Записки научных
семинаров ПОМИ
Том 489, 2020 г.

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ON TYPE I BLOWUPS OF SUITABLE WEAK SOLUTIONS TO NAVIER–STOKES EQUATIONS NEAR BOUNDARY

Abstract. In this note, boundary Type I blowups of suitable weak solutions to the Navier–Stokes equations are discussed. In particular, it has been shown that, under certain assumptions, the existence of non-trivial mild bounded ancient solutions in half space leads to the existence of suitable weak solutions with Type I blowup on the boundary.

§1. Introduction

The aim of the note is to study conditions under which solutions to the Navier–Stokes equations undergo Type I blowups on the boundary.

Consider the classical Navier–Stokes equations

$$\partial_t v + v \cdot \nabla v - \Delta v = -\nabla q, \qquad \operatorname{div} v = 0 \tag{1.1}$$

in the space time domain $Q^+ = B^+ \times]-1,0[$, where $B^+ = B^+(1)$ and $B^+(r) = \{x \in \mathbb{R}^3 : |x| < r,\, x_3 > 0\}$ is a half ball of radius r centred at the origin $x = 0$. It is supposed that v satisfies the homogeneous Dirichlet boundary condition

$$v(x', 0, t) = 0 \tag{1.2}$$

for all $|x'| < 1$ and $-1 < t < 0$. Here, $x' = (x_1, x_2)$ so that $x = (x', x_3)$ and $z = (x, t)$.

Our goal is to understand how to determine whether or not the origin $z = 0$ is a singular point of the velocity field v. We say that $z = 0$ is a regular point of v if there exists $r > 0$ such that $v \in L_\infty(Q^+(r))$ where $Q^+(r) = B^+(r) \times]-r^2, 0[$. It is known, see [4] and [5], that the velocity v is Hölder continuous in a parabolic vicinity of $z = 0$ if $z = 0$ is a regular point. However, further smoothness even in spatial variables does not follow in the regular case, see [3] and [7] for counter-examples.

Key words and phrases: Navier–Stokes equations, suitable weak solutions, ancient solutions, Type I blowups.

This work is supported by the grant RFBR 20-01-00397.

The class of solutions to be studied is as follows.

Definition 1.1. *A pair of functions v and q is called a suitable weak solution to the Navier–Stokes equations in Q^+ near the boundary if and only if the following conditions hold:*

$$v\in L_{2,\infty}(Q^+)\cap L_2(-1,0;W^1_2(Q^+)),\qquad q\in L_{\frac32}(Q^+); \tag{1.3}$$

v and q satisfy equations (1.1) *and boundary condition* (1.2);

$$\begin{aligned}\int\limits_{B^+}\varphi(x,t)|v(x,t)|^2dx+2\int\limits_{-1}^{t}\int\limits_{B^+}\varphi|\nabla v|^2dxdt\\ \leqslant\int\limits_{-1}^{t}\int\limits_{B^+}(|v|^2(\partial_t\varphi+\Delta\varphi)+v\cdot\nabla v(|v|^2+2q))dxdt\end{aligned} \tag{1.4}$$

for all non-negative functions $\varphi\in C^\infty_0(B\times]-1,1[)$ such that $\varphi|_{x_3=0}=0$.

In what follows, some statements will be expressed in terms of scale invariant quantities (invariant with respect to the Navier–Stokes scaling: $\lambda v(\lambda x,\lambda^2t)$ and $\lambda^2q(\lambda x,\lambda^2t)$). Here, they are:

$$A(v,r)=\sup_{-r^2<t<0}\frac1r\int\limits_{B^+(r)}|v(x,t)|^2dx,\qquad E(v,r)=\frac1r\int\limits_{Q^+(r)}|\nabla v|^2dz,$$

$$C(v,r)=\frac1{r^2}\int\limits_{Q^+(r)}|v|^3dz,\qquad D_0(q,r)=\frac1{r^2}\int\limits_{Q^+(r)}|q-[q]_{B^+(r)}|^{\frac32}dz,$$

$$D_2(q,r)=\frac{1}{r^{\frac{13}{8}}}\int\limits_{-r^2}^{0}\Big(\int\limits_{B^+(r)}|\nabla q|^{\frac{12}{11}}dx\Big)^{\frac{11}{8}}dt,$$

where

$$[f]_\Omega=\frac1{|\Omega|}\int\limits_\Omega fdx.$$

We also introduce the following values:

$$g(v):=\min\{\sup_{0<R<1}A(v,R),\sup_{0<R<1}C(v,R),\sup_{0<R<1}E(v,R)\}$$

and, given $r>0$,

$$G_r(v,q):=$$

$$\max\{\sup_{0<R<r} A(v,R), \sup_{0<R<r} C(v,R), \sup_{0<R<r} E(v,R), \sup_{0<R<r} D_0(q,R)\}.$$

Relationships between $g(v)$ and $G_1(v,q)$ is described in the following proposition.

Proposition 1.2. *Let v and q be a suitable weak solution to the Navier–Stokes equations in Q^+ near the boundary. Then, G_1 is bounded if and only if g is bounded.*

If $z=0$ is a singular point of v and $g(v)<\infty$, then $z=0$ is called a Type I singular point or a Type I blowup point.

Now, we are ready to state the main results of the paper.

Definition 1.3. *A function $u: Q^+_- := \mathbb{R}^3_+ \times]-\infty, 0[\to \mathbb{R}^3$ is called a local energy ancient solution if there exists a function $p: Q^+_- \to \mathbb{R}$ such that the pair u and p is a suitable weak solution in $Q^+(R)$ for any $R>0$. Here, $\mathbb{R}^3_+ := \{x \in \mathbb{R}^3 : \, x_3 > 0\}$.*

Theorem 1.4. *There exists a suitable weak solution v and q with Type* I *blowup at the origin $z=0$ if and only if there exists a non-trivial local energy ancient solution u such that u and the corresponding pressure p have the following prosperities:*

$$G_\infty(u,p) := \max\{\sup_{0<R<\infty} A(u,R), \sup_{0<R<\infty} E(u,R),$$
$$\sup_{0<R<\infty} C(u,R), \sup_{0<R<\infty} D_0(p,R)\} < \infty \tag{1.5}$$

and

$$\inf_{0<a<1} C(u,a) \geqslant \varepsilon_1 > 0. \tag{1.6}$$

Remark 1.5. *According to* (1.5) *and* (1.6)*, the origin $z=0$ is Type* I *blowup of the velocity u.*

There is another way to construct a suitable weak solution with Type I blowup. It is motivated by the recent result in [1] for the interior case. Now, the main object is related to the so-called mild bounded ancient solutions in a half space, for details see [8] and [2].

Definition 1.6. *A bounded function u is a mild bounded ancient solution if and only if there exists a pressure $p = p^1 + p^2$, where the even extension of p^1 in x_3 to the whole space $\mathbb{R}^3$ is a $L_\infty(-\infty, 0; BMO(\mathbb{R}^3))$-function,*

$$\Delta p^1 = \operatorname{divdiv} u \otimes u$$

in Q^+_- *with* $p^1_{,3}(x',0,t)=0$*, and* $p^2(\cdot,t)$ *is a harmonic function in* $\mathbb{R}^3_+$*, whose gradient satisfies the estimate*

$$|\nabla p^2(x,t)|\leqslant \ln(2+1/x_3)$$

for all $(x,t)\in Q^+_-$ *and has the property*

$$\sup_{x'\in\mathbb{R}^2}|\nabla p^2(x,t)|\to 0$$

as $x_3\to\infty$*; functions* u *and* p *satisfy:*

$$\int\limits_{Q^+_-} u\cdot\nabla q dz=0$$

for all $q\in C^\infty_0(Q_-:=\mathbb{R}^3\times]-\infty,0[)$ *and, for any* $t<0$*,*

$$\int\limits_{Q^+_-}\Big(u\cdot(\partial_t\varphi+\Delta\varphi)+u\otimes u:\nabla\varphi+p\mathrm{div}\,\varphi\Big)dz=0$$

for and $\varphi\in C^\infty_0(Q_-)$ *with* $\varphi(x',0,t)=$ *for all* $x'\in\mathbb{R}^2$*.*

As it has been shown in [2], any mild bounded ancient solution u in a half space is infinitely smooth up to the boundary and $u|_{x_3}=0$.

Theorem 1.7. *Let* u *be a mild bounded ancient solution such that* $|u|\leqslant 1$ *and* $|u(0,a,0)|=1$ *for a positive number* a *and such that* (1.5) *holds. Then there exists a suitable weak solution in* Q^+ *having Type* I *blowup at the origin* $z=0$*.*

§2. Basic Estimates

In this section, we are going to state and prove certain basic estimates for arbitrary suitable weak solutions near the boundary.

For our purposes, the main estimate of the convective term can be derived as follows. First, we apply Hölder inequality in spatial variables:

$$\||v||\nabla v|\|^{\frac32}_{\frac{12}{11},\frac32,Q^+(r)}=\int\limits_{-r^2}^0\Big(\int\limits_{B^+(r)}|v|^{\frac{12}{11}}|\nabla v|^{\frac{12}{11}}dx\Big)^{\frac{11}{8}}dt$$

$$\leqslant\int\limits_{-r^2}^0\Big(\int\limits_{B^+(r)}|\nabla v|^2dx\Big)^{\frac34}\Big(\int\limits_{B^+(r)}|v|^{\frac{12}{5}}dx\Big)^{\frac58}dt.$$

Then, by interpolation, since $\frac{12}{5} = 2 \cdot \frac{3}{5} + 3 \cdot \frac{2}{5}$, we find

$$\Big(\int\limits_{B^+(r)} |v|^{\frac{12}{5}} dx \Big)^{\frac{5}{8}} \leqslant \Big(\int\limits_{B^+(r)} |v|^2 dx \Big)^{\frac{3}{8}} \Big(\int\limits_{B^+(r)} |v|^3 dx \Big)^{\frac{1}{4}}.$$

So,

$$\begin{aligned}
&\| |v| |\nabla v| \|^{\frac{3}{2}}_{\frac{12}{11}, \frac{3}{2}, Q^+(r)} \\
&\leqslant \int\limits_{-r^2}^{0} \Big(\int\limits_{B^+(r)} |\nabla v|^2 dx \Big)^{\frac{3}{4}} \Big(\int\limits_{B^+(r)} |v|^2 dx \Big)^{\frac{3}{8}} \Big(\int\limits_{B^+(r)} |v|^3 dx \Big)^{\frac{1}{4}} dt \\
&\leqslant \sup_{-r^2<t<0} \Big(\int\limits_{B^+(r)} |v|^2 dx \Big)^{\frac{3}{8}} \Big(\int\limits_{Q^+(r)} |\nabla v|^2 dx dt \Big)^{\frac{3}{4}} \Big(\int\limits_{Q^+(r)} |v|^3 dx dt \Big)^{\frac{1}{4}} \\
&\leqslant r^{\frac{3}{8}} r^{\frac{3}{4}} r^{\frac{1}{2}} A^{\frac{3}{8}}(v,r) E^{\frac{3}{4}}(v,r) C^{\frac{1}{4}}(v,r) \\
&= r^{\frac{13}{8}} A^{\frac{3}{8}}(v,r) E^{\frac{3}{4}}(v,r) C^{\frac{1}{4}}(v,r).
\end{aligned} \tag{2.1}$$

Two other estimates are well known and valid for any $0 < r \leqslant 1$:

$$C(v,r) \leqslant c A^{\frac{3}{4}}(v,r) E^{\frac{3}{4}}(v,r) \tag{2.2}$$

and

$$D_0(q,r) \leqslant c D_2(q,r). \tag{2.3}$$

Next, one more estimate immediately follows from the energy inequality (2.4) for a suitable choice of cut-off function φ:

$$A(v,\tau R) + E(v,\tau R) \leqslant c(\tau) \Big[C^{\frac{2}{3}}(v,R) + C^{\frac{1}{3}}(v,R) D_0^{\frac{2}{3}}(q,R) + C(v,R) \Big] \tag{2.4}$$

for any $0 < \tau < 1$ and for all $0 < R \leqslant 1$.

The last two estimates are coming out from the linear theory. Here, they are:

$$\begin{aligned}
D_2(q,r) &\leqslant c \Big(\frac{r}{\varrho} \Big)^2 \Big[D_2(q,\varrho) + E^{\frac{3}{4}}(v,\varrho) \Big] \\
&\quad + c \Big(\frac{\varrho}{r} \Big)^{\frac{13}{8}} A^{\frac{3}{8}}(v,\varrho) E^{\frac{3}{4}}(v,\varrho) C^{\frac{1}{4}}(v,\varrho)
\end{aligned} \tag{2.5}$$

for any $0 < r < \varrho \leqslant 1$ and

$$\begin{aligned}\|\partial_t v\|_{\frac{12}{11},\frac{3}{2},Q^+(\tau R)} + \|\nabla^2 v\|_{\frac{12}{11},\frac{3}{2},Q^+(\tau R)} + \|\nabla q\|_{\frac{12}{11},\frac{3}{2},Q^+(\tau R)} \\ \leqslant c(\tau) R^{\frac{13}{12}} \Big[D_0^{\frac{2}{3}}(q,R) + C^{\frac{1}{3}}(v,R) + E^{\frac{1}{2}}(v,R) \\ + (A^{\frac{3}{8}}(v,R) E^{\frac{3}{4}}(v,R) C^{\frac{1}{4}}(v,R))^{\frac{2}{3}} \Big]\end{aligned} \tag{2.6}$$

for any $0 < \tau < 1$ and for all $0 < R \leqslant 1$.

Estimate (2.6) follows from bound (2.1), from the local regularity theory for the Stokes equations (linear theory), see paper [5], and from scaling. Estimate (2.5) will be proven in the next section.

§3. Proof of (2.5)

Here, we follows paper [4]. We let $\widetilde{f} = -v \cdot \nabla v$ and observe that

$$\frac{1}{r} \|\nabla v\|_{\frac{12}{11},\frac{3}{2},Q^+(r)} \leqslant r^{\frac{13}{12}} E^{\frac{1}{2}}(v,r) \tag{3.1}$$

and, see (2.1),

$$\|\widetilde{f}\|_{\frac{12}{11},\frac{3}{2},Q^+(r)} \leqslant c r^{\frac{13}{12}} (A^{\frac{3}{8}}(v,r) E^{\frac{3}{4}}(v,r) C^{\frac{1}{4}}(v,r))^{\frac{2}{3}}. \tag{3.2}$$

Next, we select a convex domain with smooth boundary so that

$$B^+(1/2) \subset \widetilde{B} \subset B^+$$

and, for $0 < \varrho < 1$, we let

$$\widetilde{B}(\varrho) = \{x \in \mathbb{R}^3 : x/\varrho \in \widetilde{B}\}, \qquad \widetilde{Q}(\varrho) = \widetilde{B}(\varrho) \times]-\varrho^2, 0[.$$

Now, consider the following initial boundary value problem:

$$\partial_t v^1 - \Delta v^1 + \nabla q^1 = \widetilde{f}, \qquad \operatorname{div} v^1 = 0 \tag{3.3}$$

in $\widetilde{Q}(\varrho)$ and

$$v^1 = 0 \tag{3.4}$$

on parabolic boundary $\partial' \widetilde{Q}(\varrho)$ of $\widetilde{Q}(\varrho)$. It is also supposed that $[q^1]_{\widetilde{B}(\varrho)}(t) = 0$ for all $-\varrho^2 < t < 0$.

Due to estimate (3.2) and due to the Navier–Stokes scaling, a unique solution to problem (3.3) and (3.4) satisfies the estimate

$$\begin{aligned}\frac{1}{\varrho^2}\|v^1\|_{\frac{12}{11},\frac32,\widetilde{Q}(\varrho)}+\frac{1}{\varrho}\|\nabla v^1\|_{\frac{12}{11},\frac32,\widetilde{Q}(\varrho)}+\|\nabla^2 v^1\|_{\frac{12}{11},\frac32,\widetilde{Q}(\varrho)}\\ +\frac{1}{\varrho}\|q^1\|_{\frac{12}{11},\frac32,\widetilde{Q}(\varrho)}+\|\nabla q^1\|_{\frac{12}{11},\frac32,\widetilde{Q}(\varrho)}\\ \leqslant c\|\widetilde{f}\|_{\frac{12}{11},\frac32,\widetilde{Q}(\varrho)}\leqslant c\varrho^{\frac{12}{11}}(A^{\frac38}(v,\varrho)E^{\frac34}(v,\varrho)C^{\frac14}(v,\varrho))^{\frac23},\end{aligned}\tag{3.5}$$

where a generic constant c is independent of ϱ.

Regarding $v^2=v-v^1$ and $q^2=q-[q]_{B_+(\varrho/2)}-q^1$, one can notice the following:

$$\partial_t v^2-\Delta v^2+\nabla q^2=0,\qquad \operatorname{div} v^2=0\tag{3.6}$$

in $\widetilde{Q}(\varrho)$ and

$$v^2|_{x_3=0}=0.\tag{3.7}$$

As it was indicated in [5], functions v^2 and q^2 obey the estimate

$$\|\nabla^2 v^2\|_{9,\frac32,Q^+(\varrho/4)}+\|\nabla q^2\|_{9,\frac32,Q^+(\varrho/4)}\leqslant \frac{c}{\varrho^{\frac{29}{12}}}L,\tag{3.8}$$

where

$$L:=\frac{1}{\varrho^2}\|v^2\|_{\frac{12}{11},\frac32,Q^+(\varrho/2)}+\frac{1}{\varrho}\|\nabla v^2\|_{\frac{12}{11},\frac32,Q^+(\varrho/2)}+\frac{1}{\varrho}\|q^2\|_{\frac{12}{11},\frac32,Q^+(\varrho/2)}.$$

As to an evaluation of L, we have

$$\begin{aligned}L\leqslant&\Big[\frac{1}{\varrho^2}\|v\|_{\frac{12}{11},\frac32,Q^+(\varrho/2)}+\frac{1}{\varrho}\|\nabla v\|_{\frac{12}{11},\frac32,Q^+(\varrho/2)}+\frac{1}{\varrho}\|q-[q]_{B^+(\varrho/)}\|_{\frac{12}{11},\frac32,Q^+(\varrho/2)}\\ &+\frac{1}{\varrho^2}\|v^1\|_{\frac{12}{11},\frac32,Q^+(\varrho/2)}+\frac{1}{\varrho}\|\nabla v^1\|_{\frac{12}{11},\frac32,Q^+(\varrho/1)}+\frac{1}{\varrho}\|q^1\|_{\frac{12}{11},\frac32,Q^+(\varrho/2)}\Big]\\ \leqslant&\Big[\frac{1}{\varrho}\|\nabla v\|_{\frac{12}{11},\frac32,Q^+(\varrho/2)}+\frac{1}{\varrho}\|\nabla q\|_{\frac{12}{11},\frac32,Q^+(\varrho/2)}\\ &+\frac{1}{\varrho}\|\nabla v^1\|_{\frac{12}{11},\frac32,Q^+(\varrho/2)}+\frac{1}{\varrho}\|q^1\|_{\frac{12}{11},\frac32,Q^+(\varrho/2)}\Big].\end{aligned}$$

So, by (3.1), by (2.6) with $R=\varrho$ and $\tau=\frac12$, and by (3.5), one can find the following bound

$$\begin{aligned}\|\nabla q^2\|_{9,\frac32,Q^+(\varrho/4)}\leqslant \frac{c}{\varrho^{\frac43}}\Big[E^{\frac12}(v,\varrho)+D_2^{\frac23}(q,\varrho)\\ +(A^{\frac38}(\varrho)E^{\frac34}(v,\varrho)C^{\frac14}(v,\varrho))^{\frac23}\Big].\end{aligned}\tag{3.9}$$

Now, assuming $0 < r < \varrho/4$, we can derive from (3.5) and from (3.9) the estimate

$$
\begin{aligned}
D_2(r) &\leqslant \frac{c}{r^{\frac{13}{8}}}\Big[\int\limits_{-r^2}^{0}\Big(\int\limits_{B^+(r)}|\nabla q^1|^{\frac{12}{11}}dx\Big)^{\frac{11}{8}}dt+\int\limits_{-r^2}^{0}\Big(\int\limits_{B^+(r)}|\nabla q^2|^{\frac{12}{11}}dx\Big)^{\frac{11}{8}}dt\Big]\\
&\leqslant \frac{c}{r^{\frac{13}{8}}}\int\limits_{-r^2}^{0}\Big(\int\limits_{B^+(r)}|\nabla q^1|^{\frac{12}{11}}dx\Big)^{\frac{11}{8}}dt+cr^2\int\limits_{-r^2}^{0}\Big(\int\limits_{B^+(r)}|\nabla q^1|^{9}dx\Big)^{\frac{1}{6}}dt\\
&\leqslant c\Big(\frac{\varrho}{r}\Big)^{\frac{13}{8}}A^{\frac{3}{8}}(v,\varrho)E^{\frac{3}{4}}(v,\varrho)C^{\frac{1}{4}}(v,\varrho)+c\Big(\frac{r}{\varrho}\Big)^{2}\Big[E^{\frac{3}{4}}(v,\varrho)+D_2(q,\varrho)\\
&\qquad +A^{\frac{3}{8}}(v,\varrho)E^{\frac{3}{4}}(v,\varrho)C^{\frac{1}{4}}(v,\varrho)\Big]
\end{aligned}
$$

and thus

$$
D_2(q,r)\leqslant c\Big(\frac{r}{\varrho}\Big)^{2}\Big[E^{\frac{3}{4}}(v,\varrho)+D_2(q,\varrho)\Big]+c\Big(\frac{\varrho}{r}\Big)^{\frac{13}{8}}A^{\frac{3}{8}}(v,\varrho)E^{\frac{3}{4}}(v,\varrho)C^{1}4(v,\varrho)
$$

for $0 < r < \varrho/4$. The latter implies estimate (2.5).

§4. Proof of Proposition 1.2

Proof. We let $g = g(v)$ and $G = G_1(v,q)$.

Let us assume that $g < \infty$. Our aim is to show that $G < \infty$. There are three cases:

Case 1. Suppose that

$$
C_0 := \sup_{0<R<1} C(v,R) < \infty. \tag{4.1}
$$

Then, from (2.4), one can deduce that

$$
A(v,R/2)+E(v,R/2)\leqslant c_1(1+D_0^{\frac{2}{3}}(q,R)).
$$

Here and in what follows in this case, c_1 is a generic constant depending on C_0 only.

Now, let us use (2.3), (2.5) with $\varrho = R/2$, and the above estimate. As a result, we find

$$
\begin{aligned}
D_2(q,r) &\leqslant c\Big(\frac{r}{R}\Big)^{2}D_2(q,R/2)+c_1\Big(\frac{R}{r}\Big)^{\frac{13}{8}}[E^{\frac{3}{4}}(v,R/2)+1+D_2^{\frac{3}{4}}(q,R)]\\
&\leqslant c\Big(\frac{r}{R}\Big)^{2}D_2(q,R)+c_1\Big(\frac{R}{r}\Big)^{\frac{13}{8}}[1+D_2(q,R)^{\frac{2}{3}}]
\end{aligned}
$$

for all $0 < r < R/2$. So, by Young's inequality,

$$D_2(q,r) \leqslant c\Big(\frac{r}{R}\Big)^2 D_2(q,R) + c_1\Big(\frac{R}{r}\Big)^{\frac{71}{8}} \tag{4.2}$$

for all $0 < r < R/2$. If $R/2 \leqslant r \leqslant R$, then

$$D_2(q,r) \leqslant \frac{1}{(R/2)^{\frac{13}{8}}} \int\limits_{-R^2}^{0} \Big(\int\limits_{B^+(R)} |\nabla q|^{\frac{12}{11}} dx\Big)^{\frac{11}{8}} dt \leqslant 2^{\frac{13}{8}} D_2(q,R)\Big(\frac{2r}{R}\Big)^2.$$

So, estimate (4.2) holds for all $0 < r < R < 1$.

Now, for μ and R in $]0,1[$, we let $r = \mu R$ in (4.2) and find

$$D_2(q,\mu R) \leqslant c\mu^2 D_2(q,R) + c_1 \mu^{-\frac{71}{8}}.$$

Picking μ up so small that $2c\mu \leqslant 1$, we show that

$$D_2(q,\mu R) \leqslant \mu D_2(q,R) + c_1$$

for any $0 < R < 1$. One can iterate the last inequality and get the following:

$$D_2(q,\mu^{k+1}R) \leqslant \mu^{k+1} D_2(q,R) + c_1(1+\mu+...+\mu^k)$$

for all natural numbers k. The latter implies that

$$D_2(q,r) \leqslant c_1 \frac{r}{R} D_2(q,R) + c_1 \tag{4.3}$$

for all $0 < r < R < 1$. And we can deduce from (2.3) and from the above estimate that

$$\max\{\sup_{0<R<1} D_0(q,R), \sup_{0<R<1} D_2(q,\tau R)\} < \infty$$

for any $0 < \tau < 1$. Uniform boundedness of $A(R)$ and $E(R)$ follows from the energy estimate (2.4) and from the assumption (4.1).

Case 2. Assume now that

$$A_0 := \sup_{0<R<1} A(v,R) < \infty. \tag{4.4}$$

Then, from (2.2), it follows that

$$C(v,r) \leqslant c A_0^{\frac{3}{4}} E^{\frac{3}{4}}(v,r)$$

for any $0 < r < 1$ and thus

$$A(v,\tau\varrho) + E(v,\tau\varrho) \leqslant c_3(A_0,\tau)\Big[E^{\frac{1}{2}}(v,\varrho) + E^{\frac{1}{4}}(v,\varrho) D_0^{\frac{2}{3}}(q,\varrho) + E^{\frac{3}{4}}(v,\varrho)\Big].$$

for any $0 < \tau < 1$ and $0 < \varrho < 1$.

Our next step is an estimate for the pressure quantity:

$$D_2(q,r) \leqslant c\Big(\frac{r}{\varrho}\Big)^2\Big[D_2(q,\varrho)+E^{\frac34}(v,\varrho)\Big]+c_2\Big(\frac{\varrho}{r}\Big)^{\frac{13}{8}}E^{\frac{15}{16}}(v,\varrho)$$
$$\leqslant c\Big(\frac{r}{\varrho}\Big)^2 D_2(q,\varrho)+c_2\Big(\frac{\varrho}{r}\Big)^{\frac{13}{8}}(E^{\frac{15}{16}}(v,\varrho)+1)$$

for any $0<r<\varrho<1$. Here, a generic constant, depending on A_0 only, is denoted by c_2.

Letting $r=\tau R$ and $\mathcal{E}(r):=A(v,r)+D_2(q,r)$, one can deduce from latter inequalities, see also (2.3), the following estimates:

$$\begin{aligned}\mathcal{E}(\tau\varrho) &\leqslant c\tau^2 D_2(q,\varrho)+c_2\Big(\frac{1}{\tau}\Big)^{\frac{13}{8}}(E^{\frac{15}{16}}(v,\varrho)+1)\\ &\quad + c_3(A_0,\tau)\Big[E^{\frac12}(v,\varrho)+E^{\frac14}(v,\varrho)D_2^{\frac23}(\varrho)+E^{\frac34}(v,\varrho)\Big]\\ &\leqslant c\tau^2 D_2(q,\varrho)+c_2\Big(\frac{1}{\tau}\Big)^{\frac{13}{8}}(E^{\frac{15}{16}}(v,\varrho)+1)\\ &\quad + c_3(A_0,\tau)\Big(\frac{1}{\tau}\Big)^4E^{\frac34}(v,\varrho)+c_3(A_0,\tau)\Big[E^{\frac12}(v,\varrho)+E^{\frac34}(v,\varrho)\Big]\\ &\leqslant c\tau^2\mathcal{E}(\varrho)+c_3(A_0,\tau).\end{aligned}$$

The rest of the proof is similar to what has been done in Case 1, see derivation of (4.3).

CASE 3. Assume now that

$$E_0:=\sup_{0<R<1}E(v,R)<\infty. \tag{4.5}$$

Indeed,

$$C(v,r)\leqslant cE_0^{\frac34}A^{\frac34}(v,r)$$

for all $0<r\leqslant 1$. As to the pressure, we can find

$$D_2(\tau\varrho)\leqslant c\tau^2D_2(\varrho)+c_4(E_0,\tau)A^{\frac{9}{16}}(\varrho)$$

for any $0<\tau<1$ and for any $0<\varrho<1$. In turn, the energy inequality gives:

$$\begin{aligned}A(v,\tau\varrho)&\leqslant c_5(E_0,\tau)\Big[A^{\frac12}(v,\varrho)+A^{\frac14}(v,\varrho)D_0^{\frac23}(q,\varrho)+A^{\frac34}(v,\varrho)\Big]\\ &\leqslant c_5(E_0,\tau)\Big[A^{\frac12}(v,\varrho)+A^{\frac14}(v,\varrho)D_2^{\frac23}(q,\varrho)+A^{\frac34}(v,\varrho)\Big]\end{aligned}$$

for any $0 < \tau < 1$ and for any $0 < \varrho < 1$. Similar to Case 2, one can introduce the quantity $\mathcal{E}(r) = A(v, r) + D_2(q, r)$ and find the following inequality for it:

$$\begin{aligned}\mathcal{E}(\tau\varrho) &\leqslant c\tau^2 D_2(q,\varrho) + c_4(E_0,\tau)A^{\frac{9}{16}}(v,\varrho)\\ &\quad + c_5(E_0,\tau)\Big[A^{\frac{1}{2}}(v,\varrho) + A^{\frac{1}{4}}(v,\varrho)D_2^{\frac{2}{3}}(q,\varrho) + A^{\frac{3}{4}}(v,\varrho)\Big]\\ &\leqslant c\tau^2\mathcal{E}(\varrho) + c_5(E_0,\tau)\end{aligned}$$

for any $0 < \tau < 1$ and for any $0 < \varrho < 1$. The rest of the proof is the same as in Case 2.

□

§5. Proof of Theorem 1.4

Assume that v and q is a suitable weak solution in Q^+ with Type I blow up at the origin so that

$$g = g(v) = \min\{\sup_{0<R<1} A(v,R), \sup_{0<R<1} E(v,R) \sup_{0<R<1} C(v,R)\} < \infty. \tag{5.1}$$

By Theorem 1.2,

$$\begin{aligned}G_1 = G_1(v,q) := \max\{&\sup_{0<R<1} A(v,R), \sup_{0<R<1} E(v,R),\\ &\sup_{0<R<1} C(v,R), \sup_{0<R<1} D_0(v,R)\} < \infty.\end{aligned} \tag{5.2}$$

We know, see Theorem 2.2 in [6], that there exists a positive number $\varepsilon_1 = \varepsilon_1(G_1)$ such that

$$\inf_{0<R<1} C(v,R) \geqslant \varepsilon_1 > 0. \tag{5.3}$$

Otherwise, the origin $z = 0$ is a regular point of v.

Let $R_k \to 0$ and $a > 0$ and let

$$u^{(k)}(y,s) = R_k v(x,t), \qquad p^{(k)}(y,s) = R_k^2 q(x,t),$$

where $x = R_k y$, $t = R_k^2 s$. Then, we have

$$A(v,aR_k) = A(u^{(k)},a) \leqslant G_1, \qquad E(v,aR_k) = E(u^{(k)},a) \leqslant G_1,$$

$$C(v,aR_k) = C(u^{(k)},a) \leqslant G_1, \qquad D_0(q,u^{(k)}) = D_0(p^{(k)},a) \leqslant G_1.$$

Thus, by (2.6),

$$\|\partial_t u^{(k)}\|_{\frac{12}{11},\frac{3}{2},Q^+(a)} + \|\nabla^2 u^{(k)}\|_{\frac{12}{11},\frac{3}{2},Q^+(a)} + \|\nabla p^{(k)}\|_{\frac{12}{11},\frac{3}{2},Q^+(a)} \leqslant c(a,G_1)$$

Moreover, the well known multiplicative inequality implies the following bound:

$$\sup_k \int\limits_{Q^+} |u^{(k)}|^{\frac{10}{3}} dz \leqslant c(a, G_1).$$

Using known arguments, one can select a subsequence (still denoted in the same way as the whole sequence) such that, for any $a > 0$,

$$u^{(k)} \to u$$

in $L_3(Q^+(a))$,

$$\nabla u^{(k)} \rightharpoonup \nabla u$$

in $L_2(Q^+(a))$,

$$p^{(k)} \rightharpoonup p$$

in $L_{\frac{3}{2}}(Q^+(a))$. The first two statements are well known and we shall comment on the last one only.

Without loss of generality, we may assume that

$$\nabla p^{(k)} \rightharpoonup w$$

in $L_{\frac{12}{11}}(Q^+(a))$ for all positive a.

We let $p_1^{(k)}(x,t) = p^{(k)}(x,t) - [p^{(k)}]_{B^+(1)}(t)$. Then, there exists a subsequence $\{k_j^1\}_{j=1}^\infty$ such that

$$p_1^{(k_j^1)} \rightharpoonup p_1$$

in $L_{\frac{3}{2}}(Q^+(1))$ as $j \to \infty$. Indeed, it follows from Poincaré-Sobolev inequality

$$\|p_1^{(k_j^1)}\|_{\frac{3}{2}, Q^+(1)} \leqslant c \|\nabla p^{(k_j^1)}\|_{\frac{12}{11}, \frac{3}{2}, Q^+(1)} \leqslant c(1, G_1).$$

Moreover, one has $\nabla p_1 = w$ in $Q^+(1)$.

Our next step is to define $p_2^{(k_j^1)}(x,t) = p^{(k_j^1)}(x,t) - [p^{(k_j^1)}]_{B^+(2)}(t)$. For the same reason as above, there is a subsequence $\{k_j^2\}_{j=1}^\infty$ of the sequence $\{k_j^1\}_{j=1}^\infty$ such that

$$p_2^{(k_j^2)} \rightharpoonup p_2$$

in $L_{\frac{3}{2}}(Q^+(2))$ as $j \to \infty$. Moreover, we claim that $\nabla p_2 = w$ in $Q^+(2)$ and

$$p_2(x,t) - p_1(x,t) = [p_2]_{B^+(1)}(t) - [p_1]_{B^+(1)}(t) = [p_2]_{B^+(1)}(t)$$

for $x \in B^+(1)$ and $-1 < t < 0$, i.e., in $Q^+(1)$.

After s steps, we arrive at the following: there exists a subsequence $\{k^s_j\}_{j=1}^\infty$ of the sequence $\{k^{s-1}_j\}_{j=1}^\infty$ such that $p_s^{(k^s_j)}(x,t) = p^{(k^s_j)}(x,t) - [p^{(k^s_j)}]_{B^+(s)}(t)$ in $Q^+(s)$ and

$$p_s^{(k^s_j)} \rightharpoonup p_s$$

in $L_{\frac{3}{2}}(Q^+(s))$ as $j\to\infty$. Moreover, $\nabla p_s = w$ in $Q^+(s)$ and

$$p_s(x,t) = p_{s-1}(x,t) + [p_s]_{B^+(s-1)}(t)$$

in $Q^+(s-1)$. And so on.

The following function p is going to be well defined: $p = p_1$ in $Q^+(1)$ and

$$p(x,t) = p_{s+1}(x,t) - \sum_{m=1}^{s} [p_{m+1}]_{B^+(m)}(t)\chi_{]-m^2,0[}(t)$$

in $Q^+(s+1)$, where $\chi_\omega(t)$ is the indicator function of the set $\omega\in\mathbb{R}$. Indeed, to this end, we need to verify that

$$\begin{aligned} p_{s+1}(x,t) - \sum_{m=1}^{s} [p_{m+1}]_{B^+(m)}(t)\chi_{]-m^2,0[}(t) \\ = p_s(x,t) - \sum_{m=1}^{s-1} [p_{m+1}]_{B^+(m)}(t)\chi_{]-m^2,0[}(t) \end{aligned}$$

in $Q^+(s)$. The latter is an easy exercise.

Now, let us fix s and consider the sequence

$$p^{(k^s_j)}(x,t) = p_s^{(k^s_j)}(x,t) - \sum_{m=1}^{s-1} [p_{m+1}^{(k^s_j)}]_{B^+(m)}(t)\chi_{]-m^2,0[}(t)$$

in $Q^+(s)$. Then, since the sequence $\{k^s_j\}_{j=1}^\infty$ is a subsequence of all sequences $\{k^{m+1}_j\}_{j=1}^\infty$ with $m\leqslant s-1$, one can easily check that

$$p^{(k^s_j)} \rightharpoonup p$$

in $L_{\frac{3}{2}}(Q^+(s))$. It remains to apply the diagonal procedure of Cantor.

Having in hands the above convergences, we can conclude that the pair u and p is a local energy ancient solution in Q^+_- and (1.5) and (1.6) hold.

The inverse statement is obvious.

§6. Proof of Theorem 1.7

The proof is similar to the proof of Theorem 1.4. We start with scaling $u^\lambda(y,s)=\lambda u(x,t)$ and $p^\lambda(y,s)=\lambda^2 p(x,t)$ where $x=\lambda y$ and $t=\lambda^2 s$ and $\lambda\to\infty$. We know

$$|u^\lambda(0,y_{3\lambda},0)|=\lambda|u(0,a,0)|=\lambda$$

and so that $y_{3\lambda}\to 0$ as $\lambda\to\infty$.

For any $R>0$, by the invariance with respect to the scaling, we have

$$A(u^\lambda,R)=A(u,\lambda R)\leqslant G(u,p)=:G_0,\qquad E(u^\lambda,R)=E(u,\lambda R)\leqslant G_0,$$

$$C(u^\lambda,R)=C(u,\lambda R)\leqslant G_0,\qquad D_0(p^\lambda,R)=D_0(p,\lambda R)\leqslant G_0.$$

Now, one can apply estimate (1.5) and get the following:

$$\|\partial_t u^\lambda\|_{\frac{12}{11},\frac32,Q^+(R)}+\|\nabla^2 u^\lambda\|_{\frac{12}{11},\frac32,Q^+(R)}+\|\nabla p^\lambda\|_{\frac{12}{11},\frac32,Q^+(aR}\leqslant c(R,G_0).$$

Without loss of generality, we can deduce from the above estimates that, for any $R>0$,

$$u^{(k)}\to v$$

in $L_3(Q^+(R))$,

$$\nabla u^{(k)}\rightharpoonup \nabla v$$

in $L_2(Q^+(R))$,

$$p^{(k)}\rightharpoonup q$$

in $L_{\frac32}(Q^+(R))$. Passing to the limit as $\lambda\to\infty$, we conclude that v and q are a local energy ancient solution in Q^+_- for which $G(v,q)<\infty$.

Now, our goal is to prove that $z=0$ is a singular point of v. We argue ad absurdum. Assume that the origin is a regular point, i.e., there exist numbers $R_0>0$ and $A_0>0$ such that

$$|v(z)|\leqslant A_0$$

for all $z\in Q^+(R_0)$. Hence,

$$C(v,R)=\frac{1}{R^2}\int\limits_{Q^+(R)}|v|^3dz\leqslant cA_0^3R^3 \tag{6.1}$$

for all $0<R\leqslant R_0$. Moreover,

$$C(u^\lambda,R)\to C(v,R) \tag{6.2}$$

as $\lambda\to\infty$. By weak convergence,

$$D_0(q,R)\leqslant G_0$$

for all $R > 0$. Now, we can calculate positive numbers $\varepsilon(G_0)$ and $c(G_0)$ of Theorem 2.2 in [6]. Then, let us fix $0 < R_1 < R_0$, see (6.1), so that $C(v, R_1) < \varepsilon(G_0)/2$. According to (6.2), one can find a number $\lambda_0 > 0$ such that

$$G(u^\lambda, R_1) < \varepsilon(G_0)$$

for all $\lambda > \lambda_0$. By Theorem 2.2 of [6],

$$\sup_{z\in Q^+(R_1/2)} |u^\lambda(z)| < \frac{c(G_0)}{R_1}$$

for all $\lambda > \lambda_0$. It remains to select $\lambda_1 > \lambda_0$ such that $y_{3\lambda} = a/\lambda < R_1/2$ and $\lambda_1 > c(G_0)/R_1$. Then

$$|u^{\lambda_1}(0, y_{3\lambda_1}, 0)| = \lambda_1 \leqslant \sup_{z\in Q^+(R_1/2)} |u^{\lambda_1}(z)| < \frac{c(G_0)}{R_1}.$$

This is a contradiction.

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Поступило 29 октября 2019 г.