk}^0(\lambda)|_{\mathcal{H}_\mp^0}]\mathbf{F}$ be the Lions extension of $\mathcal{T}_{jk}^0(\lambda)F|_{\Gamma_\mp}$ into $\mathcal{H}_{\mp j}^0$ such that

$$\begin{aligned}&\sum_{k=0}^{2} |\lambda|^{k/2}\|E_\pm[\mathcal{T}_{jk}^0(\lambda)|_{\mathcal{H}_{\mp j}^0}]F\|_{H_q^{2-k}(\mathcal{H}_j^0)} \leqslant C\kappa\|F\|_{\mathcal{Y}_q(\mathcal{H}_j^0)},\\ &\partial_x^\alpha(E_\pm[\mathcal{T}_{jk}^0(\lambda)|_{\mathcal{H}_{\mp j}^0}]F)|_\Gamma = \partial_x^\alpha(\mathcal{T}_{jk}^0(\lambda)F)|_{\Gamma_j\mp} \quad (|\alpha| \leqslant 2),\end{aligned} \tag{4.10}$$

where

$$f|_{\Gamma_j\mp}(x_0) = \lim_{\substack{x\in\mathcal{H}_{\mp j}^0\\ x\to x_0}} f(x) \quad \text{for } x_0 \in \Gamma_j.$$

Using these notations, we obtain

$$\begin{aligned}[][[\alpha^{-1}\mathrm{curl}\,(\zeta_j^0\mathcal{T}_j^0(\lambda)F)]]\mathbf{n}_j^0 &= \zeta_j^0[[\alpha^{-1}\mathrm{curl}\,\mathcal{T}_j^0(\lambda)F]]\mathbf{n}_j^0 + R_{curl,j}^0(\lambda)F,\\ [[\beta\mathrm{div}\,(\zeta\mathcal{T}_j^0(\lambda)F)]] &= \zeta_j^0[[\beta\mathrm{div}\,\mathcal{T}_j^0(\lambda)F]] + R_{\mathrm{div}\,,j}^0(\lambda)F,\end{aligned} \tag{4.11}$$

where

$$\begin{aligned}R_{curl,j}^0(\lambda)F|_{(k,\ell)} &= \Big(\frac{\partial\zeta_j^0}{\partial x_k}\Big)\{\alpha_+^{-1}E_-[\mathcal{T}_{j\ell}^0(\lambda)|_{\mathcal{H}_{+j}^0}]F - \alpha_-^{-1}E_+[\mathcal{T}_{j\ell}^0(\lambda)|_{\mathcal{H}_{-j}^0}]F\}|_\Gamma\\ &- \Big(\frac{\partial\zeta_j^0}{\partial x_\ell}\Big)\{\alpha_+^{-1}E_-[\mathcal{T}_{jk}^0(\lambda)|_{\mathcal{H}_{+j}^0}]F - \alpha_-^{-1}E_+[\mathcal{T}_{jk}^0(\lambda)|_{\mathcal{H}_{-j}^0}]F\}|_\Gamma,\\ R_{div,j}^0(\lambda)F &= \sum_{k=1}^{N}\Big(\frac{\partial\zeta_j^0}{\partial x_k}\Big)\{\alpha_+^{-1}E_-[\mathcal{T}_{jk}^0(\lambda)|_{\mathcal{H}_{+j}^0}]F - \alpha_-^{-1}E_+[\mathcal{T}_{jk}^0(\lambda)|_{\mathcal{H}_{-j}^0}]F\}|_\Gamma.\end{aligned}$$

Also we have the relation

$$\mathrm{curl}\,(\zeta_j^1\mathcal{T}_j^1(\lambda)F) = \zeta_j^1\mathrm{curl}\,\mathcal{T}_j^1(\lambda)F + R_{curl,j}^1(\lambda)F$$

with

$$R_{curl,j}^1(\lambda)F|_{(k,\ell)} = \Big(\frac{\partial\zeta_j^1}{\partial x_k}\Big)\mathcal{T}_{j\ell}^1(\lambda)F - \Big(\frac{\partial\zeta_j^1}{\partial x_\ell}\Big)\mathcal{T}_{jk}^1(\lambda)F.$$

Proposition 4.2 and estimates (4.8) imply $\mathbf{U}(\lambda)\mathbf{F} \in H_q^2(\dot\Omega)^N$. Inserting $\mathbf{H} = \mathbf{U}(\lambda)\mathbf{F}$ into (1.8) and taking into account that $\mathbf{n}_0 = \mathbf{n}_j^0$ on $\mathrm{supp}\,\zeta_j^0\cap\Gamma$, $\mathbf{n} = \mathbf{n}_j^1$ on $\mathrm{supp}\,\zeta_j^1 \cap S$, we arrive at

$$\begin{aligned}\mu\lambda\mathbf{H} - \alpha^{-1}\Delta\mathbf{H} &= \mathbf{f} - \mathbf{V}^1(\lambda)\mathbf{F} && \text{in } \dot\Omega,\\ ([[\alpha^{-1}\mathrm{curl}\,\mathbf{H}]]\mathbf{n}_0, \quad [[\beta\mathrm{div}\,\mathbf{H}]]) &= \mathbf{g} - \mathbf{V}^2(\lambda)\mathbf{F} && \text{on } \Gamma,\\ [[\mathbf{H}- <\mathbf{n}_0, \mathbf{H}> \mathbf{n}_0]] = \mathbf{h}', \quad [[\beta\mathbf{n}_0\cdot\mathbf{H}]] &= h_N && \text{on } \Gamma,\end{aligned}$$

$$[[\mathrm{curl}\,\mathbf{H}_-]]\mathbf{n} = \mathbf{g}'_- - \mathbf{V}^3(\lambda)\mathbf{F}, \quad \mathbf{H}_- \cdot \mathbf{n} = h_- \qquad \text{on } S, \tag{4.12}$$

where

$$\begin{aligned}
\mathbf{V}^1(\lambda)\mathbf{F} &= \alpha^{-1}\{\sum_{j=1}^{\infty}[2(\nabla\zeta_j^0) : (\nabla\mathcal{T}_j^0(\lambda)F_\lambda^0(\widetilde{\zeta}_j^0\mathbf{f}, \widetilde{\zeta}_j^0\mathbf{g}, \widetilde{\zeta}_j^0\mathbf{h}) \\
&\quad + (\Delta\zeta_j^0)\mathcal{T}_j^0(\lambda)F_\lambda^0(\widetilde{\zeta}_j^0\mathbf{f}, \widetilde{\zeta}_j^0\mathbf{g}, \widetilde{\zeta}_j^0\mathbf{h})] \\
&\quad + \alpha^{-1}\{\sum_{j=1}^{\infty}[2(\nabla\zeta_j^1) : (\nabla\mathcal{T}_j^1(\lambda)F_\lambda^1\widetilde{\zeta}_j^1(\mathbf{f}, \mathbf{g}'_-, h_-) \\
&\quad + (\Delta\zeta_j^1)\mathcal{T}_j^1(\lambda)F_\lambda^1\widetilde{\zeta}_j^1(\mathbf{f}, \mathbf{g}'_-, h_))] \qquad (4.13) \\
&\quad + \alpha^{-1}\sum_{\pm}\sum_{j=1}^{\infty}[2(\nabla\zeta_{\pm j}^2) : (\nabla\mathcal{T}_{\pm j}^2(\lambda)\widetilde{\zeta}_{\pm j}^2\mathbf{f}) + (\Delta\zeta_{\pm j}^2)\mathcal{T}_{\pm j}^2(\lambda)\widetilde{\zeta}_{\pm j}^2\mathbf{f}] \\
\mathbf{V}^2(\lambda)\mathbf{F} &= \sum_{j=1}^{\infty}(R_{curl,j}^0(\lambda)F_\lambda^0(\widetilde{\zeta}_j^0\mathbf{f}, \widetilde{\zeta}_j^0\mathbf{g}, \widetilde{\zeta}_j^0\mathbf{h}), R_{div,j}^0(\lambda)F_\lambda^0(\widetilde{\zeta}_j^0\mathbf{f}, \widetilde{\zeta}_j^0\mathbf{g}, \widetilde{\zeta}_j^0\mathbf{h})), \\
\mathbf{V}^3(\lambda)\mathbf{F} &= \sum_{j=1}^{\infty}R_{curl,j}^1(\lambda)F_\lambda^1(\widetilde{\zeta}_j^1\mathbf{f}, \widetilde{\zeta}_j^1\mathbf{g}'_-, \widetilde{\zeta}_j^1 h_-).
\end{aligned}$$

We set

$$\mathbf{V}(\lambda)\mathbf{F} = (\mathbf{V}^1(\lambda)\mathbf{F}, \mathbf{V}^2(\lambda)\mathbf{F}, 0, \mathbf{V}^3(\lambda)\mathbf{F}, 0).$$

Proposition 4.2 and estimates (4.8) imply that $\mathbf{V}(\lambda) \in Z_q(\Omega)$,

$$\|F_\lambda\mathbf{V}(\lambda)\mathbf{F}\|_{\mathcal{Z}_q(\Omega)} \leqslant C\lambda_1^{-1/2}\|F_\lambda\mathbf{F}\|_{\mathcal{Z}_q(\Omega)} \tag{4.14}$$

for any $\lambda \in \Sigma_{\epsilon,\lambda_1}$, $\lambda_1 \geqslant \lambda_0 \geqslant 1$, where F_λ is the operator given in (1.11) in Theorem 1.5. Since $\|F_\lambda\mathbf{F}\|_{\mathcal{Z}_q(\Omega)}$, $\lambda \neq 0$ are equivalent norms of $Z_q(\Omega)$, we can choose $\lambda_1 \geqslant \lambda_0$ so large that in (4.14) $C\lambda_1^{-1/2} \leqslant 1/2$. We see that there exists $(\mathbf{I} - \mathbf{V}(\lambda))^{-1} \in \mathcal{L}(Z_q(\Omega))$ and $\mathbf{H} = \mathbf{U}(\lambda)(\mathbf{I} - \mathbf{V}(\lambda))^{-1}\mathbf{F}$ is a solution of (1.8). The uniqueness follows from the existence theorem of dual problem.

4.4. Construction of $\mathcal{R}$-bounded solution operators. For $F = (F_0, F_1, \ldots, F_{10}) \in \mathbf{Z}_q(\Omega)$, $F^0 = (F_0, F_1, F_2, F_3, F_4, F_5) \in \mathcal{Y}_q(\Omega)$, $F^1 = (F_0|_{\Omega_-}, F_6, F_7, F_8, F_9, F_{10}) \in \mathcal{X}_q(\Omega)$, we define the following operators:

$$\mathcal{U}(\lambda)\mathbf{F} = \sum_{i=0}^{1}\sum_{j=1}^{\infty}\zeta_j^i\mathcal{T}_j^i(\lambda)F^i + \sum_{\pm}\sum_{j=1}^{\infty}\zeta_{\pm j}^2\mathcal{T}_{\pm j}^2(\lambda)F_0,$$

$$
\begin{aligned}
\mathcal{V}^1(\lambda)F &= \alpha^{-1}\sum_{j=1}^\infty [2(\nabla\zeta^0_j):(\nabla\mathcal{T}^0_j(\lambda)\widetilde{\zeta}^0_j F^0 + (\Delta\zeta^0_j)\mathcal{T}^0_j(\lambda)\widetilde{\zeta}^0_j F^0] \\
&+ \alpha^{-1}\sum_{j=1}^\infty [2(\nabla\zeta^1_j):(\nabla\mathcal{T}^1_j(\lambda)\widetilde{\zeta}^1_j F^+ + (\Delta\zeta^1_j)\mathcal{T}^1_j(\lambda)\widetilde{\zeta}^1_j F^1] \\
&+ \alpha^{-1}\sum_{\pm}\sum_{j=1}^\infty [2(\nabla\zeta^2_{\pm j}):(\nabla\mathcal{T}^2_{\pm j}(\lambda)\widetilde{\zeta}^2_{\pm j} F_0) + (\Delta\zeta^2_{\pm j})\mathcal{T}^2_{\pm j}(\lambda)\widetilde{\zeta}^2_{\pm j} F_0],
\end{aligned}
$$

$$
\mathcal{V}^2(\lambda)\mathbf{F} = \sum_{j=1}^\infty (R^0_{curl,j}(\lambda)\widetilde{\zeta}^0_j F^0, R^0_{div,j}(\lambda)\widetilde{\zeta}^0_j F^0),
$$

$$
\mathcal{V}^3(\lambda)\mathbf{F} = \sum_{j=1}^\infty R^1_{curl,j}(\lambda)\widetilde{\zeta}^1_j F^1,
$$

$$
\mathcal{V}(\lambda)F = (\mathcal{V}^1(\lambda)F, \mathcal{V}^2(\lambda)F, 0, \mathcal{V}^3(\lambda)F, 0). \tag{4.15}
$$

Obviously, $\mathbf{U}(\lambda)\mathbf{F} = \mathcal{U}(\lambda)F_\lambda\mathbf{F}$ and $\mathbf{V}(\lambda)\mathbf{F} = \mathcal{V}(\lambda)F_\lambda\mathbf{F}$. By (4.5), (4.7) and Proposition 4.2, we see that

$$
\begin{aligned}
\mathcal{U}(\lambda) &\in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_1}, \mathcal{L}(\mathcal{Z}_q(\Omega), H^2_q(\dot\Omega)^N)),\\
\mathcal{V}(\lambda) &\in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_1}, \mathcal{L}(\mathcal{Z}_q(\Omega), Z_q(\Omega))).
\end{aligned}
$$

Moreover, by (4.7) and Proposition 4.2 we have

$$
\begin{aligned}
&\mathcal{R}_{\mathcal{L}(\mathcal{Z}_q(\Omega), H^{2-k}_q(\dot\Omega)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{k/2}\mathcal{U}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,M}\}) \leqslant C\kappa \\
&\quad (k=0,1,2), \\
&\mathcal{R}_{\mathcal{L}(\mathcal{Z}_q(\Omega))}(\{(\tau\partial_\tau)^\ell F_\lambda\mathcal{V}(\lambda) \mid \lambda \in \Sigma_{\epsilon,M}\}) \leqslant CM^{-1/2}\kappa \quad (\ell = 0,1)
\end{aligned} \tag{4.16}
$$

for any $M \geqslant \lambda_1$. By (4.16), $\mathcal{A}(\lambda)F = \mathcal{U}(\lambda)(\mathbf{I} - \mathcal{V}(\lambda))^{-1}F$ exists and

$$
\mathcal{R}_{\mathcal{Z}_q(\Omega), H^{2-k}_q(\dot\Omega)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{k/2}\mathcal{A}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,M}\}) \leqslant C\kappa
$$

for $\ell = 0,1$ and $k = 0,1,2$. Since $\mathcal{V}(\lambda)F_\lambda\mathbf{F} = \mathbf{V}(\lambda)\mathbf{F}$, we have

$$
\begin{aligned}
F_\lambda(\mathbf{I} - \mathbf{V}(\lambda))^{-1} &= \sum_{j=0}^\infty F_\lambda\mathbf{V}(\lambda)^j = \sum_{j=0}^\infty F_\lambda(\mathcal{V}(\lambda)F_\lambda)^j \\
&= \sum_{j=0}^\infty (F_\lambda\mathcal{V}(\lambda))^j F_\lambda = (\mathbf{I} - F_\lambda\mathcal{V}(\lambda))^{-1}F_\lambda,
\end{aligned}
$$

consequently,

$$\begin{aligned}\mathbf{H} &= \mathbf{U}(\lambda)(\mathbf{I}-\mathbf{V}(\lambda))^{-1}\mathbf{F} = \mathcal{U}(\lambda)F_\lambda(\mathbf{I}-\mathbf{V}(\lambda))^{-1}\mathbf{F}\\ &= \mathcal{U}(\lambda)(\mathbf{I}-F_\lambda\mathcal{V}(\lambda))^{-1}F_\lambda\mathbf{F} = \mathcal{A}(\lambda)F_\lambda\mathbf{F}.\end{aligned}$$

This completes the proof of Theorem 1.5.

§5. Proof of Theorem 1.2

Since $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$, $\mathbf{g}_-$, and h_- in the right-hand side of Eq. (1.4) are defined for $t \in \mathbb{R}$, we can divide a solution to problem (1.4) into two parts: $\mathbf{H}_1 = \mathbf{H}_{1\pm}$ and $\mathbf{H}_2 = \mathbf{H}_{\pm 2}$, where $\mathbf{H}_i$ $(i = 1, 2)$ are solutions to the following problems:

$$\begin{aligned} \mu\partial_t\mathbf{H}_1 - \alpha^{-1}\Delta\mathbf{H}_1 &= \mathbf{f} && \text{in } \dot\Omega\times\mathbb{R},\\ [[\alpha^{-1}\mathrm{curl}\,\mathbf{H}_1]]\mathbf{n}_0 = \mathbf{g}', \quad [[\beta\mathrm{div}\,\widehat{\mathbf{H}}_1]] &= g_N && \text{on } \Gamma\times\mathbb{R},\\ [[\mathbf{H}_1 - <\mathbf{H}_1, \mathbf{n}_0> \mathbf{n}_0]] = \mathbf{h}', \quad [[\beta\mathbf{H}_1\cdot\mathbf{n}_0]] &= h_N && \text{on } \Gamma\times\mathbb{R},\\ (\mathrm{curl}\,\mathbf{H}_{1-})\mathbf{n} = \mathbf{g}_-, \quad \mathbf{n}\cdot\mathbf{H}_{1-} &= h_- && \text{on } S\times\mathbb{R}; \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} \mu\partial_t\mathbf{H}_2 - \alpha^{-1}\Delta\mathbf{H}_2 &= 0 && \text{in } \dot\Omega\times(0,\infty),\\ [[\alpha^{-1}\mathrm{curl}\,\mathbf{H}_2]]\mathbf{n}_0 = 0, \quad [[\beta\mathrm{div}\,\widehat{\mathbf{H}}_2]] &= 0 && \text{on } \Gamma\times(0,\infty),\\ [[\mathbf{H}_2 - <\mathbf{H}_2, \mathbf{n}_0> \mathbf{n}_0]] = 0, \quad [[\beta\mathbf{H}_2\cdot\mathbf{n}_0]] &= 0 && \text{on } \Gamma\times(0,\infty),\\ (\mathrm{curl}\,\mathbf{H}_{2-})\mathbf{n} = 0, \quad \mathbf{n}\cdot\mathbf{H}_{2-} &= 0 && \text{on } S\times(0,\infty),\\ \mathbf{H}_2|_{t=0} &= \mathbf{H}_0 - \mathbf{H}_1|_{t=0} && \text{in } \dot\Omega. \end{aligned} \tag{5.2}$$

Applying the Laplace transform to Eq. (5.1), we arrive at

$$\begin{aligned} \mu\lambda\widehat{\mathbf{H}}_1 - \alpha^{-1}\Delta\widehat{\mathbf{H}}_1 &= L[\mathbf{f}] && \text{in } \dot\Omega,\\ [[\alpha^{-1}\mathrm{curl}\,\widehat{\mathbf{H}}_1]]\mathbf{n}_0 = L[\mathbf{g}'], \quad [[\beta\mathrm{div}\,\widehat{\mathbf{H}}_1]] &= L[g_N] && \text{on } \Gamma,\\ [[\widehat{\mathbf{H}}_1 - <\widehat{\mathbf{H}}_1, \mathbf{n}_0> \mathbf{n}_0]] = L[\mathbf{h}'], \quad [[\beta\widehat{\mathbf{H}}_1\cdot\mathbf{n}_0]] &= L[h_N] && \text{on } \Gamma,\\ (\mathrm{curl}\,\widehat{\mathbf{H}}_{1-})\mathbf{n} = L[\mathbf{g}_-], \quad \mathbf{n}\cdot\widehat{\mathbf{H}}_{1-} &= L[h_-] && \text{on } S. \end{aligned} \tag{5.3}$$

By Theorem 1.5, we have $\widehat{\mathbf{H}}_1 = [\mathcal{A}(\lambda)\mathbf{G}(\lambda)]$, where

$$\begin{aligned}\mathbf{G}(\lambda) = (&L[\mathbf{f}], \lambda^{1/2}L[\mathbf{g}], L[\mathbf{g}], \lambda L[\mathbf{h}], \lambda^{1/2}L[\mathbf{h}], L[\mathbf{h}],\\ &\lambda^{1/2}L[\mathbf{g}_-], L[\mathbf{g}_-], \lambda L[h_-], \lambda^{1/2}L[h_-], L[h_-])\end{aligned}$$

for $\lambda = \gamma + i\tau \in \Sigma_{\epsilon,\lambda_1}$. The function $\mathbf{H}_1 = L_\lambda^{-1}[\widehat{\mathbf{H}}_1]$ is a solution to the non-stationary problem (5.1). To estimate $\mathbf{H}_1$, we use the Weis operator valued Fourier multiplier theorem [26] stated as follows:

Theorem 5.1. *Let X and Y be two UMD Banach spaces and $1 < p < \infty$. Let M be a function in $C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(X, Y))$ such that*

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\{(\rho\partial_\rho)^\ell M(\rho) \mid \rho \in \mathbb{R} \setminus \{0\}\}) = \kappa_\ell < \infty \quad (\ell = 0, 1).$$

Let T_M be the operator defined by $T_M\phi = \mathcal{F}^{-1}[M\mathcal{F}[\phi]]$ for any ϕ with $\mathcal{F}[\phi] \in \mathcal{D}(\mathbb{R}, X)$. Then, T_M is extended to a bounded linear operator from $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, Y)$. Moreover, denoting this extension also by T_M, we have

$$\|T_M\phi\|_{L_p(\mathbb{R},Y)} \leqslant C(\kappa_0 + \kappa_1)\|\phi\|_{p(\mathbb{R},X)}$$

for any $\phi \in L_p(\mathbb{R}, X)$ with some positive constant C depending on p.

Since any Lebesgue space and Sobolev space on domains in $\mathbb{R}^N$ are UMD space (cf. Amann [1]), applying Theorem 5.1 and taking into account that $L[f] = \mathcal{F}[e^{-\gamma t} f]$ and $e^{-\gamma t} L_\lambda^{-1}[g] = \mathcal{F}_\tau^{-1}[g]$, we have

$$\begin{aligned}
&\|e^{-\gamma t}\partial_t\mathbf{H}_1\|_{L_p(\mathbb{R},L_q(\dot\Omega))} + \|e^{-\gamma t}\mathbf{H}_1\|_{L_p(\mathbb{R},H^2_q(\dot\Omega))} \leqslant C\{\|e^{-\gamma t}\mathbf{f}\|_{L_p(\mathbb{R},L_q(\dot\Omega))} \\
&+ \|e^{-\gamma t}\mathbf{g}\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))} + \|e^{-\gamma t}\mathbf{g}\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))} + \|e^{-\gamma t}\partial_t\mathbf{h}\|_{L_p(\mathbb{R},L_q(\dot\Omega))} \\
&+ \|e^{-\gamma t}\mathbf{h}\|_{L_p(\mathbb{R},H^2_q(\dot\Omega))} + \|e^{-\gamma t}\mathbf{g}_-\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega_-))} + \|e^{-\gamma t}\mathbf{g}_-\|_{L_p(\mathbb{R},H^1_q(\Omega_-))} \\
&+ \|e^{-\gamma t}\partial_t h_-\|_{L_p(\mathbb{R},L_q(\Omega_-))} + \|e^{-\gamma t}h_-\|_{L_p(\mathbb{R},H^2_q(\Omega_-))}\}. \qquad (5.4)
\end{aligned}$$

Here, we have used the fact that

$$\|e^{-\gamma t}f\|_{H^{1/2}_p(\mathbb{R},H^1_q(\dot\Omega))} \leqslant C\{\|e^{-\gamma t}\partial_t f\|_{L_p(\mathbb{R},L_q(\dot\Omega))} + \|e^{-\gamma t}f\|_{L_p(\mathbb{R},H^2_q(\dot\Omega))}\}.$$

To solve problem (5.2), we use the semi-group approach. Let us consider the resolvent problem:

$$\begin{cases}
\lambda\mathbf{H} - (\alpha\mu)^{-1}\Delta\mathbf{H} = \mathbf{f} & \text{in } \dot\Omega, \\
[[\alpha^{-1}\mathrm{curl}\,\mathbf{H}]]\mathbf{n}_0 = 0, \quad [[\beta\mathrm{div}\,\widehat{\mathbf{H}}]] = 0 & \text{on } \Gamma, \\
[[\mathbf{H} - <\mathbf{H}, \mathbf{n}_0> \mathbf{n}_0]] = 0, \quad [[\beta\mathbf{H}\cdot\mathbf{n}_0]] = 0, & \text{on } \Gamma, \\
(\mathrm{curl}\,\mathbf{H}_-)\mathbf{n} = 0, \quad \mathbf{n}\cdot\mathbf{H}_- = 0 & \text{on } S.
\end{cases} \qquad (5.5)$$

Let

$$\begin{aligned}
\mathcal{D}_q(\dot\Omega) = \{\mathbf{H} \in H^2_q(\dot\Omega)^N \mid\ & [[\alpha^{-1}\mathrm{curl}\,\mathbf{H}]]\mathbf{n}_0 = 0, \quad [[\beta\mathrm{div}\,\widehat{\mathbf{H}}]] = 0 \quad \text{on } \Gamma, \\
& [[\mathbf{H} - <\mathbf{H}, \mathbf{n}_0> \mathbf{n}_0]] = 0, \quad [[\beta\mathbf{H}\cdot\mathbf{n}_0]] = 0 \quad \text{on } \Gamma,
\end{aligned}$$

$$(\operatorname{curl}\mathbf{H}_-)\mathbf{n} = 0, \quad \mathbf{n}\cdot\mathbf{H}_- = 0 \quad \text{on } S\}.$$

We set $\mathbf{A}\mathbf{H} = (\mu\alpha)^{-1}\Delta\mathbf{H}$ for $\mathbf{H}\in\mathcal{D}_q(\dot\Omega)$.

Then, problem (5.5) takes the form

$$(\lambda - \mathbf{A})\mathbf{H} = \mathbf{f} \quad \text{for } \mathbf{H}\in\mathcal{D}_q(\dot\Omega).$$

Since $\mathcal{R}$-boundedness implies usual boundedness, $\rho(\mathbf{A})\supset\Sigma_{\epsilon,\lambda_1}$, and

$$|\lambda|\|(\mu\lambda-\mathbf{A})^{-1}\mathbf{f}\|_{L_q(\dot\Omega)} + \|(\mu\lambda-\mathbf{A})^{-1}\mathbf{f}\|_{H^2_q(\dot\Omega)} \leqslant C\|\mathbf{f}\|_{L_q(\dot\Omega)}.$$

Thus, the operator $\mathbf{A}$ generates C_0 analytic semi-group $\{T(t)\}t\geqslant 0$ associated with problem (5.2). Moreover, if we define

$$\mathcal{D}_{q,p}(\dot\Omega) = (L_q(\dot\Omega), \mathcal{D}_q(\dot\Omega))_{1-1/p,p},$$

where $(\cdot,\cdot)_{1-1/p,p}$ denotes a real interpolation functor, we have the following maximal regularity result.

Theorem 5.2. *Let $< q < \infty$ and let $\{T(t)\}_{t\geqslant 0}$ be the C_0 analaytic semi-group defined above. Then,*

$$\|e^{-\gamma t}\partial_t T(t)\mathbf{f}\|_{L_p(\mathbb{R}_+,L_q(\Omega))} + \|e^{-\gamma t}T(t)\mathbf{f}\|_{L_p(\mathbb{R}_+,H^2_q(\dot\Omega))} \leqslant C_\gamma\|\mathbf{f}\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)}$$

for any $\mathbf{f}\in\mathcal{D}_{q,p}(\dot\Omega)$ and $\gamma\geqslant\lambda_1$ with some constant $C>0$, where λ_1 is the same constant as in Theorem 1.5.

We see that

$$\begin{aligned}\mathcal{D}_{q,p}(\dot\Omega) = \{&\mathbf{H}=\mathbf{H}_\pm\in B^{2(1-1/p)}_{q,p}(\dot\Omega) \mid \\ &[[\alpha^{-1}\operatorname{curl}\mathbf{H}]]=0, [[\beta\operatorname{div}\mathbf{H}]]=0 \ \ [[\mathbf{H}-<\mathbf{H},\mathbf{n}_0>\mathbf{n}_0]]=0,\\ &[[\beta\mathbf{H}\cdot\mathbf{n}_0]]=0 \ \text{ on } \Gamma, \ \ (\operatorname{curl}\mathbf{H}_-)\mathbf{n}=0, \ \ \mathbf{n}\cdot\mathbf{H}_-=0 \ \text{ on } S\}\end{aligned}$$

provided that $2/p+1/q<1$;

$$\begin{aligned}\mathcal{D}_{q,p}(\dot\Omega) = \{&\mathbf{H}=\mathbf{H}_\pm\in B^{2(1-1/p)}_{q,p}(\dot\Omega) \mid \\ &[[\mathbf{H}-<\mathbf{H},\mathbf{n}_0>\mathbf{n}_0]]=0, \ \ [[\beta\mathbf{H}\cdot\mathbf{n}_0]]=0 \ \text{ on } \Gamma, \quad \mathbf{n}\cdot\mathbf{H}_-=0 \ \text{ on } S\}\end{aligned}$$

provided that $1<2/p+1/q<2$, and $\mathcal{D}_{q,p}(\dot\Omega)=B^{2(1-1/p)}_{q,p}(\dot\Omega)$ provided that $2/p+1/q>2$. Let $\mathbf{H}_2 = T(t)(\mathbf{H}_0-\mathbf{H}_1|_{t=0})$. By the compatibility condition (1.7), $\mathbf{H}_0-\mathbf{H}_1|_{t=0}\in\mathcal{D}_{q,p}(\dot\Omega)$, consequently, by Theorem 5.2, $\mathbf{H}_2$ satisfies the estimate:

$$\begin{aligned}&\|e^{-\gamma t}\partial_t\mathbf{H}_2\|_{L_p((\mathbb{R}_+,L_q(\dot\Omega))} + \|e^{-\gamma t}\mathbf{H}_2\|_{L_p(\mathbb{R}_+,H^2_q(\dot\Omega))}\\ &\leqslant C(\|\mathbf{H}_0\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)} + \|\mathbf{H}_1|_{t=0}\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)}).\end{aligned}$$

We know that

$$H^1_p(\mathbf{R}_+, L_q(\dot\Omega)) \cap L_p(\mathbb{R}_+, H^2_q(\dot\Omega)) \subset BUC([0,\infty), B^{2(1-1/p)}_{q,p}(\dot\Omega))$$

where the inclusion is continuous and BUC is the space of bounded uniformly continuous functions (cf. Tanabe [24, (1.18)]), therefore

$$\|\mathbf{H}_1|_{t=0}\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)} \leqslant C(\|e^{-\gamma t}\partial_t\mathbf{H}_1\|_{L_p(\mathbb{R}_+,L_q(\dot\Omega))} + \|e^{-\gamma t}\mathbf{H}_1\|_{L_p(\mathbb{R}_+,H^2_q(\dot\Omega))}).$$

Thus, $\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2$ is a required solution of Eq. (1.4). The uniqueness follows from the existence of C_0 analytic semi-group $\{T(t)\}_{t\geqslant 0}$. Theorem 1.2 is proved.

Appendix §A. Divergence free condition

In this appendix, we show that if $\mathbf{v}$, $\mathfrak{p}$, and $\mathbf{H}$ is a solution to problem (1.3) with $\operatorname{div}\mathbf{H}|_{t=0} = 0$ in $\dot\Omega$, then $\operatorname{div}\mathbf{H} = 0$ on $\dot\Omega$ as long as the solution exists. Let $\mathbf{K} = (K_{\pm ij}) = \mathbf{K}_\pm$ be an $N\times N$ antisymmetric matrix of functions from $H^1_q(\Omega_{t\pm})$ and $\mathbf{G} = (G_1,\ldots,G_N)^\top = \mathbf{G}_\pm \in H^1_q(\dot\Omega)^N$. If $\mathbf{K}$ and $\mathbf{G}$ satisfy the conditions:

$$[[\mathbf{K}\mathbf{n}_t]] = 0, \quad [[< \mathbf{G}, \tau_{\mathbf{t}} >]] = \mathbf{0} \qquad \text{on } \Gamma_t, \quad \mathbf{K}_-\mathbf{n}|_{\mathbf{S}} = \mathbf{0},$$

where $\{\tau_{t1},\ldots,\tau_{tN-1}\}$ is a orthogonal base of tangent space of Γ_t, then the divergence theorem of Gauss implies

$$(\operatorname{Div}\mathbf{K}, \mathbf{G})_{\dot\Omega_t} = \frac{1}{2}(\mathbf{K}, \operatorname{curl}\mathbf{G})_{\dot\Omega_t}. \tag{A.1}$$

Let $T_0 \in (0,T]$, and that $\psi = \psi_\pm(x) \in L_q(\dot\Omega)$ be an arbitrary function. We consider the following problem:

$$\begin{cases} \partial_t\varphi + (\alpha\mu)^{-1}\Delta\varphi = 0 & \text{in } \dot Q_{T_0}, \\ [[\varphi]] = 0, \quad [[(\alpha\mu)^{-1}\mathbf{n}_t\cdot\nabla\varphi]] = 0 & \text{on } G_T, \\ \mathbf{n}\cdot(\nabla\varphi_-) = 0 & \text{on } S\times(0,T_0), \\ \varphi|_{t=T_0} = \psi & \text{in } \dot\Omega_{T_0}. \end{cases} \tag{A.2}$$

Let $\varphi = \varphi_\pm$ be a solution to (A.2), $\mathbf{H}$ a solution to (1.3), and $\operatorname{div}\mathbf{H}|_{t=0} = 0$. From $[[\varphi]] = 0$ it follows that $[[< \nabla\varphi, \tau_{\mathbf{t}} >]] = \mathbf{0}$. Consequently, with the help of (1.2) and (A.1), we obtain

$$\begin{aligned}(\mu\partial_t\mathbf{H}, \nabla\varphi)_{\dot\Omega_t} &= (\alpha^{-1}\Delta\mathbf{H} + \operatorname{Div}\mu(\mathbf{v}\otimes\mathbf{H} - \mathbf{H}\otimes\mathbf{v}), \nabla\varphi)_{\dot\Omega_t} \\ &= -\frac{1}{2}(\alpha^{-1}\operatorname{curl}\mathbf{H} - \mu(\mathbf{v}\otimes\mathbf{H} - \mathbf{H}\otimes\mathbf{v}), \operatorname{curl}\nabla\varphi)_{\dot\Omega_t} - (\mu\operatorname{div}\mathbf{H}, (\alpha\mu)^{-1}\Delta\varphi)_{\dot\Omega_t}.\end{aligned}$$

Since $(\alpha\mu)^{-1}\Delta\varphi = -\partial_t\varphi$ in $\dot\Omega_t$ and $\operatorname{curl}\nabla\varphi = 0$, we have

$$(\mu\partial_t\mathbf{H}, \nabla\varphi)_{\dot\Omega_t} = (\mu\operatorname{div}\mathbf{H}, \partial_t\varphi)_{\dot\Omega_t}.$$

Since $[[(\mu\partial_t\mathbf{H})\cdot\mathbf{n}_t]] = 0$ as follows from $[[\mu\mathbf{H}\cdot\mathbf{n}_t]] = 0$ and since $[[\varphi]] = 0$, we arrive at

$$(\mu\partial_t\mathbf{H}, \nabla\varphi)_{\dot\Omega_t} = -(\partial_t(\mu\operatorname{div}\mathbf{H}), \varphi)_{\dot\Omega_t}.$$

Combination of these two formulas gives us the relation

$$\begin{aligned} 0 &= (\partial_t(\mu\operatorname{div}\mathbf{H}), \varphi)_{\dot\Omega_t} + (\mu\operatorname{div}\mathbf{H}, \partial_t\varphi)_{\dot\Omega_t} \\ &= \frac{d}{dt}\int\limits_{\dot\Omega_t}(\mu\operatorname{div}\mathbf{H})\varphi\,dx - \int\limits_{\dot\Omega_t}\mathbf{v}\cdot\nabla((\mu\operatorname{div}\mathbf{H})\varphi)\,dx. \end{aligned}$$

As we know that $\operatorname{div}\mathbf{v} = 0$ on $\dot\Omega_t$, we have

$$\begin{aligned} \int\limits_{\dot\Omega_t}\mathbf{v}\cdot\nabla((\mu\operatorname{div}\mathbf{H})\varphi)\,dx &= \int\limits_{\dot\Omega_t}\operatorname{div}(\mathbf{v}(\mu\operatorname{div}\mathbf{H})\varphi)\,dx \\ &= \int\limits_{\Gamma_t}[[\mathbf{n}_t\cdot\mathbf{v}(\mu\operatorname{div}\mathbf{H})\varphi]]\,d\sigma + \int\limits_S \mathbf{v}_-\cdot\mathbf{n}(\mu_-\operatorname{div}\mathbf{H}_-)\varphi_-\,d\sigma. \end{aligned}$$

Since $\mathbf{v}_-\cdot\mathbf{n} = 0$ on S and $[[\mathbf{n}_t\cdot\mathbf{v}]] = 0$, we deduce

$$\int\limits_{\dot\Omega_t}\mathbf{v}\cdot\nabla((\mu\operatorname{div}\mathbf{H})\varphi)\,dx = \int\limits_{\Gamma_t}(\mathbf{n}_t\cdot\mathbf{v}_+)(\mu_+\operatorname{div}\mathbf{H}_+\varphi_+ - \mu_-\operatorname{div}\mathbf{H}_-\varphi_-)\,d\sigma = 0,$$

because $[[\mu\operatorname{div}\mathbf{H}]] = 0$ and $[[\varphi]] = 0$. Thus, we have

$$\frac{d}{dt}\int\limits_{\dot\Omega_t}(\mu\operatorname{div}\mathbf{H})\varphi\,dx = 0.$$

Integrating this formula from $t = 0$ to $t = T_0$ and taking into account that $\operatorname{div}\mathbf{H}(\cdot,0) = \operatorname{div}\mathbf{H}_0 = 0$ on $\dot\Omega$, we have

$$\int\limits_{\dot\Omega_{T_0}}(\mu\operatorname{div}\mathbf{H}(\cdot,T_0)\psi\,dx = 0.$$

By the arbitrary choice of ψ, we have $\operatorname{div}\mathbf{H}(x,T_0) = 0$ for $x \in \Omega_{T_0}$. This shows that

$$\operatorname{div}\mathbf{H} = 0 \quad \text{in } \dot Q_T.$$

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