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LOCAL SOLVABILITY OF FREE BOUNDARY PROBLEM FOR VISCOUS COMPRESSIBLE AND INCOMPRESSIBLE FLUIDS IN THE SPACES $W_p^{2+l,1+l/2}(Q_T)$, $p > 2$

Abstract. We prove local in time solvability of the free boundary problem for two phase viscous compressible and incompressible fluids in the spaces $W_p^{2+l,1+l/2}(Q_T)$ with $p > 2$, $l \in (1/p, 2/p)$.

§1. Introduction

The present paper is a continuation of the articles [1,2], where the evolutionary free boundary problem for two phase viscous fluids of different types was studied in the spaces $W_2^{2+l,1+l/2}$, $l \in (1/2, 1)$. Our aim is to extend the solvability theorem of this problem to the case $p > 2$. The problem has the form

$$\begin{cases} \rho^-(\mathcal{D}_t\boldsymbol{v}^- + (\boldsymbol{v}^-\cdot\nabla)\boldsymbol{v}^-) - \nabla\cdot\mathbb{T}^-(\boldsymbol{v}^-) + \nabla p^- = \rho^-\boldsymbol{f}, \\ \nabla\cdot\boldsymbol{v}^- = 0 \quad \text{in} \quad \Omega_t^-, \\ \rho^+(\mathcal{D}_t\boldsymbol{v}^+ + (\boldsymbol{v}^+\cdot\nabla)\boldsymbol{v}^+) - \nabla\cdot\mathbb{T}^+(\boldsymbol{v}^+) + \nabla p(\rho^+) = \rho^+\boldsymbol{f}, \\ \mathcal{D}_t\rho^+ + \nabla\cdot(\rho^+\boldsymbol{v}^+) = 0 \quad \text{in} \quad \Omega_t^+, \\ \boldsymbol{v}^\pm|_{t=0} = \boldsymbol{v}_0^\pm \quad \text{in} \quad \Omega_0^\pm, \quad \rho^+|_{t=0} = \rho_0^+ \quad \text{in} \quad \Omega_0^+, \\ \boldsymbol{v}^+|_\Sigma = 0, \quad [\boldsymbol{v}]|_{\Gamma_t} = 0, \quad V_n = \boldsymbol{v}\cdot\boldsymbol{n}|_{\Gamma_t}, \\ (-p(\rho^+) + p^-)\boldsymbol{n} + [\mathbb{T}(\boldsymbol{u})\boldsymbol{n}] = -\sigma H\boldsymbol{n} \quad \text{on} \quad \Gamma_t. \end{cases} \tag{1.1}$$

It is assumed that the incompressible fluid fills the variable unknown domain Ω_t^- that is a strictly interior subdomain of a container $\Omega \subset \mathbb{R}^3$ and the compressible fluid is contained in the domain $\Omega_t^+ = \Omega\setminus\overline{\Omega_t^-}$ surrounding Ω_t^-. The surface $\Gamma_t = \partial\Omega_t^-$ is a free interface between $\Omega_t^\pm$. The unknown functions are the velocities $\boldsymbol{v}^\pm(x,t)$, $x \in \Omega_t^\pm$, the pressure $p^-(x,t)$ of the incompressible fluid and the density $\rho^+(x,t)$ of the compressible one, and

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$p(\rho^+)$ is a positive smooth strictly increasing function representing the pressure of the compressible fluid. By $\mathbb{T}^{\pm}$ we mean viscous parts of the stress tensors

$$\mathbb{T}^-(\boldsymbol{v}^-) = \mu^-\mathbb{S}(\boldsymbol{v}^-), \quad \mathbb{T}^+(\boldsymbol{v}^+) = \mu^+\mathbb{S}(\boldsymbol{v}^+) + \mu_1^+\mathbb{I}\nabla\cdot\boldsymbol{v}^+,$$

where $\mu^{\pm} > 0$, $\mu_1^+ > -2\mu^+/3$ are constant viscosity coefficients,

$$\mathbb{S}(\boldsymbol{w}) = (\nabla\otimes\boldsymbol{w}) + (\nabla\otimes\boldsymbol{w})^T$$

is the doubled rate-of-strain tensor, the superscript T means transposition, $\mathbb{I}$ is the identity matrix, σ is a positive constant coefficient of the surface tension, H is the doubled mean curvature of Γ_t, V_n is the velocity of evolution of Γ_t in the direction of $\boldsymbol{n}$, the exterior normal to Γ_t with respect to Ω_t^-, $[u]|_{\Gamma_t}$ is the jump of the functions $u^{\pm}$ given in $\Omega_t^{\pm}$ on the surface Γ_t, i.e.,

$$[u]|_{\Gamma_t} = u^+|_{\Gamma_t} - u^-|_{\Gamma_t}.$$

We consider Problem (1.1) in the Lagrangian coordinates $y\in\Omega_0^+\cup\Gamma_0\cup\Omega_0^-$ connected with the Eulerian coordinates $x\in\Omega_t^+\cup\Gamma_t\cup\Omega_t^-$ by the equation

$$x = y + \int_0^t \boldsymbol{u}(y,\tau)d\tau \equiv X_{\boldsymbol{u}}(y,t), \tag{1.2}$$

where $\boldsymbol{u}(y,\tau)$ is the velocity vector field written as a function of the Lagrangian coordinates. We also represent ρ^+ in the form $\rho^+ = \overline{\rho}^+ + \vartheta^+(x,t)$, where $\overline{\rho}^+ = M^+/|\Omega_t|$ is the mean value of ρ^+ and $M^+ = \int\limits_{\Omega_t^+}\rho^+\,\mathrm{d}x$ is total mass of the compressible fluid. It is clear that $\int\limits_{\Omega_t^+}\vartheta^+(x,t)\,\mathrm{d}x = 0$ and $|\Omega_t^{\pm}|$ are independent of t. In addition, we define $\vartheta^-(x,t) = p^-(x,t) - p(\overline{\rho}^+)$, $\theta^{\pm}(y,t) = \vartheta^{\pm}(X_{\boldsymbol{u}}(y,t),t)$ and $\widehat{\boldsymbol{f}}(y,t) = \boldsymbol{f}(X_{\boldsymbol{u}}(y,t),t)$. Then Problem (1.1) is converted into

$$\begin{cases} \rho^-\mathcal{D}_t\boldsymbol{u}^- - \nabla_{\boldsymbol{u}}\cdot\mathbb{T}_{\boldsymbol{u}}^-(\boldsymbol{u}^-) + \nabla_{\boldsymbol{u}}\theta^- = \rho^-\widehat{\boldsymbol{f}}, \\ \nabla_{\boldsymbol{u}}\cdot\boldsymbol{u}^- = 0 \quad \text{in} \quad \Omega_0^-, \\ (\overline{\rho}^+ + \theta^+)\mathcal{D}_t\boldsymbol{u}^+ - \nabla_{\boldsymbol{u}}\cdot\mathbb{T}_{\boldsymbol{u}}^+(\boldsymbol{u}^+) + \nabla_{\boldsymbol{u}}p(\overline{\rho}^+ + \theta^+) = (\overline{\rho}^+ + \theta^+)\widehat{\boldsymbol{f}}, \\ \mathcal{D}_t\theta^+ + (\overline{\rho}^+ + \theta^+)\nabla_{\boldsymbol{u}}\cdot\boldsymbol{u}^+ = 0, \quad \theta^+|_{t=0} = \theta_0^+ = \rho_0^+ - \overline{\rho}^+ \text{ in } \Omega_0^+, \\ \boldsymbol{u}^{\pm}|_{t=0} = \boldsymbol{u}_0^{\pm} \equiv \boldsymbol{v}_0^{\pm} \quad \text{in} \quad \Omega_0^{\pm}, \quad \boldsymbol{u}^+|_{\Sigma} = 0, \quad [\boldsymbol{u}]|_{\Gamma_0} = 0, \\ (-p(\overline{\rho}^+ + \theta^+) + p(\overline{\rho}^+) + \theta^-)\boldsymbol{n} + [\mathbb{T}_u(\boldsymbol{u})\boldsymbol{n}] = -\sigma\widehat{H}\boldsymbol{n} \quad \text{on} \quad \Gamma_0, \end{cases} \tag{1.3}$$

where $\nabla_{\boldsymbol{u}} = \mathbb{L}^{-1T}\nabla_y = \mathbb{L}^{-T}\nabla_y$ is the transformed gradient ∇_x, $\mathbb{L} = \left(\frac{\partial x}{\partial y}\right)$ is the Jacobi matrix of the transformation (1.2), $\widehat{\mathbb{L}} = \mathbb{L}^{-T}L$, $L = \det\mathbb{L}$, $L = 1$ in Ω_t^-, $\mathbb{S}_{\boldsymbol{u}}(\boldsymbol{u}) = \nabla_{\boldsymbol{u}} \otimes \boldsymbol{u} + (\nabla_{\boldsymbol{u}} \otimes \boldsymbol{u})^T$ is the transformed doubled rate-of-strain tensor,

$$\mathbb{T}^-_{\boldsymbol{u}}(\boldsymbol{u}^-) = \mu^-\mathbb{S}_{\boldsymbol{u}}(\boldsymbol{u}^-), \quad \mathbb{T}^+_{\boldsymbol{u}}(\boldsymbol{u}^+) = \mu^+\mathbb{S}_{\boldsymbol{u}}(\boldsymbol{u}^+) + \mu_1^+\mathbb{I}\nabla_{\boldsymbol{u}}\cdot\boldsymbol{u}^+, \quad \widehat{H} = H(X_{\boldsymbol{u}}, t).$$

The elements of the transposed co-factors matrix $\widehat{\mathbb{L}}^T$ are given by

$$(\widehat{\mathbb{L}}^T)_{im} = (\nabla X_j \times \nabla X_k)_m, \tag{1.4}$$

where $X_j = (X_{\boldsymbol{u}})_j$ and (i,j,k) is a cyclic permutation of $(1,2,3)$. The kinematic condition $V_n = \boldsymbol{u}\cdot\boldsymbol{n}$ is fulfilled automatically. The normal $\boldsymbol{n}(X_{\boldsymbol{u}}, t)$ to Γ_t is connected with the normal $\boldsymbol{n}_0$ to Γ_0 by the formula

$$\boldsymbol{n} = \frac{\widehat{\mathbb{L}}^T\boldsymbol{n}_0(y)}{|\widehat{\mathbb{L}}^T\boldsymbol{n}_0(y)|}. \tag{1.5}$$

Since $H\boldsymbol{n} = \Delta(t)X_u$, where $\Delta(t)$ is the Laplace–Beltrami operator on Γ_t, it can be shown that the corresponding linear problem has the form

$$\begin{cases} \overline{\rho}^+\mathcal{D}_t\boldsymbol{v}^+ - \mu^+\nabla^2\boldsymbol{v}^+ - (\mu^+ + \mu_1^+)\nabla(\nabla\cdot\boldsymbol{v}^+) + p_1\nabla\theta^+ = \boldsymbol{f}^+, \\ \mathcal{D}_t\theta^+ + \overline{\rho}^+\nabla\cdot\boldsymbol{v}^+ = h^+ \quad \text{in} \quad \Omega_0^+, \\ \rho^-\mathcal{D}_t\boldsymbol{v}^- - \mu^-\nabla^2\boldsymbol{v}^- + \nabla\theta^- = \boldsymbol{f}^-, \quad \nabla\cdot\boldsymbol{v}^- = h^- \quad \text{in} \quad \Omega_0^-, \\ \boldsymbol{v}^\pm|_{t=0} = \boldsymbol{v}_0^\pm \quad \text{in} \quad \Omega_0^\pm, \quad \theta^+|_{t=0} = \theta_0^+ \quad \text{in} \quad \Omega_0^+, \\ [\boldsymbol{v}]|_{\Gamma_0} = 0, \quad [\mu\Pi_0\mathbb{S}(\boldsymbol{v})\boldsymbol{n}_0]|_{\Gamma_0} = \boldsymbol{b}, \\ -p_1\theta^+ + \theta^- + [\boldsymbol{n}_0\cdot\mathbb{T}(\boldsymbol{v})\boldsymbol{n}_0] + \sigma\boldsymbol{n}_0\displaystyle\int_0^t \Delta(0)\boldsymbol{v}(y,\tau)\,\mathrm{d}\tau|_{\Gamma_0} = b + \sigma\int_0^t B\,\mathrm{d}\tau, \end{cases} \tag{1.6}$$

where $\boldsymbol{f}^\pm$, $h^\pm$, $\boldsymbol{b}$, b, B, $\boldsymbol{v}_0^\pm$, θ_0^+ are some given functions and $p_1 = p'(\overline{\rho}^+) > 0$.

In the paper [3], the following theorem is proved.

Theorem 1. *Let* $\Sigma, \Gamma_0 \in W_p^{2+l-1/p}$, $p > 2$, $l \in (1/p, 2/p)$. *For arbitrary* $\boldsymbol{f} \in W_p^{l,l/2}(\cup Q_T^\pm)$, $h^- \in W_p^{l+1,(l+1)/2}(Q_T^-)$ *such that* $\mathcal{D}_t h^- = \nabla\cdot\boldsymbol{H} + H_1$, $\boldsymbol{H}, H_1 \in W_p^{0,l/2}(Q_T^-)$, $h^+ \in W_p^{l+1,0}(Q_T^+) \cap W_p^{l/2}((0,T); W_p^1(\Omega_0^+))$,

$$\boldsymbol{b} \in W_p^{l+1-1/p, l/2+1/2-1/2p}(G_T),$$

$$b \in W_2^{l+1-1/p,0}(G_T) \cap \widehat{W}_p^{l/2}((0,T), W_p^{1-1/p}(\Gamma_0)),$$

$B \in \widehat{W}_p^{l-1/p,l/2-1/2p}(G_T)$, $\boldsymbol{v}_0^{\pm} \in W_p^{l+2-2/p}(\Omega_0^{\pm})$, $\theta_0^+ \in W_p^{l+1}(\Omega_0^+)$, *satisfying the compatibility conditions*

$$\begin{aligned} &\nabla \cdot \boldsymbol{v}_0^-(y) = \boldsymbol{h}_0^-(y) \text{ in } \Omega_0^-, \ \ [\mu \Pi_0 \mathbb{S}(\boldsymbol{v}_0)\boldsymbol{n}_0]|_{\Gamma_0} = \boldsymbol{b}(y,0), \ \ \boldsymbol{b}\cdot\boldsymbol{n}_0 = 0, \\ &\quad [\boldsymbol{v}_0]|_{\Gamma_0} = 0, \quad \boldsymbol{v}_0|_{\Sigma} = 0, \end{aligned} \tag{1.7}$$

problem (1.6) *has a unique solution in an arbitrary finite time interval* $(0,T)$, *and the inequality*

$$\begin{aligned} &\|\boldsymbol{v}\|_{\widehat{W}_p^{2+l,1+l/2}(\cup Q_T^{\pm})} + \|\theta^-\|_{\widehat{W}_p^{l/2}((0,T);W_p^1(\Omega_0^-))} + \|\theta^-\|_{W_p^{l+1,0}(Q_T^-)} \\ &+ \|\theta^+\|_{\widehat{W}_p^{l/2}((0,T);W_p^1(\Omega_0^+))} + \|\theta^+\|_{W_p^{l+1,0}(Q_T^+)} \\ &+ \|\mathcal{D}_t\theta^+\|_{\widehat{W}_p^{l/2}((0,T);W_p^1(\Omega_0^+))} + \|\mathcal{D}_t\theta^+\|_{W_p^{l+1,0}(Q_T^+)} \\ &\leqslant c(T)\big(\|\widehat{\boldsymbol{f}}\|_{W_p^{l,l/2}(\cup Q_T^{\pm})} + \|h^-\|_{W_p^{l+1,0}(Q_T^-)} + \|\boldsymbol{H}\|_{\widehat{W}_p^{0,l/2}(Q_T^-)} + \|H_1\|_{\widehat{W}_p^{0,l/2}(Q_T^-)} \\ &+ \|h^+\|_{W_p^{l+1,0}(Q_T^+)} + \|h^+\|_{\widehat{W}_p^{l/2}((0,T);W_2^1(\Omega_0^+))} + \|\boldsymbol{b}\|_{W_p^{l+1-1/p,l/2+1-1/2p}(G_T)} \\ &+ \sup_{t<T}\|\boldsymbol{b}(\cdot,t)\|_{W_p^{l+1-3/p}(\Gamma_0)} + \|b\|_{W_p^{l+1-1/p,0}(G_T)} + \|b\|_{\widehat{W}_p^{1/2}((0,T);W_p^{1-1/p}(\Gamma_0))} \\ &+ \|B\|_{\widehat{W}_p^{l-1/p,l/2-1/2p}(G_T)} + \|\boldsymbol{v}_0\|_{W_p^{l+2-2/p}(\cup\Omega_0^{\pm})} + \|\theta_0^+\|_{W_p^{l+1}(\Omega_0^+)}\big) \end{aligned} \tag{1.8}$$

holds, where $c(T)$ *is a bounded non-decreasing function of* T.

We recall (see [3]) that the norms in the spaces $W_p^r(\Omega)$ and $W_p^{r,r/2}(Q)$ where $\Omega \subset \mathbb{R}^n$, $Q_T = \Omega \times (0,T)$ are defined by

$$\|u\|^p_{W_p^r(\Omega)} = \sum_{|j|\leqslant r} \int_\Omega |D^j u(x)|^p dx, \quad \text{if } r \text{ is an integer},$$

and

$$\|u\|^p_{W_p^r(\Omega)} = \sum_{|j|=[r]} \int_\Omega \int_\Omega \frac{|D^j u(x) - D^j u(y)|^p}{|x-y|^{n+p\rho}} dy, \quad \text{if } r = [r]+\rho,\ 0<\rho<1,$$

$$\begin{aligned} \|u\|^p_{W_p^{r,r/2}(Q_T)} &= \|u\|^p_{W_p^{r,0}(Q_T)} + \|u\|^p_{W_p^{0,r/2}(Q_T)} \\ &= \int_0^T \|u(\cdot,t)\|^p_{W_p^r(\Omega)}\,\mathrm{d}t + \int_\Omega \|u(x,\cdot)\|^p_{W_p^{r/2}(0,T)}\,\mathrm{d}x. \end{aligned}$$

In addition, if $\Omega = \Omega^+ \cup \Omega^-$, then we set

$$\|u\|^p_{W_p^r(\cup\Omega^{\pm})} = \|u\|^p_{W_p^r(\Omega^+)} + \|u\|^p_{W_p^r(\Omega^-)},$$

and define the norms

$$\|\boldsymbol{v}\|^p_{\widehat{W}_p^{2+l,1+l/2}(\cup Q_T^\pm)} = \|\boldsymbol{v}\|^p_{W_p^{2+l,0}(\cup Q_T^\pm)} + \|\mathcal{D}_t\boldsymbol{v}\|^p_{\widehat{W}_p^{l,l/2}(\cup Q_T^\pm)} + \sup_{t<T}\|\boldsymbol{v}(\cdot,t)\|^p_{W_p^{2+l-2/p}(\cup\Omega_0^\pm)},$$

$$\|\boldsymbol{w}\|^p_{\widehat{W}_p^{l,l/2}(\cup Q_T^\pm)} = \|\boldsymbol{w}\|^p_{W_p^{l,l/2}(\cup Q_T^\pm)} + T^{-pl/2}\int\limits_0^T \|\boldsymbol{w}(\cdot,t)\|^p_{L_p(\Omega)}\,\mathrm{d}t,$$

$$\|\boldsymbol{b}\|^p_{\widehat{W}_p^{1+l-1/p,1/2+l/2-1/2p}(G_T)} = \|\boldsymbol{b}\|^p_{W_p^{1+l-1/p,1/2+l/2-1/2p}(G_T)} + \sup_{t<T}\|\boldsymbol{b}(\cdot,t)\|^p_{W_p^{1+l-3/p}(\Gamma_0)}.$$

The imbedding and trace theorems for the spaces defined above can be found in [4]. Now we state the main result of the paper.

Theorem 2. *Assume that* $\Gamma_0 \in W_p^{l+3-1/p}$, $\Sigma \in W_2^{l+2-1/p}$, $l \in (1/p, 2/p)$, $p(\rho^+)$ *is* C^2*-function with Lipschitz continuous second derivatives, and the compatibility conditions*

$$\nabla\cdot\boldsymbol{u}_0 = 0, \quad [\mu\Pi_0\mathbb{S}(\boldsymbol{u}_0)\boldsymbol{n}_0]|_{\Gamma_0} = 0, \quad [\boldsymbol{u}_0]|_{\Gamma_0} = 0, \quad \boldsymbol{u}_0|_\Sigma = 0$$

are satisfied. Then there exists such $T > 0$ *that for arbitrary* $\boldsymbol{f}$, $\nabla\boldsymbol{f} \in W_p^{l,l/2}(Q_T)$, $\mathcal{D}_y^j\boldsymbol{f} \in L_p(Q_T)$, $|j| = 1,2$, *where* $Q_T = \Omega\times(0,T)$, *Problem* (1.3) *has a unique solution* $(\boldsymbol{u}^\pm, \theta^\pm)$ *such that* $\boldsymbol{u} \in W_p^{2+l,1+l/2}(\cup Q_T^\pm)$, $\theta^+, \mathcal{D}_t\theta^+ \in W_2^{l+1,0}(Q_T^+) \cap W_2^{l/2}((0,T); W_2^1(\Omega_0^+))$, $\theta^- \in W_p^{l+1,0}(Q_T^-) \cap W_p^{l/2}((0,T); W_2^1(\Omega_0^-))$ *and the inequality*

$$\begin{aligned}
Y(\boldsymbol{u},\theta) \equiv{}& \|\boldsymbol{u}\|_{\widehat{W}_p^{2+l,1+l/2}(\cup Q_T^\pm)} + \|\theta^-\|_{\widehat{W}_p^{l/2}((0,T);W_p^1(\Omega_0^-))} \\
&+ \|\theta^-\|_{W_p^{l+1,0}(Q_T^-)} + \|\theta^+\|_{W_p^{l+1,0}(Q_T^+)} + \|\theta^+\|_{\widehat{W}_p^{l/2}((0,T);W_p^1(\Omega_0^+))} \\
&+ \|\mathcal{D}_t\theta^+\|_{W_p^{l+1,0}(Q_T^+)} + \|\mathcal{D}_t\theta^+\|_{\widehat{W}_p^{l/2}((0,T);W_p^1(\Omega_0^+))} \\
\leqslant{}& c(T)\big(\|\boldsymbol{u}_0\|_{W_p^{l+2-1/2p}(\cup\Omega_0^\pm)} + \sigma\|H_0\|_{W_p^{l+1-1/p}(\Gamma_0)} \\
&+ \|\theta_0^+\|_{W_p^{l+1}(\Omega_0^+)} + T^{1/p-l/2}\|\theta_0^+\|^2_{W_p^{l+1}(\Omega_0^+)} + \|\boldsymbol{f}\|_{\widehat{W}_2^{l,l/2}(Q_T)}\big)
\end{aligned} \tag{1.9}$$

holds, where $c(T)$ *is a non-decreasing function of* T.

§2. Proof of solvability of Problem (1.3)

By separating linear and nonlinear terms we transform (1.3) into

$$\begin{cases}\rho^-\mathcal{D}_t\boldsymbol{u}^- - \nabla\cdot\mathbb{T}^-(\boldsymbol{u}^-) + \nabla\theta^- = \boldsymbol{l}_1^-(\boldsymbol{u}^-,\theta^-) + \rho^-\widehat{\boldsymbol{f}},\\ \nabla\cdot\boldsymbol{u}^- = l_2^-(\boldsymbol{u}^-) \quad \text{in } \Omega_0^-,\ \ t>0,\\ \overline{\rho}^+\mathcal{D}_t\boldsymbol{u}^+ - \nabla\cdot\mathbb{T}^+(\boldsymbol{u}^+) + p_1\nabla\theta^+ = \boldsymbol{l}_1^+(\boldsymbol{u}^+,\theta^+) + (\overline{\rho}^+ + \theta^+)\widehat{\boldsymbol{f}},\\ \mathcal{D}_t\theta^+ + \overline{\rho}^+\nabla\cdot\boldsymbol{u} = l_2^+(\boldsymbol{u}^+,\theta^+) \quad \text{in} \quad \Omega_0^+, \quad t>0,\\ \boldsymbol{u}^\pm|_{t=0} = \boldsymbol{u}_0^\pm \quad \text{in} \quad \Omega_0^\pm, \quad \theta^+|_{t=0} = \theta_0^+ = \rho_0^+ - \overline{\rho}^+,\\ [\boldsymbol{u}]|_{\Gamma_0} = 0, \quad [\mu\Pi_0\mathbb{S}(\boldsymbol{u})\boldsymbol{n}_0]|_{\Gamma_0} = \boldsymbol{l}_3(\boldsymbol{u})|_{\Gamma_0},\\ -p_1\theta^+ + \theta^- + [\boldsymbol{n}_0\cdot\mathbb{T}(\boldsymbol{u})\boldsymbol{n}_0]\big|_{\Gamma_0} + \sigma\boldsymbol{n}_0\cdot\displaystyle\int_0^t \Delta(0)\boldsymbol{u}(y,\tau)\,\mathrm{d}\tau\big|_{\Gamma_0}\\ = l_4(\boldsymbol{u}) - \displaystyle\int_0^t (l_5(\boldsymbol{u}) + l_6(\boldsymbol{u}))\,\mathrm{d}\tau - \sigma H_0, \quad \boldsymbol{u}|_\Sigma = 0,\end{cases} \tag{2.1}$$

where $H_0 = H|_{t=0}$,

$$\begin{aligned}
&\boldsymbol{l}_1^-(\boldsymbol{u},\theta) = \nabla_{\boldsymbol{u}}\cdot\mathbb{T}_{\boldsymbol{u}}^-(\boldsymbol{u}^-) - \nabla\cdot\mathbb{T}^-(\boldsymbol{u}^-) + (\nabla - \nabla_{\boldsymbol{u}})\theta^-,\\
&\boldsymbol{l}_1^+(\boldsymbol{u},\theta) = \nabla_{\boldsymbol{u}}\cdot\mathbb{T}_{\boldsymbol{u}}^+(\boldsymbol{u}^+) - \nabla\cdot\mathbb{T}^+(\boldsymbol{u}^+)\\
&\ + p_1(\nabla - \nabla_{\boldsymbol{u}})\theta^+ - \nabla_{\boldsymbol{u}}\big(p(\overline{\rho}^+ + \theta^+) - p(\overline{\rho}^+) - p_1\theta^+\big) - \theta^+\mathcal{D}_t\boldsymbol{u}^+,\\
&l_2^-(\boldsymbol{u}) = (\nabla - \nabla_{\boldsymbol{u}})\cdot\boldsymbol{u}^- = \nabla\cdot\boldsymbol{L}(\boldsymbol{u}^-),\\
&\boldsymbol{L}(\boldsymbol{u}^-) = (\mathbb{I} - \mathbb{L}^{-1})\boldsymbol{u}^- = (\mathbb{I} - \widehat{\mathbb{L}})\boldsymbol{u}^-,\\
&l_2^+(\boldsymbol{u},\theta) = \overline{\rho}^+(\nabla - \nabla_{\boldsymbol{u}})\cdot\boldsymbol{u}^+ - \theta^+\nabla_{\boldsymbol{u}}\cdot\boldsymbol{u}^+,\\
&\boldsymbol{l}_3(\boldsymbol{u}) = [\mu\Pi_0(\Pi_0\mathbb{S}(\boldsymbol{u})\boldsymbol{n}_0 - \Pi\mathbb{S}_{\boldsymbol{u}}(\boldsymbol{u})\boldsymbol{n})]\big|_{\Gamma_0},\\
&l_4(\boldsymbol{u}) = [\boldsymbol{n}_0\cdot\mathbb{T}(\boldsymbol{u})\boldsymbol{n}_0 - \boldsymbol{n}\cdot\mathbb{T}_{\boldsymbol{u}}(\boldsymbol{u})\boldsymbol{n}] - (p(\overline{\rho}^+ + \theta^+) - p^+(\overline{\rho}^+) - p_1\theta^+)\big|_{\Gamma_0},\\
&l_5(\boldsymbol{u}) = \sigma\mathcal{D}_t(\boldsymbol{n}\Delta(t))\cdot\int_0^t \boldsymbol{u}(y,\tau)\,\mathrm{d}\tau + \sigma(\boldsymbol{n}\cdot\Delta(t) - \boldsymbol{n}_0\cdot\Delta(0))\boldsymbol{u},\\
&l_6(\boldsymbol{u}) = \sigma(\dot{\boldsymbol{n}}\Delta(t) + \boldsymbol{n}\dot{\Delta}(t))\cdot\boldsymbol{y}\Big|_{\Gamma_0}, \quad \dot{\boldsymbol{n}} = \mathcal{D}_t\boldsymbol{n}, \quad \dot{\Delta}(t) = \mathcal{D}_t\Delta(t),\\
&\Pi_0\boldsymbol{g} = \boldsymbol{g} - \boldsymbol{n}_0(\boldsymbol{n}_0\cdot\boldsymbol{g}), \quad \Pi\boldsymbol{g} = \boldsymbol{g} - \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{g}).
\end{aligned} \tag{2.2}$$

The operator $\Delta(t)$ is given by

$$\Delta(t) = \frac{1}{\sqrt{g}} \sum_{\alpha,\beta=1}^{2} \frac{\partial}{\partial s_\alpha} g^{\alpha\beta} \sqrt{g} \frac{\partial}{\partial s_\beta}, \tag{2.3}$$

where $g = \det(g_{\alpha\beta})$, $\alpha, \beta = 1, 2$, $g_{\alpha\beta} = \frac{\partial X_{\boldsymbol{u}}}{\partial s_\alpha} \cdot \frac{\partial X_{\boldsymbol{u}}}{\partial s_\beta}$ are elements of the metric tensor on Γ_t, $g^{\alpha\beta}$ and $\widehat{g}_{\alpha\beta}$ are elements of the inverse and transposed co-factors matrices to $(g_{\alpha\beta})$, respectively. We assume that (s_1, s_2) are local Cartesian coordinates on the tangential plane to Γ_0 with the origin at the point $y_0 \equiv 0$. Let $\Gamma_0' \subset \Gamma_0$ be a neighborhood of the origin defined by the equation

$$s_3 = \phi(s_1, s_2) \in W_p^{l+3-\frac{1}{p}}(K), \quad K = \{s_1^2 + s_2^2 \leqslant d^2\},$$

the y_3-axis being directed along $\boldsymbol{n}_0(y_0)$. Then the set $\Gamma_t' = X_{\boldsymbol{u}} \Gamma_0' \subset \Gamma_t$ is given by the equations

$$\begin{aligned} z_\gamma &= s_\gamma + \int_0^t u_\gamma(s_1, s_2, \phi(s_1, s_2), \tau)\, \mathrm{d}\tau \quad \gamma = 1, 2, \\ z_3 &= \phi(s_1, s_2) + \int_0^t u_3(s_1, s_2, \phi(s_1, s_2), \tau)\, \mathrm{d}\tau, \end{aligned} \tag{2.4}$$

where u_i are projections of $\boldsymbol{u}$ on the s_i-axes and

$$\begin{aligned} g_{\alpha\beta} &= \sum_{i=1}^{3} \frac{\partial z_i}{\partial s_\alpha} \frac{\partial z_i}{\partial s_\beta} = \delta_{\alpha\beta} + \phi_\alpha \phi_\beta + \phi_\alpha U_{3\beta} + \phi_\beta U_{3\alpha} + U_{\alpha\beta} \\ &\quad + U_{\beta\alpha} + \sum_{i=1}^{3} U_{i\alpha} U_{i\beta}, \\ U_{i\alpha} &= \int_0^t \Big(\frac{\partial u_i}{\partial s_\alpha} + \phi_\alpha \frac{\partial u_i}{\partial s_3} \Big)\, \mathrm{d}\tau, \quad \phi_\alpha = \frac{\partial \phi}{\partial s_\alpha}. \end{aligned} \tag{2.5}$$

The time derivative $\dot{\Delta}(t)$ of $\Delta(t)$ is given by

$$\dot{\Delta}(t) = -\frac{\dot{g}}{2g} \Delta(t) + \frac{1}{\sqrt{g}} \sum_{\alpha,\beta=1}^{2} \frac{\partial}{\partial s_\alpha} \widetilde{g}_{\alpha\beta} \frac{\partial}{\partial s_\beta}, \tag{2.6}$$

where $\widetilde{g}_{\alpha\beta} = \mathcal{D}_t \frac{\widehat{g}_{\alpha\beta}}{\sqrt{g}}$, $\dot{g} = \mathcal{D}_t g$.

The proof of Theorem 2 rests on Theorem 1 and on the estimates of nonlinear expressions (2.2).

Proposition 1. *Let*

$$\begin{aligned}Z(\boldsymbol{u},\theta) &= \|\boldsymbol{l}_1^{\pm}\|_{\widehat{W}_p^{l,l/2}(\cup Q_T^{\pm})} + \|l_2^{\pm}\|_{W_p^{l+1,0}(\cup Q_T^{\pm})} + \|l_2^{\pm}\|_{\widehat{W}_p^{l/2}((0,T),W_p^1(\Omega_0^{\pm}))}\\ &+ \|\mathcal{D}_t\boldsymbol{L}(\boldsymbol{u})\|_{\widehat{W}_p^{0,l/2}(Q_T^-)} + \|\boldsymbol{l}_3(\boldsymbol{u})\|_{\widehat{W}_p^{l+1-1/p,l/2+1/2-1/2p}(G_T)}\\ &+ \|l_4(\boldsymbol{u})\|_{\widehat{W}_p^{l/2}((0,T);W_p^{1-1/p}(\Gamma_0))}\\ &+ \|l_4(\boldsymbol{u})\|_{\widehat{W}_p^{l+1-1/p,0}(G_T)} + \|l_5(\boldsymbol{u})\|_{\widehat{W}_p^{l-1/p,l/2-1/2p}(G_T)}.\end{aligned}$$

If $\boldsymbol{u}_0=0$, $\theta_0^+=0$ *and*

$$\big(T^{1/p}+T^{1/p'}\big)Y(\boldsymbol{u},\theta)\leqslant\delta\ll 1,\quad 1/p'=1-1/p, \tag{2.7}$$

where $Y(\boldsymbol{u},\theta)$ *is defined in* (1.9), *then*

$$Z(\boldsymbol{u},\theta)\leqslant c\delta Y(\boldsymbol{u},\theta) \tag{2.8}$$

and

$$\|l_6(\boldsymbol{u})\|_{\widehat{W}_p^{l-1/p,l/2-1/2p}(G_T)}\leqslant\epsilon_1\|\boldsymbol{u}\|_{W_p^{2+l,1+l/2}(\cup Q_T^{\pm})}+c(\epsilon_1)\|\boldsymbol{u}\|_{L_p(Q_T)},\quad \epsilon_1\ll 1. \tag{2.9}$$

If $\boldsymbol{f}\in W_p^{l,l/2}(Q_T)$ *and* $\nabla\boldsymbol{f}\in L_p(Q_T)$, *then*

$$\|\widehat{\boldsymbol{f}}\|_{W_p^{l,l/2}(Q_T)}\leqslant c(\|\boldsymbol{f}\|_{W_p^{l,l/2}(Q_T)}+\|\nabla\boldsymbol{f}\|_{L_p(Q_T)}\sup_{Q_T}|\boldsymbol{u}(y,t)|). \tag{2.10}$$

Proof. We invoke some auxiliary inequalities (cf. [5] for $p=2$), namely,

$$\begin{aligned}&\|uv\|_{W_p^l(\mathfrak{Q})}\leqslant c\|u\|_{W_p^l(\mathfrak{Q})}\|v\|_{W_p^s(\mathfrak{Q})},\\ &\|uv\|_{L_p(\mathfrak{Q})}\leqslant c\|u\|_{W_p^l(\mathfrak{Q})}\|v\|_{W_p^{n/p-l}(\mathfrak{Q})},\ \text{if } l<n/p,\\ &\|uv\|_{W_p^l(\mathfrak{Q})}\leqslant c\big(\|u\|_{W_p^l(\mathfrak{Q})}\|v\|_{W_p^s(\mathfrak{Q})}+\|v\|_{W_p^l(\mathfrak{Q})}\|u\|_{W_p^s(\mathfrak{Q})}\big),\ \text{if } l\geqslant n/p,\end{aligned} \tag{2.11}$$

where $\mathfrak{Q}$ is a bounded domain in $\mathbb{R}^n$, $n=2,3$, $s>n/p$. If u,v depend also on $t\in(0,T)$, then (2.11) imply

$$\begin{aligned}&\|uv\|_{W_p^{l,0}(\mathfrak{Q}_T)}\leqslant c\|u\|_{W_p^{l,0}(\mathfrak{Q}_T)}\sup_{t\in(0,T)}\|v(\cdot,t)\|_{W_p^{n/p+\varkappa}(\mathfrak{Q})},\\ &\mathfrak{Q}_T=\mathfrak{Q}\times(0,T),\end{aligned} \tag{2.12}$$

where $l < n/p$, $\varkappa \in (0, l-1/p)$. In addition, from

$$\|\Delta_t(-h)uv\|_{L_p(\mathfrak{Q})} \leqslant \sup_{\mathfrak{Q}} |v(y,t)| \|\Delta_t(-h)u(\cdot,t)\|_{L_p(\mathfrak{Q})} + \|\Delta_t(-h)v\|_{L_{q_0}(\mathfrak{Q})}\|u\|_{L_{q_1}(\mathfrak{Q})},$$

$$T^{-pl/2}\int\limits_0^T \|uv(\cdot,t)\|^p_{L_p(\mathfrak{Q})}\,\mathrm{d}t \leqslant T^{-pl/2}\int\limits_0^T \|u(\cdot\,,t)\|^p_{L_p(\mathfrak{Q})} dt \sup_{Q_T}|v|^p$$

it follows that

$$\|uv\|_{\widehat{W}^{0,l/2}_p(\mathfrak{Q}_T)} \leqslant c \sup_{\mathfrak{Q}_T}|v(y,t)|\|u\|_{\widehat{W}^{0,l/2}_p(\mathfrak{Q}_T)} + c\|v\|_{\widehat{W}^{l/2}_p((0,T);W^{n/p-l}_p(\mathfrak{Q}))} \sup_{t<T}\|u(\cdot,t)\|_{W^l_p(\mathfrak{Q})}, \tag{2.13}$$

where $l - n/p + n/q_1 = 0$, $1/q_0 = 1/p - 1/q_1$, $l < n/p$. If $l > n/p$, then

$$\|uv\|_{\widehat{W}^{0,l/2}_p(\mathfrak{Q}_T)} \leqslant c\big(\sup_{t<T}|u(y,t)|\|v\|_{\widehat{W}^{0,l/2}_p(\mathfrak{Q}_T)} + \sup_{\mathfrak{Q}_T}|v(y,t)|\|u\|_{\widehat{W}^{0,l/2}_p(\mathfrak{Q}_T)}\big). \tag{2.14}$$

In view of (2.7) and (1.4), we have

$$\begin{gathered}
\|\widehat{\mathbb{L}} - \mathbb{I}\|_{W^{1+l}_p(\cup\Omega^{\pm}_0)} + \|\boldsymbol{n} - \boldsymbol{n}_0\|_{W^{l+1-1/p}_p(\Gamma_0)} \leqslant cT^{1/p'}\|\nabla\boldsymbol{u}\|_{W^{l+1,0}_p(\cup Q^{\pm}_T)} \leqslant c\delta,\\
\|\mathcal{D}_t\widehat{\mathbb{L}}\|_{W^{l+1}_p(\cup\Omega^{\pm}_0)} \leqslant c\|\nabla\boldsymbol{u}\|_{W^{l+1}_p(\cup\Omega^{\pm}_0)},\\
\|\theta^+(\cdot,t)\|_{W^{l+1}_p(\Omega^+_0)} \leqslant \|\theta^+_0\|_{W^{l+1}_p(\Omega^+_0)} + \int\limits_0^t \|\mathcal{D}_\tau\theta^+\|_{W^{l+1,0}_p(Q^+_0)}\,\mathrm{d}\tau, \quad \forall t < T.
\end{gathered} \tag{2.15}$$

Hence the expressions $\boldsymbol{l}^{\pm}_1(\boldsymbol{u}, \theta^{\pm})$, $l^{\pm}_2$, (except for $\boldsymbol{P}$),

$$\nabla_{\boldsymbol{u}} \cdot \mathbb{T}^+_{\boldsymbol{u}}(\boldsymbol{u}^+) - \nabla \cdot \mathbb{T}^+(\boldsymbol{u}^+), \quad \theta^+\mathcal{D}_t\boldsymbol{u},$$

as well as $\boldsymbol{l}_3$, $[\boldsymbol{n}_0 \cdot \mathbb{T}(\boldsymbol{u})\boldsymbol{n}_0 - \boldsymbol{n} \cdot \mathbb{T}_{\boldsymbol{u}}(\boldsymbol{u})\boldsymbol{n})]_{\Gamma_0}$ are estimated by the same arguments as in [5] in the case $p = 2$, by $c\delta Y$, i.e., the norms of all these expressions satisfy (2.8). The $W^{l,l/2}_p(Q_T)$-norm of $\widehat{\boldsymbol{f}}$ is estimated as in [2], i.e., by passing to the Eulerian coordinates and by using the relations

$$\boldsymbol{f}(X_{\boldsymbol{u}}(y,t),t) - \boldsymbol{f}(X_{\boldsymbol{u}}(y,t-\tau),t) = -\int\limits_0^1 \nabla\boldsymbol{f}(X_{\boldsymbol{u}}(y,t-\lambda\tau),t)\boldsymbol{u}(y,t-\lambda\tau)\tau\,\mathrm{d}\lambda,$$

$$\int_0^T \mathrm{d}t \int_0^t \frac{\mathrm{d}\tau}{\tau^{1+pl/2}} \int_\Omega |\boldsymbol{f}(X_{\boldsymbol{u}}(y,t),t) - \boldsymbol{f}(X_{\boldsymbol{u}}(y,t-\tau),t)|^p \,\mathrm{d}y$$

$$\leqslant cT^{p-1-pl/2} \|\nabla \boldsymbol{f}\|^p_{L_p(Q_T)} \int_0^T \sup_{Q_T} |\boldsymbol{u}(y,t)|^p \,\mathrm{d}t.$$

Let us consider the term

$$\boldsymbol{P} \equiv \nabla_{\boldsymbol{u}}(p(\overline{\rho}^+ + \theta^+) - p(\overline{\rho}^+) - p_1\theta^+) = \mathbb{L}^{-T}(p'(\overline{\rho}^+ + \theta^+) - p'(\overline{\rho}^+))\nabla\theta^+.$$

Since $p \in C^{2+1}(\min \rho^+, \max \rho^+)$, we have

$$|p'(\overline{\rho}^+ + \theta^+) - p'(\overline{\rho}^+)| \leqslant c|\theta^+|,$$
$$|p'(\overline{\rho}^+ + \theta^+(y+z)) - p'(\overline{\rho}^+ + \theta^+(y))| \leqslant c|\theta^+(y+z) - \theta^+(y)|,$$

and, in view of (2.7),

$$\|\boldsymbol{P}\|_{W^l_p(\Omega_0^+)} \leqslant c\|\theta^+\|^2_{W^{l+1}_p(\Omega_0^+)},$$

$$\|\boldsymbol{P}\|_{W^{l,0}_p(Q_T^+)} \leqslant T^{1/p}(\|\theta_0^+\|^{2p}_{W^{l+1}_p(\Omega_0^+)}$$
$$+ 2p \sup_{t<T} \|\theta^+(\cdot,t)\|^p_{W^{l+1}_p(\Omega_0^+)} \int_0^T \|\theta^+(\cdot,t)\|^{p-1}_{W^{l+1}_p(\Omega_0^+)} \|\mathcal{D}_t\theta^+(\cdot,t)\|_{W^{l+1}_p(\Omega_0^+)} \,\mathrm{d}t)^{1/p}$$
$$\leqslant cT^{1/p}(\|\theta_0^+\|^{2p}_{W^{l+1}_p(\Omega_0^+)} + Y^{2p}(\boldsymbol{u},\theta))^{1/p} \leqslant c(T^{1/p}\|\theta_0^+\|^2_{W^{l+1}_p(\Omega_0^+)} + \delta Y(u,\theta)),$$

$$\frac{1}{T^{pl/2}} \int_0^T \|\boldsymbol{P}\|^p_{L_p(\Omega_0^+)} \,\mathrm{d}t \leqslant cT^{1-pl/2} \sup_{t<T} \|\nabla\theta^+(\cdot,t)\|^p_{L_p(\Omega_0^+)} \sup_{Q_T^+} |\theta^+(y,t)|^p,$$
$$\leqslant cT^{1-pl/2}(\|\theta_0^+\|^{2p}_{W^{l+1}_p(\Omega_0^+)} + Y^{2p}(u,\theta)),$$

$$\|\Delta_t(-h)\boldsymbol{P}\|_{L_p(\Omega_0^+)} \leqslant c(\|\Delta_t(-h)\nabla\theta^+\|_{L_p(\Omega_0^+)} \sup_{Q_T^+} |\theta^+(y,t)|$$
$$+ \|\Delta_t(-h)\theta^+\|_{L_{q_0}(\Omega_0^+)} \|\nabla\theta^+\|_{L_{q_1}(\Omega_0^+)}),$$

$$\|\boldsymbol{P}\|_{W^{0,l/2}_p(Q_T^+)} \leqslant cT^{1/p-l/2} \|D_t\theta^+\|_{W^{l+1,0}_p(Q_T^+)} \sup_{t<T} \|\theta^+(\cdot,t)\|_{W^{l+1}_p(\Omega_0^+)}$$

we obtain

$$\|\boldsymbol{P}\|_{\widehat{W}^{l,l/2}_p(Q_T^+)} \leqslant c(T^{1/p-l/2}(\|\theta_0^+\|^2_{W^{l+1}_p(\Omega_0^+)} + \delta Y(\boldsymbol{u},\theta)). \tag{2.16}$$

The term in l_4 containing the expression $(p(\overline{\rho}^+ + \theta^+) - p(\overline{\rho}^+) - p_1\theta^+)\big|_{\Gamma_0}$ is estimated in a similar way.

We proceed with the estimates of $l_5(\boldsymbol{u})$ and $l_6(\boldsymbol{u})$. From (2.3)–(2.7) it follows that the coefficients $g_{\alpha\beta}$ in $\Delta(t)$ are uniformly bounded and coefficients $\dot g_{\alpha\beta}$ in $\dot\Delta(t)$ are controlled by $|\nabla\boldsymbol{u}|$. By (2.2), l_5 is equal to the sum $l_5=l_{51}+l_{52}$ with $l_{51}=\sigma\mathcal{D}_t(\boldsymbol{n}\Delta(t))\int\limits_0^t\boldsymbol{u}(y,\tau)\,\mathrm{d}\tau$, whence

$$
\begin{aligned}
\|l_{51}\|_{W_p^{l-1/p}(\Gamma_0)} &\leqslant c\|\nabla\boldsymbol{u}\|_{W_p^{l+1-1/p-\varkappa}(\Gamma_0)}\int\limits_0^t\|\boldsymbol{u}\|_{W_p^{2+l-1/p}(\Gamma_0)}\,\mathrm{d}\tau\\
&\leqslant c\delta\|\nabla\boldsymbol{u}\|_{W_p^{l+1-1/p-\varkappa}(\Gamma_0)},\quad \varkappa\in(0,l-1/p),\\
\|\Delta_t(-h)l_{51}\|_{L_p(\Gamma_0)} &\leqslant c\|\Delta_t(-h)\nabla\boldsymbol{u}\|_{L_p(\Gamma_0)}\int\limits_0^t\sup_{\Gamma_0}|\boldsymbol{u}(y,t-\tau)|\,\mathrm{d}\tau\\
&\quad+\|\nabla\boldsymbol{u}\|_{W_p^{l+1-1/p-\varkappa}(\Gamma_0)}\int\limits_0^h\|\boldsymbol{u}(\cdot,\tau-h)\|_{W_p^{2+l-1/p}(\Gamma_0)}\,\mathrm{d}\tau,\\
\|l_{51}\|_{\widehat{W}_p^{l-1/p,l/2-1/2p}(G_T)} &\leqslant c\delta(\|\nabla\boldsymbol{u}\|_{\widehat{W}_p^{l/2-1/2p}((0,T);W_p^1(\Gamma_0))}\\
&\quad+\|\boldsymbol{u}\|_{W_p^{l+1-1/p-\varkappa,0}(G_T)}).
\end{aligned}
$$

□

The expression $l_{52}=\sigma\int\limits_0^t\mathcal{D}_\tau(\boldsymbol{n}\Delta(\tau)\,\mathrm{d}\tau\cdot\boldsymbol{u}$ is estimated in the same way.

It remains to estimate $l_6(\boldsymbol{u})$. We have

$$
\begin{aligned}
&\|l_6\|_{W_p^{l-1/p}(\Gamma_0)}\leqslant c\|\nabla\boldsymbol{u}\|_{W_p^{l+1-1/p-\varkappa}(\Gamma_0)}\|\boldsymbol{y}\|_{W_p^{2+l-1/p}(\Gamma_0)}\leqslant c\|\nabla\boldsymbol{u}\|_{W_p^{l+1-1/p-\varkappa}(\Gamma_0)},\\
&\frac{1}{T^{pl/2-1/2}}\int\limits_0^T\|l_6\|^p_{L_p(\Gamma_0)}\,\mathrm{d}t\leqslant cT^{3/2-pl/2}\sup_{t<T}\|\nabla\boldsymbol{u}\|^p_{L_p(\Gamma_0)}\|\boldsymbol{y}\|^p_{W_p^{2+l-1/p}(\Gamma_0)},\\
&\|\triangle_t(-h)l_6\|_{L_p(\Gamma_0)}\leqslant c(\|\triangle_t(-h)\nabla\boldsymbol{u}\|_{L_p(\Gamma_0)}+\|\nabla\boldsymbol{u}\|_{W_p^{l+1-1/p-\varkappa}(\Gamma_0)}\\
&\qquad\times\sqrt{h}\|\nabla\boldsymbol{u}\|_{W_p^{l+1-1/p-\varkappa,0}(G_{t-h,t})})\|\boldsymbol{y}\|_{W_p^{l+2-1/2p}(\Gamma_0)},
\end{aligned}\tag{2.17}
$$

hence

$$
\begin{aligned}
\|l_6\|_{\widehat{W}_p^{l-1/p,l/2-1/2p}(G_T)}\leqslant c(&\|\nabla\boldsymbol{u}\|_{\widehat{W}_p^{l/2-1/2p}((0,T);W_p^1(\Gamma_0))}\\
&+\sup_{t<T}\|\nabla\boldsymbol{u}\|_{W_p^{l+1/p-\varkappa}(\Gamma_0)}).
\end{aligned}
$$

The estimates of the expressions (2.2) obtained above imply inequalities (2.8), (2.9) with the constants bounded for small T. This completes the proof of Proposition 1.

Scheme of proof of Theorem 2.

We seek the solution of Problem (1.3) in the form

$$\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{w}, \quad \theta = \theta_1 + \theta_2,$$

where $\boldsymbol{u}_1, \theta_1$ and $\boldsymbol{w}, \theta_2$ are defined as the solutions of

$$\begin{cases} \rho^- \mathcal{D}_t \boldsymbol{u}_1^- - \nabla \cdot \mathbb{T}^-(\boldsymbol{u}_1^-) + \nabla \theta_1^- = 0, \quad \nabla \cdot \boldsymbol{u}_1^- = 0 \quad \text{in} \quad \Omega_0^-, \\ \overline{\rho}^+ \mathcal{D}_t \boldsymbol{u}_1^+ - \nabla \cdot \mathbb{T}^+(\boldsymbol{u}_1^+) + p_1 \nabla \theta_1^+ = 0, \\ \mathcal{D}_t \theta_1^+ + \overline{\rho}^+ \nabla \cdot \boldsymbol{u}_1^+ = 0 \quad \text{in} \quad \Omega_0^+, \quad t > 0, \\ \boldsymbol{u}_1^+|_\Sigma = 0, \quad \boldsymbol{u}_1^\pm(y,0) = \boldsymbol{u}_0^\pm(y) \quad \text{in} \quad \Omega_0^\pm, \quad \theta_1^+(y,0) = \theta_0^+(y) \quad \text{in} \quad \Omega_0^+, \\ [\boldsymbol{u}_1]|_{\Gamma_0} = 0, \quad [\mu \Pi_0 \mathbb{S}(\boldsymbol{u}_1)\boldsymbol{n}_0]|_{\Gamma_0} = 0, \\ -p_1\theta_1^+ + \theta_1^- + [\boldsymbol{n}_0 \cdot \mathbb{T}(\boldsymbol{u}_1)\boldsymbol{n}_0] + \sigma \boldsymbol{n}_0 \cdot \Delta(0) \int\limits_0^t \boldsymbol{u}_1(\xi,\tau)\,\mathrm{d}\tau|_{\Gamma_0} = -\sigma H|_{t=0}, \end{cases} \tag{2.18}$$

$$\begin{cases} \rho^- \mathcal{D}_t \boldsymbol{w}^- - \nabla \cdot \mathbb{T}^-(\boldsymbol{w}^-) + \nabla \theta_2 = \boldsymbol{l}_1^-(\boldsymbol{u}^-, \theta^-) + \rho^- \widehat{\boldsymbol{f}}, \ \ \nabla \cdot \boldsymbol{w} = l_2(\boldsymbol{u}^-) \text{ in } \Omega_0^-, \\ \overline{\rho}^+ \mathcal{D}_t \boldsymbol{w}_t^+ - \nabla \cdot \mathbb{T}^+(\boldsymbol{w}^+) + p_1 \nabla \theta_2^+ = \boldsymbol{l}_1^+(\boldsymbol{u}^+, \theta^+) + (\overline{\rho}^+ + \theta^+)\widehat{\boldsymbol{f}}, \\ \mathcal{D}_t \theta_2^+ + \overline{\rho}^+ \nabla \cdot \boldsymbol{w}^+ = l_2^+(\boldsymbol{u}^+, \theta^+), \ \ \theta_2^+(y,0) = 0 \ \text{ in } \ \Omega_0^+, \\ \boldsymbol{w}^+|_\Sigma = 0, \quad \boldsymbol{w}(y,0) = 0 \text{ in } \Omega_0^\pm, \\ [\boldsymbol{w}]|_{\Gamma_0} = 0, \quad [\mu \Pi_0 \mathbb{S}(\boldsymbol{w})\boldsymbol{n}_0]|_{\Gamma_0} = \boldsymbol{l}_3(\boldsymbol{u}), \\ -p_1\theta_2^+ + \theta_2^- + [\boldsymbol{n}_0 \cdot \mathbb{T}(\boldsymbol{w})\boldsymbol{n}_0] + \sigma \boldsymbol{n}_0 \cdot \Delta(0) \int\limits_0^t \boldsymbol{w}(y,\tau) d\tau|_{\Gamma_0} \\ = l_4(\boldsymbol{u}) - \int\limits_0^t (l_5(\boldsymbol{u}) + l_6(\boldsymbol{u}))d\tau, \end{cases} \tag{2.19}$$

By Theorem 1, problem (2.18) is uniquely solvable and the solution satisfies the inequality

$$Y(\boldsymbol{u}_1, \theta_1) \leqslant c(T)(\|\boldsymbol{u}_0\|_{W_p^{1+2=2/p}(\cup \Omega_0^\pm)} + \|\theta_0^+\|_{W_p^{l+1}(\Omega_0^+)} + \sigma \|H_0\|_{W_p^{l+1-1/p}(\Gamma_0)}). \tag{2.20}$$

The solution of Problem (2.19) can be constructed by iterations:

$$\begin{cases} \rho^- \mathcal{D}_t \boldsymbol{w}^-_{m+1} - \nabla \cdot \mathbb{T}^-(\boldsymbol{w}^-_{m+1}) + \nabla\theta^-_{2,m+1} = \boldsymbol{l}^-_1(\boldsymbol{u}^-_m, \theta^-_m) + \rho^- \widehat{\boldsymbol{f}}_m, \\ \nabla \cdot \boldsymbol{w}^-_{m+1} = l^-_2(\boldsymbol{u}^-_m) \quad \text{in} \quad \Omega^-_0, \\ \overline{\rho}^+ \mathcal{D}_t \boldsymbol{w}^+_{m+1} - \nabla \cdot \mathbb{T}^+(\boldsymbol{w}^+_{m+1}) + p_1 \nabla\theta^+_{2,m+1} = \boldsymbol{l}^+_1(\boldsymbol{u}^+_m, \theta^+_m) + (\overline{\rho}^+ + \theta^+_m)\widehat{\boldsymbol{f}}_m, \\ \mathcal{D}_t \theta^+_{2,m+1} + \overline{\rho}^+ \nabla \cdot \boldsymbol{w}^+_{m+1} = l^+_2(\boldsymbol{u}^+_m, \theta^+_m) \quad \text{in} \quad \Omega^+_0, \\ \boldsymbol{w}^+_{m+1}|_\Sigma = 0, \quad \boldsymbol{w}^\pm_{m+1}(y,0) = 0 \quad \text{in} \quad \Omega^\pm_0, \quad \theta^+_{2,m+1}(y,0) = 0 \quad \text{in} \quad \Omega^+_0, \\ [\boldsymbol{w}_{m+1}]|_{\Gamma_0} = 0, \quad [\mu \Pi_0 \mathbb{S}(\boldsymbol{w}_{m+1})\boldsymbol{n}_0]|_{\Gamma_0} = \boldsymbol{l}_3(\boldsymbol{u}_m), \\ -p_1\theta^+_{2,m+1} + \theta^-_{2,m+1} + [\boldsymbol{n}_0 \cdot \mathbb{T}(\boldsymbol{w}_{m+1})\boldsymbol{n}_0]|_{\Gamma_0} - \sigma \boldsymbol{n}_0 \cdot \Delta(0) \int\limits_0^t \boldsymbol{w}_{m+1}(y,\tau)\,\mathrm{d}\tau|_{\Gamma_0} \\ \quad = l_4(\boldsymbol{u}_m) - \int\limits_0^t (l_5(\boldsymbol{u}_m) + l_6(\boldsymbol{u}_m))\,\mathrm{d}\tau, \end{cases} \tag{2.21}$$

where $m = 1, 2, \dots$, $\boldsymbol{u}_m = \boldsymbol{u}_1 + \boldsymbol{w}_m$, $\theta_m = \theta_1 + \theta_{2,m}$, $\widehat{\boldsymbol{f}}_m = \boldsymbol{f}(X_{\boldsymbol{u}_m}, t)$, $\boldsymbol{w}_1 = 0$, $\theta_{2,1} = 0$.

By Theorem 1 and Proposition 1, problem (2.21) with given $\boldsymbol{w}_m \in W^{2+l,1+l/2}_p(Q^\pm_T)$, $\theta_{2,m} \in \widehat{W}^{0,l/2}_2(G_T)$, $\nabla\theta_{2,m} \in \widehat{W}^{l,l/2}_p(Q^\pm_T)$, is uniquely solvable and the solution satisfies the inequality

$$\begin{aligned} &Y(\boldsymbol{w}_{m+1}, \theta_{2,m+1}) \leqslant c(T)(Z(\boldsymbol{u}_m, \theta_m) + \|l_4(\boldsymbol{u}_m, \theta_m)\|_{W^{l/2}_p((0,T), W^{1-1/p}_p(\Gamma_0))} \\ &+ \|l_5(\boldsymbol{u}_m)\|_{W^{l-1/p, l/2-1/2p}_p(G_T)} + \|l_6(\boldsymbol{u}_m)\|_{W^{l-1/p, l/2-1/2p}_p(G_T)} \\ &+ \|\widehat{\boldsymbol{f}}_m\|_{W^{l,l/2}_p(Q_T)} + \|\theta^+_m \widehat{\boldsymbol{f}}_m\|_{W^{l,l/2}_p(Q_T)}) \\ &\leqslant c(T)(\delta Y(\boldsymbol{u}_m, \theta_m) + \|\widehat{\boldsymbol{f}}_m\|_{W^{l,l/2}_p(Q_T)}) \\ &+ \epsilon_1 Y(\boldsymbol{u}_m, \theta_m) + c_1(\epsilon_1)\|\boldsymbol{u}_m\|_{L_p(Q_T)} + T^{1/p}\|\theta_0\|^2_{W^{l+1}_p(\Omega^+_0)}). \end{aligned} \tag{2.22}$$

Hence

$$\begin{aligned} &Y(\boldsymbol{u}_{m+1}, \theta_{m+1}) \leqslant Y(\boldsymbol{u}_1, \theta_1) + Y(\boldsymbol{w}_{m+1}, \theta_{2,m+1}) \leqslant c(T)(\delta_1 Y(\boldsymbol{u}_m, \theta_m) \\ &+ \|\nabla \boldsymbol{f}_m\|_{Q_T} \sup_{Q_T} |\boldsymbol{u}_m| + \epsilon_1 Y(\boldsymbol{u}_m, \theta_m) + c_1(\epsilon_1)\|\boldsymbol{u}_m\|_{Q_T} + cF(T)) \\ &\leqslant c(T)(\delta_1 Y(\boldsymbol{u}_m, \theta_m) + c_1(\epsilon_1)\|\boldsymbol{u}_m\|_{Q_T} + F(T)), \end{aligned} \tag{2.23}$$

where $\delta_1 = \delta + \epsilon_1$,

$$F(T) = \|\boldsymbol{u}_0\|_{W_p^{l+2-2/p}(\cup\Omega_0^\pm)} + \|\theta_0^+\|_{W_p^{l+1}(\Omega_0^+)} + T^{1/p}\|\theta_0^+\|^2_{W_p^{l+1}(\Omega_0^+)}$$
$$+ \|H_0\|_{W_p^{l+1-1/p}(\Gamma_0)} + \|\widehat{\boldsymbol{f}}\|_{W_p^{l,l/2}(Q_T)}.$$

We obtain a uniform estimate for $Y_m(T) = Y(\boldsymbol{u}_m, \theta_m)$. Inequality (2.23) holds for arbitrary $t < T$, hence

$$Y^p_{m+1}(t) \leqslant \delta_2 Y^p_m(t) + c_1 \int_0^t Y^p_m(\tau)\,\mathrm{d}\tau + c_2 F^p(t), \tag{2.24}$$

because

$$\|\boldsymbol{u}_m(\cdot,t)\|^p_{L_p(\Omega_0)} \leqslant \|\boldsymbol{u}_0\|^p_{L_p(\Omega_0)} + p\int_0^t \|\boldsymbol{u}_m\|^{p-1}_{L_p(\Omega_0)}\|\mathcal{D}_\tau \boldsymbol{u}_m\|_{L_p(\Omega_0)}\,\mathrm{d}\tau$$
$$\leqslant \|\boldsymbol{u}_0\|^p_{L_p(\Omega_0)} + pY^p_m(t)$$

and

$$\int_0^t \|\boldsymbol{u}_m\|^p_{L_p(\Omega)}\,\mathrm{d}\tau \leqslant t\|\boldsymbol{u}_0\|^p_{L_p(\Omega_0)} + p\int_0^t Y^p_m(\tau)\,\mathrm{d}\tau.$$

For $m = 0$, (1.9) reduces to (1.8). If (1.9) holds for all $Y_j(T)$, $j = 2,\dots,m$, then (2.24) yields

$$Y^p_{m+1}(t) \leqslant c_4 F^p(t) + \mathcal{A}(Y^p_{m-1} + c_2 F^p(t))$$
$$\leqslant \mathcal{A}^{m+1} Y^p_0(t) + c_2(F^p + \mathcal{A}F^p + \dots + \mathcal{A}^m F^p(t),$$

where

$$\mathcal{A}f(t) = \delta_2 f(t) + c_3\int_0^t f(\tau)\,\mathrm{d}\tau.$$

As is shown in [2], this implies

$$Y^p_{m+1}(t) \leqslant c(T)(\delta_2 Y^p_0(t) + F^p(t))).$$

hence inequality (1.9) for Y_{m+1} follows.

The convergence of the approximations $\boldsymbol{u}_m, \theta_m$ to the solution of (1.3) and the uniqueness of the solution is verified as in [2]. The solution satisfies inequality (1.9) and, for small T, also (2.7).

It follows that the surface

$$\Gamma_t = \Big\{x = y + \int\limits_0^t \boldsymbol{u}(y,\tau)\,\mathrm{d}\tau,\ y \in \Gamma_0,\Big\}$$

is contained in the $\delta-$ neighborhood of Γ_0 and belongs to the class $W_p^{2+l-1/p}$. Equation

$$-\big(p(\overline{\rho}^+ + \theta^+) - p(\overline{\rho}^+)\big) + \theta^- + [\boldsymbol{n}\cdot\mathbb{T}_{\boldsymbol{u}}(\boldsymbol{u})\boldsymbol{n}] = -\sigma H \tag{2.25}$$

allows us to show that Γ_t is more regular. Let $\mathcal{G}$ be a smooth closed surface also located in the $\delta-$ neighborhood of Γ_0, so that Γ_t is given by the equation

$$x = \eta + \boldsymbol{N}(\eta)r(\eta,t) \equiv \mathcal{X}(\eta,t), \tag{2.26}$$

where $\boldsymbol{N}$ is the normal to $\mathcal{G}$ and $r \in W_p^{2+l-1/p}(\mathcal{G})$. Assume that near a certain point $\eta_0 \in \mathcal{G}$ the surface Γ_t is given by the equation $\eta_3 = \psi(\eta_1,\eta_2)$, where $\eta' = (\eta_1,\eta_2) \in \mathfrak{Q}$ belongs to the tangential plane to $\mathcal{G}$ at the point η_0 and the η_3- axis is directed along $\boldsymbol{N}(\eta_0)$. In this coordinates, Equation (2.25) takes the form

$$d(\eta,t) = \sigma \sum_{\alpha=1,2} \frac{\partial}{\partial\eta_\alpha} \frac{\mathcal{D}_{\eta_\alpha}\psi}{\sqrt{1+|\nabla'\psi|^2}}, \tag{2.27}$$

where $d(\eta,t) = g(x,t)|_{x=\mathcal{X}(\eta,t)}$,

$$\begin{aligned} g(x,t) = &-(p(\overline{\rho}^+ + \vartheta(x,t)) - p(\overline{\rho}^+)) + \vartheta^-(x,t) \\ &+ [\boldsymbol{n}\cdot\mathbb{T}(\boldsymbol{v})\boldsymbol{n}] \in W_p^{l+1-1/p,0}(\Gamma_t \times (0,T)), \end{aligned}$$

which implies $d(\eta,t) \in W_p^{l+1-1/p,0}(\mathfrak{G}_T)$, $\mathfrak{G}_T = \mathcal{G}o \times (0,T)$. Equation (2.27) is nonlinear elliptic whose coefficients are expressed in terms of the derivatives of ψ and their norms are controlled by $\|\nabla\boldsymbol{u}\|_{W_p^{l+1-1/p}(\Gamma_0)}$. From the regularity theorem for the solutions of elliptic equations one can conclude that $\psi \in W_p^{3+l-1/p,0}(\mathfrak{Q}' \times (0,T))$, $\mathfrak{Q}' \subset \mathfrak{Q}$; since η_0 is arbitrary we have $r \in W_p^{3+l-1/p,0}(\mathfrak{G}_T)$ and

$$\|r\|_{W_p^{3+l-1/p,0}(\mathfrak{G}_T)} \leqslant c(Y(\boldsymbol{u},\theta) + F(T)) \leqslant cF(T). \tag{2.28}$$

In addition, from $\mathcal{D}_t X_{\boldsymbol{u}} \in W_p^{2+l-1/p,0}(G_T)$ it follows that $\mathcal{D}_t r(\cdot,t) \in W_p^{2+l-1/p,0}(\mathfrak{G}_T)$ and

$$\|\mathcal{D}_t r\|_{W_p^{2+l-1/p,0}(\mathfrak{G}_T)} \leqslant c\|\boldsymbol{u}\|_{W_p^{2+l-1/p,0}(\mathfrak{G}_T)}, \tag{2.29}$$

hence

$$\|r(\cdot,t)\|_{W_p^{3+l-2/p}(\mathfrak{G})} \leqslant c\big(\|r\|_{W_p^{3+l-1/p,0}(\mathfrak{G}_T)} + \|\mathcal{D}_t r\|_{W_p^{2+l-1/p,0}(\mathfrak{G}_T)}\big) \leqslant cF, \quad (2.30)$$

i.e., $\Gamma_t \in W_p^{3+l-2/p}(\mathfrak{G}), \quad \forall t \in [0,T]$.

Under certain not too strong restrictions on $\boldsymbol{f}$ it can be shown that for $0 < t \leqslant T$ $\Gamma_t \in W_p^{3+l-1/p}(\mathfrak{G})$ (cf. [2] for $p = 2$). This is a consequence of the following proposition.

Proposition 2. *Assume that* $p \in C^{3+1}(\min\rho, \max\rho)$ *and* $\boldsymbol{f}$ *satisfies additional restrictions:*

$$\boldsymbol{f} \in W_p^{\alpha_1}\big((t_0,T); W_p^l(\Omega)\big) \cap W_p^{0,\alpha_1+l/2}(Q_{t_0,T})$$

with $\alpha_1 \in (1/p,1)$, $\nabla \boldsymbol{f} \in W_p^{l,l/2}(Q_T)$. *Then* $\boldsymbol{u}^{(s)}(y,t) = \boldsymbol{u}(y,t) - \boldsymbol{u}(y,t-s)$ *and* $\theta^{(s)}(y,t) = \theta(y,t) - \theta(y,t-s)$ *satisfy the inequality*

$$\|u^{(s)}\|_{W_p^{2+l,1+l/2}(\cup Q^{\pm}_{t_2,t_1})} + \sum_{\pm}(\|\nabla\theta^{\pm(s)}\|_{W_p^{l,l/2}(Q^{\pm}_{t_2,t_1})} + \|\theta^{\pm(s)}\|_{W_p^{0,l/2}(Q^{\pm}_{t_2,t_1})}) \leqslant C(\boldsymbol{u},\theta,r)s^a,$$

where $a > 1/p$, $0 < t_0 < t_1 < T$, $t_2 = (t_1 - (t_1 - t_0)/4)$, $0 < s < \min(t_1 - t_2, t_0)$, $Q^{\pm}_{t_2,t_1} = \Omega_0^{\pm} \times (t_2,t_1)$ *and* C *is a constant dependent on the norms of the solution of* (1.3).

In the case $p = 2$ this result is obtained in [2], Theorem 6.

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