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ESTIMATES OF THE DISTANCE TO THE SOLUTION OF AN EVOLUTIONARY PROBLEM OBTAINED BY LINEARIZATION OF THE NAVIER–STOKES EQUATION

Abstract. The paper is concerned with a linearization of the Navier–Stokes equation in the space-time cylinder Q_T. The main goal is to deduce computable estimates of the distance between the exact solution and a function in the energy admissible class of vector valued functions. First, the estimates are derived for the case, where this class contains only divergence free (solenoidal) functions. In the next section, estimates of the distance to sets of divergence free functions depending on the space and time variables are considered. These results are used to extend earlier derived estimates to non–solenoidal approximations. The corresponding estimates contain an additional term, which can be viewed as a penalty for the violation of the divergence free condition.

§1. Introduction

We consider a linearization of the classical Navier–Stokes equations in a bounded Lipschitz domain Ω, which is to find a vector valued function $u(x,t)$ (velocity) and a scalar valued function $p(x,t)$ (pressure) such that

$$u_t - \mathrm{Div}\sigma + (\mathrm{a}\cdot\nabla)u = f \quad \text{in } Q_T := \Omega\times(0,T),\ \ T>0, \tag{1.1}$$

$$\mathrm{div} u = 0 \quad \text{in } Q_T, \tag{1.2}$$

$$\sigma = \nu\varepsilon(u) - p\,\mathbb{I} \quad \text{in } Q_T, \tag{1.3}$$

$$u(x,0) = \phi(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } S_T := \partial\Omega\times(0,T). \tag{1.4}$$

Here $\phi \in L_2(\Omega,\mathbb{R}^d)$ is a given divergence free vector valued function, $x\in\Omega$ is the Cartesian coordinate, ν is a positive constant (viscosity), $f\in L_2(Q_T,\mathbb{R}^d)$, div and Div denote the spatial divergence operators for the vector and tensor valued functions, respectively, ∇ denotes the spatial gradient operator, and $\varepsilon(u)$ is the symmetric part of ∇u.

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Below we use standard notation for the Bochner spaces, namely, for a separable Banach space X endowed with the norm $\|\cdot\|_X$, $L_2(0,T;X)$ denotes the space of functions with the norm

$$\|v\|^2_{L_2(0,T;X)} := \int\limits_0^T \|\nabla v\|^2_X \, dt < \infty.$$

For the scalar and vector valued functions in Ω, we use Lebesgue and Sobolev spaces $L_p(\Omega)$ and $W^l_p(\Omega)$ (where $l,p \geqslant 1$) and mark them above by $\circ$ if the respective functions vanish on S_T. L_2 norms of the functions in Ω and Q_T are denoted by $\|\cdot\|_\Omega$ and $\|\cdot\|_{Q_T}$, respectively. In what follows, we use the spaces

$$W^{1,0}_2(Q_T,\mathbb{R}^d) := L_2(0,T,W^1_2(\Omega,\mathbb{R}^d))$$

and

$$\overset{\circ}{W}{}^{1,0}_2(Q_T,\mathbb{R}^d) := L_2(0,T,\overset{\circ}{W}{}^{1}_2(\Omega,\mathbb{R}^d))$$

supplied with the norm

$$\|w\|_{1,0,Q_T} := \left(\int\limits_0^T (\|\nabla w\|^2_\Omega + \|w\|^2_\Omega)\, dt\right)^{1/2}.$$

For the functions in $\overset{\circ}{W}{}^{1,0}_2(Q_T,\mathbb{R}^d)$, we also use the norm

$$|\!|\!| w |\!|\!|_{1,0,Q_T} := \|\nabla w\|_{Q_T},$$

which is equivalent to $\|w\|_{1,0,Q_T}$. Next, let

$$\overset{\circ}{W}{}^{1,1}_2(Q_T,\mathbb{R}^d) := \{w \in W^1_2(Q_T),\ w = 0 \text{ on } S_T\}$$

and

$$|\!|\!| w |\!|\!|_{1,1,Q_T} := \left(\|\nabla w\|^2_{Q_T} + \|w_t\|^2_{Q_T}\right)^{1/2}.$$

The subspaces of divergence free functions in $\overset{\circ}{W}{}^1_2(\Omega)$, $\overset{\circ}{W}{}^{1,0}_2(Q_T)$, and $\overset{\circ}{W}{}^{1,1}_2(Q_T)$ are denoted by $\overset{\circ}{\mathrm{S}}{}^1(\Omega)$, $\overset{\circ}{\mathrm{S}}{}^{1,0}(Q_T)$, and $\overset{\circ}{\mathrm{S}}{}^{1,1}(Q_T)$, respectively.

For a bounded Lipschitz domain ω, $C_F(\omega)$ denotes a constant in the Friedrichs inequality $\|w\|_\omega \leqslant C_F(\omega)\|\nabla w\|_\omega$ that holds for the functions in $H^1(\omega)$ vanishing on $\partial\omega$ and $C_P(\omega)$ denotes a constant in the Poincare inequality for the functions having zero mean in ω.

Mean value of a function g in the set ω is denoted by $\{\!|g|\!\}_\omega$ and tilde is used to mark spaces containing functions having zero mean values with respect to spatial variables, i.e., $\widetilde{L}_2(0;T;X)$ is the subspace of functions in $L_2(0;T;X)$ with zero mean values on Ω for a.e. $t \in (0,T)$.

Henceforth, it is assumed that the vector valued function a in (1.1) is divergence free and sufficiently regular (e.g., $\mathrm{a} \in L^\infty(\Omega)$) and that the problem is uniquely solvable with the generalized solution $u \in \overset{\circ}{\mathrm{S}}{}^{1,1}(Q_T, \mathbb{R}^d))$ satisfying the integral relation (see, e.g., [3, 4, 12, 13])

$$\int\limits_{Q_T} (\nu\varepsilon(u) : \varepsilon(w) + (\mathrm{a}\cdot\nabla)u \cdot w) dxdt + \int\limits_{Q_T} u_t \cdot w dxdt = \int\limits_{Q_T} f \cdot w dxdt \quad \forall w \in \overset{\circ}{\mathrm{S}}{}^{1,0}(Q_T). \tag{1.5}$$

Instead of (1.5), we can also use the identity

$$\int\limits_{Q_T} (\nu\varepsilon(u) : \varepsilon(w) + (\mathrm{a}\cdot\nabla)u \cdot w)\, dxdt + [\, u\cdot w\,]_0^T - \int\limits_{Q_T} u \cdot w_t\, dxdt = \int\limits_{Q_T} f \cdot w\, dxdt, \tag{1.6}$$

where

$$[\,\zeta\,]_0^T := \int\limits_{\Omega} (\zeta(x,T) - \zeta(x,0)) dx$$

and the test functions belong to $\overset{\circ}{\mathrm{S}}{}^{1,1}(Q_T)$ (notice that for these functions we can define their traces on the faces of Q_T related to $t = 0$ and $t = T$).

Our goal is to deduce estimates, able to verify that a given function $v(x,t) \in \overset{\circ}{W}{}_2^{1,1}(Q_T, \mathbb{R}^d))$ belongs to a certain neighbourhood of u. For this purpose, we introduce a suitable measure of the distance (natural for the problem in question) and deduce majorants of the difference $e(x,t) := u(x,t) - v(x,t)$ expressed in terms of v (numerical approximation), problem data, and other quantities that are defined once a numerical approximation has been constructed. If the value of such a majorant is computed, then we obtain a guaranteed bound for the radius of the ball centred at the approximate solution that contains the exact one.

§2. The error identity and a computable error majorant

First we assume that a function v compared with u belongs to the space $\overset{\circ}{\mathrm{S}}{}^{1,1}(Q_T)$. In this case, from (1.5) it follows that

$$\int\limits_{Q_T} (\nu\varepsilon(e) : \varepsilon(w) + (\mathrm{a}\cdot\nabla)e\cdot w)\,dxdt + \int\limits_{Q_T} e_t \cdot w\,dxdt = \mathcal{L}_v(w), \qquad w \in \overset{\circ}{\mathrm{S}}{}^{1,0}(Q_T), \tag{2.1}$$

where

$$\mathcal{L}_v(w) := \int\limits_{Q_T} (f\cdot w - \nu\varepsilon(v) : \varepsilon(w) - (\mathrm{a}\cdot\nabla)v\cdot w - v_t\cdot w)\; dx\,dt$$

is a linear functional defined on $w \in \overset{\circ}{\mathrm{S}}{}^{1,0}(Q_T)$. The quantity

$$[\![\mathcal{L}_v]\!] := \sup_{w\in \overset{\circ}{\mathrm{S}}{}^{1,0}(Q_T)} \frac{|\mathcal{L}_v(w)|}{\|\nabla w\|_{Q_T}} \tag{2.2}$$

defines a norm of this functional. The functional $\mathcal{L}_v$ (residual functional) does not contain the exact solution u. It is computable for any test function w.

Consider the left hand side of (2.1). It is another linear functional $\mathcal{M}_e$, which is defined for any $w \in \overset{\circ}{\mathrm{S}}{}^{1,0}(Q_T)$ and depends on the error $e := u - v$. The corresponding norm

$$\mathbf{m}(e) := \sup_{w\in \overset{\circ}{\mathrm{S}}{}^{1,0}} \frac{|\mathcal{M}_e(w)|}{\|\nabla w\|_{Q_T}}$$

generates an error measure. Using in the above definition the norm $\|\varepsilon(w)\|_{Q_T}$ instead of $\|\nabla w\|_{Q_T}$ yields an equivalent measure, for which the estimates are quite analogous and differ only by the Korn's constant.

From (2.1) and (2.2), it follows the basic error identity [1]

$$\mathbf{m}(e) = [\![\mathcal{L}_v]\!]. \tag{2.3}$$

It is clear that $\mathbf{m}(e) \geqslant 0$ and $\mathbf{m}(e) = 0$ if $v = u$. On the other hand, if $\mathbf{m}(e) = 0$, then (2.3) implies $[\![\mathcal{L}_v]\!] = 0$, so that $\mathcal{L}_v(w) = 0$ for any

[1] Similar error identities arise in many other problems (e.g., see [8–11]). They form the basis for the derivation of computable bounds of the distance between a given function and the exact solution.

$w \in \overset{\circ}{\mathrm{S}}{}^{1,1}(Q_T)$. In view of (1.5), this means that in such a case the function v coincides with u. Thus, $\mathbf{m}(e)$ is a natural characteristic of the error e.

Notice that

$$\mathcal{M}_e(e) = \int\limits_{Q_T} (\nu\varepsilon(e) : \varepsilon(e) + (\mathrm{a}\cdot\nabla)e\cdot e)\,dxdt + \int\limits_{Q_T} e_t\cdot e\,dxdt.$$

Since a is a divergence free function, the second term of $\mathcal{M}_e(e)$ vanishes and the last term is equal to $\frac{1}{2}\left[\,\|e\|^2_\Omega\,\right]_0^T$, so that

$$\mathcal{M}_e(e) = \nu\|\varepsilon(e)\|^2_{Q_T} + \frac{1}{2}\left[\,\|e\|^2_\Omega\,\right]_0^T. \tag{2.4}$$

If $v(x,0)=\phi$, then $\left[\,\|e\|^2_\Omega\,\right]_0^T$ is reduced to $\|e(T)\|^2_\Omega$. Thus,

$$\mathbf{m}(e) \geqslant \frac{\mathcal{M}_e(e)}{\|\nabla e\|_{Q_T}} \geqslant \nu^{1/2}C_K^{-1}(\Omega)\mathcal{M}_e^{1/2}(e), \tag{2.5}$$

where $C_K(\Omega)$ is a constant in the Korn's inequality and we see that $\mathbf{m}(e)$ really controls the distance between v and u.

In view of (2.3), getting an estimate of the error measure requires computation of the norm $[\![\mathcal{L}_v]\!]$. However, in general, the quantity $[\![\mathcal{L}_v]\!]$ is not computable (because it contains supremum over an infinite amount of functions). Below we bypass this principal difficulty and deduce a computable majorants of $[\![\mathcal{L}_v]\!]$. For this purpose, we reform the functional $\mathcal{L}_v$ using a suitable integral identity that follows from an integration by parts relation.

Theorem 1. *Let* $v \in \overset{\circ}{\mathrm{S}}{}^{1,1}(Q_T)$, $q \in L_2(Q_T)$, *and*

$$\tau \in \mathrm{H}_{\mathrm{Div}}(Q_T) := \{\tau \in L_2(Q_T, \mathbb{M}^{n\times n}_{sym}) \mid \mathrm{Div}\,\tau \in L_2(Q_T,\mathbb{R}^d)\}.$$

Then

$$\nu\|\varepsilon(e)\|^2_{Q_T} + \frac{1}{2}\|e(T)\|^2_\Omega \leqslant \frac{1}{2}\|e(0)\|^2_\Omega + \frac{C_K^2(\Omega)}{\nu}\int\limits_0^T (\mathcal{R}_1(t) + \mathcal{R}_2(t))^2 dt, \tag{2.6}$$

where

$$\mathcal{R}_1(t) := \|\tau - \nu\varepsilon(v) + q\mathbb{I}\|_\Omega, \quad \mathcal{R}_2(t) := C_F(\Omega)\|\vartheta\|_\Omega,$$

and $\vartheta := f - v_t + \mathrm{Div}\tau - (\mathrm{a}\cdot\nabla)v$.

Let Ω be decomposed into a collection of disconnected open Lipschitz sets such that $\overline{\Omega} = \cup_{i=1}^N \overline{\Omega}_i$ *and in addition for almost all* $t \in [0,T]$ *it holds*

$$\{\!|\vartheta_j|\!\}_{\Omega_i} = 0, \quad \forall\, j = 1 \div d, \quad i = 1 \div N. \tag{2.7}$$

Then the estimate (2.6) *holds with* $\mathcal{R}_2(t)$ *replaced by*

$$\mathcal{R}_2^N(t) := \left(\sum_{i=1}^{N} C_P^2(\Omega_i)\mathcal{R}_{2i}^2(t)\right)^{1/2},$$

where $\mathcal{R}_{2i}(t) := \|\vartheta\|^2_{\Omega_i}$.

The right hand sides of (2.6) *vanishes if and only if*

$$v = u, \quad \tau = \widehat{\sigma} := \nu\varepsilon(u) - p\mathbb{I}, \quad \text{and} \quad q = p.$$

Proof. Notice that for any $\tau \in \mathrm{H}_{\mathrm{Div}}(Q_T)$, $q \in L_2(Q_T)$ (without a loss of generality we may assume that $\{\!|q|\!\}_\Omega = 0$ for a.a. $t \in [0, T]$), and any $w \in \overset{\circ}{\mathrm{S}}{}^{1,0}(Q_T)$, it holds

$$\int\limits_{Q_T} (\operatorname{Div} \tau \cdot w + (\tau + q\mathbb{I}) : \varepsilon(w))dxdt = 0. \tag{2.8}$$

By means of (2.8) we reform the functional $\mathcal{L}_v$ as follows:

$$\begin{aligned} \mathcal{L}_v(w) &= \int\limits_{Q_T} (\tau - \nu\varepsilon(v) + q\mathbb{I}) : \varepsilon(w)\, dxdt + \int\limits_{Q_T} \vartheta \cdot w\, dxdt \\ &\leqslant \int\limits_0^T \left(\mathcal{R}_1(t)\|\varepsilon(w)\|_\Omega + \|\vartheta\|_\Omega\|w\|_\Omega\right) dt \\ &\leqslant \int\limits_0^T \left(\mathcal{R}_1(t) + C_F\|\vartheta\|_\Omega\right) \|\nabla w\|_\Omega\, dt \\ &\leqslant \Big(\|\mathcal{R}_1(t) + \mathcal{R}_2(t)\|_{(0,T))}\Big)\|\nabla w\|_{Q_T}. \end{aligned} \tag{2.9}$$

Hence

$$|\!|\!| \mathcal{L}_v |\!|\!| \leqslant \|\mathcal{R}_1(t) + \mathcal{R}_2(t)\|_{(0,T)}.$$

In view of (2.3)–(2.5),

$$\nu C_K^{-2}\mathcal{M}_e(e) \leqslant |\!|\!| \mathcal{L}_v |\!|\!|^2,$$

and we arrive at (2.6).

If the right hand side of (2.6) vanishes, then for any $w \in \overset{\circ}{\mathrm{S}}{}^{1,0}(Q_T)$

$$0 = \int\limits_{Q_T} \vartheta \cdot w dxdt = \int\limits_{Q_T} (f \cdot w - \tau : \varepsilon(w) - (\mathrm{a} \cdot \nabla)v \cdot w - v_t \cdot w)dxdt. \tag{2.10}$$

Since $\mathcal{R}_1(t) = 0$, we have $\tau = \nu\varepsilon(v) - q\mathbb{I}$, so that

$$\int\limits_{Q_T} (f \cdot w - \nu\varepsilon(v) : \varepsilon(w) - (\mathrm{a} \cdot \nabla)v \cdot w - v_t \cdot w)dxdt = 0.$$

This identity shows (cf. (1.5)) that v is the exact solution.

Now we reform the second term of (2.9) with the help of (2.7):

$$\int\limits_{\Omega} \vartheta \cdot w dx \leqslant \sum_{i=1}^{N} \sum_{j=1}^{d} \int\limits_{\Omega_i} \vartheta_j \cdot w_j dx \leqslant \sum_{i=1}^{N} C_P(\Omega_i) \sum_{j=1}^{d} \|\vartheta_j\|_{\Omega_i} \|\nabla w_j\|_{\Omega_i}$$
$$\leqslant \sum_{i=1}^{N} C_P(\Omega_i) \|\vartheta\|_{\Omega_i} \|\nabla w\|_{\Omega_i} \leqslant \mathcal{R}_2^N(t) \|\nabla w\|_{\Omega}.$$

Hence

$$[\![\mathcal{L}_e]\!] \leqslant \|\mathcal{R}_1(t) + \mathcal{R}_2^N(t))\|_{(0,T)},$$

and by similar arguments we deduce the majorant with the term $\mathcal{R}_2^N(t)$ instead of $\mathcal{R}_2(t)$. It is clear that if we set $v = u$, $\tau = \sigma$, and $q = p$, then the majorant vanishes. On the other hand, if $\mathcal{R}_1(t) = \mathcal{R}_2^N(t) = 0$, then again (2.10) holds together with the constitutive relation, and we conclude that $u = v$. □

Remark 1. The right hand side of (2.6) contains the tensor valued function τ (which can be viewed as an approximation of the exact stress $\sigma = \nu\varepsilon(u) - q\mathbb{I}$) and the scalar valued function q (which is an approximation of the exact pressure p). It is defined by integral type norms which easily computable. If the function τ is balanced and the integral relations (2.7) hold, then the term $\mathcal{R}_2(t)$ can be replaced by $\mathcal{R}_2^N(t)$ containing constants in the Poincare inequalities for subdomains. These constants are proportional to the diameter of Ω_i (moreover for convex domains there exists a simple estimate for such a constant, see [6]) and usually are essentially smaller than the global constant $C_F(\Omega)$. Therefore, such a form is advantageous for computations.

Also, it is worth noting that the term $\mathcal{R}_2(t)$ can be represented in a somewhat different form $\mathcal{R}_2(t) := C_F(\Omega)\|\mathrm{Div}\widehat{\tau} - v_t + f\|_{\Omega}$, where $\widehat{\tau} = \tau - \mathrm{a} \otimes v$ and $\otimes$ denotes the tensor product of vectors.

Remark 2. If the flow is stationary, then all the functions do not depend on t so that $u(x,t) = u(x,T) = u(x,0)$ and $\mathcal{R}_1$ and $\mathcal{R}_2$ do not depend on t. We formally integrate in time and arrive at the estimate

$$\|\varepsilon(e)\|_\Omega \leqslant \frac{C_K}{\nu}\left(\|\tau - \nu\varepsilon(v) + q\mathbb{I}\|_\Omega + C_F(\Omega)\|\mathrm{Div}\tau - (\mathrm{a}\cdot\nabla)v + f\|_\Omega\right).$$

§3. Distance to the sets $\overset{\circ}{\mathrm{S}}{}^{1,0}$ and $\overset{\circ}{\mathrm{S}}{}^{1,1}$

Very often numerical approximations of problems related to incompressible viscous fluids do not exactly satisfy the divergence free condition. Therefore, our next goal is to extend (2.6) to a wider class of functions, which may not satisfy this condition. From now on, we assume that a function $\widehat{v}$ compared with u belongs to $\overset{\circ}{W}{}_2^{1,1}(Q_T, \mathbb{R}^d)$ (notice that approximations constructed by the majority of methods possess such a regularity). However, the function $\widehat{v}$ may not satisfy the condition $\mathrm{div}\widehat{v} = 0$.

We need easily computable majorants for the quantities

$$\mathrm{d}(\widehat{v}, \overset{\circ}{\mathrm{S}}{}^{1,0}(Q_T, \mathbb{R}^d)) := \inf_{v_\circ \in \overset{\circ}{\mathrm{S}}{}^{1,0}(Q_T,\mathbb{R}^d)} \left|\!\left| \widehat{v} - v_\circ \right|\!\right|_{1,0,Q_T},$$

$$\mathrm{d}(\widehat{v}, \overset{\circ}{\mathrm{S}}{}^{1,1}(Q_T, \mathbb{R}^d)) := \inf_{v_\circ \in \overset{\circ}{\mathrm{S}}{}^{1,1}(Q_T,\mathbb{R}^d)} \left|\!\left| \widehat{v} - v_\circ \right|\!\right|_{1,1,Q_T},$$

which define distances between the function $\widehat{v}$ and the spaces $\overset{\circ}{\mathrm{S}}{}^{1,0}(Q_T)$ and $\overset{\circ}{\mathrm{S}}{}^{1,1}(Q_T)$, respectively. Also, we estimate the distance generated by the special norm

$$|\!|\!| \, w \, |\!|\!|_{Q_T} := \left(\nu\|\nabla w\|^2_{Q_T} + \frac{1}{2}\|w(T)\|^2_\Omega\right)^{1/2},$$

i.e., the quantity

$$\widetilde{\mathrm{d}}(\widehat{v}, \overset{\circ}{\mathrm{S}}{}^{1,1}(Q_T, \mathbb{R}^d)) := \inf_{v_\circ \in \overset{\circ}{\mathrm{S}}{}^{1,1}(Q_T,\mathbb{R}^d)} |\!|\!| \, \widehat{v} - v_\circ \, |\!|\!|_{Q_T} .$$

The corresponding estimates follow from properties of the orthogonal projector $\mathbb{P}_\Omega$ that maps $\overset{\circ}{W}{}^1_2(\Omega, \mathbb{R}^d)$ to the subspace $\overset{\circ}{\mathrm{S}}{}^1(\Omega)$ containing the divergence free fields. We use the following well known result (e.g., see [7, 8]): for any $\widehat{w} \in \overset{\circ}{W}{}^1_2(\Omega)$ the projector satisfies the estimate

$$\|\nabla(\widehat{w} - \mathbb{P}_\Omega\widehat{w})\|_\Omega \leqslant \mathbb{C}_\Omega\|\mathrm{div}\widehat{w}\|_\Omega, \tag{3.1}$$

where $\mathbb{C}_\Omega$ is a constant in the inf–sup (LBB) condition (see [1, 2, 4, 5]).

In fact, this projection estimate is a form of the well known stability lemma established in the above cited publications, which says that for any $f \in L_2(\Omega)$ with zero mean, there exists a vector field $v_f \in \overset{\circ}{W}{}^1_2(\Omega, \mathbb{R}^d)$ such that

$$\mathrm{div} v_f = f \quad \text{and} \quad \|\nabla v_f\|_\Omega \leqslant \mathbb{C}_\Omega \|f\|_\Omega.$$

Moreover, the function $w_\circ = \mathbb{P}_\Omega \widehat{w} \in \overset{\circ}{\mathrm{S}}{}^1(\Omega)$ defined by the orthogonal projection operator $\mathbb{P}_\Omega$ exists and is unique. Below we use this fact to deduce the desired estimates.

Lemma 1. *For any* $\widehat{v} \in \overset{\circ}{W}{}^{1,1}_2(Q_T, \mathbb{R}^d)$

$$\mathrm{d}(\widehat{v}, \overset{\circ}{\mathrm{S}}{}^{1,0}(Q_T, \mathbb{R}^d)) \leqslant \mathbb{C}_\Omega \|\mathrm{div}\widehat{v}\|_{Q_T}, \tag{3.2}$$

If $\widehat{v}$ *is more regular, i.e.,*

$$\widehat{v} \in \overset{\circ}{V}{}^{1,1}_2(Q_T, \mathbb{R}^d) := \{\widehat{v} \in \overset{\circ}{W}{}^{1,1}_2(Q_T, \mathbb{R}^d) \mid \mathrm{div}\widehat{v}_t \in L_2(Q_T)\},$$

then

$$\mathrm{d}(\widehat{v}, \overset{\circ}{\mathrm{S}}{}^{1,1}(Q_T, \mathbb{R}^d)) \leqslant \mathbb{C}_\Omega \left(\|\mathrm{div}\widehat{v}\|^2_{Q_T} + C^2_F(\Omega)\|\mathrm{div}\widehat{v}_t\|^2_{Q_T}\right)^{1/2}, \tag{3.3}$$

$$\widetilde{\mathrm{d}}(\widehat{v}, \overset{\circ}{\mathrm{S}}{}^{1,1}(Q_T, \mathbb{R}^d)) \leqslant \mathbb{C}_\Omega \left(\nu\|\mathrm{div}\widehat{v}\|^2_{Q_T} + \frac{1}{2}C^2_F(\Omega)\|\mathrm{div}\widehat{v}(T)\|^2_\Omega\right)^{1/2}. \tag{3.4}$$

Proof. First, we consider $\widehat{v}$ of a special class. Functions in this class can be called incremental approximations because they are associated with a finite amount of time intervals (t_k, t_{k+1}), $t_{k+1} > t_k$, $t_0 = 0$, and $t_m = T$. Let the function $\widehat{v}$ be defined by the relation

$$\widehat{v}(x,t) = \lambda(t)\widehat{v}_k(x) + (1-\lambda(t))\widehat{v}_{k+1} \quad \text{for} \quad t \in [t_k, t_{k+1}], \tag{3.5}$$

where $\widehat{v}_k \in \overset{\circ}{W}{}^1_2(\Omega)$ are some functions depending on the spatial variables only and

$$\lambda(t) := \frac{t_{k+1} - t}{\delta_k}, \quad \delta_k = t_{k+1} - t_k.$$

Functions of this class form the set $V_0^{(m)}(Q_T, \mathbb{R}^d)$. We do not assume that the functions $\widehat{v}_k$ are divergence free, so that $V_0^{(m)}(Q_T, \mathbb{R}^d) \subset \overset{\circ}{W}{}^{1,1}_2(Q_T, \mathbb{R}^d)$, but in general a function $\widehat{v} \in V_0^{(m)}(Q_T, \mathbb{R}^d)$ does not satisfy the condition $\mathrm{div}\widehat{v} = 0$.

Let $v_{\circ,k} = \mathbb{P}_\Omega \widehat{v}_k$, so that (see (3.1))

$$\|\nabla(\widehat{v}_k - v_{\circ,k})\|_\Omega \leqslant \mathbb{C}_\Omega \|\mathrm{div}\widehat{v}_k\|_\Omega \qquad k = 0, 1, 2, \ldots, m. \tag{3.6}$$

Then the function $v_\circ$ defined by

$$v_\circ(x,t) = \lambda(t)v_{\circ,k}(x) + (1-\lambda(t))v_{\circ,k+1}(x), \quad t \in [t_k, t_{k+1}] \tag{3.7}$$

belongs to $\overset{\circ}{\mathrm{S}}{}^{1,1}(Q_T,\mathbb{R}^d))$ and the relation $v_\circ = \mathbb{P}_{Q_T}\widehat{v}$ defines the operator

$$\mathbb{P}_{Q_T} : \overset{\circ}{W}{}^{1,1}_2(Q_T,\mathbb{R}^d) \;\to\; \overset{\circ}{\mathrm{S}}{}^{1,1}(Q_T,\mathbb{R}^d).$$

Notice that $\mathbb{P}_\Omega$ is a linear mapping, so that $v_\circ(x,t)$ defined by (3.7) satisfies

$$\|\nabla(\widehat{v} - v_\circ)\|_\Omega(t) \leqslant \mathbb{C}_\Omega \|\mathrm{div}\widehat{v}\|_\Omega(t) \qquad \forall\, t \in [t_k, t_{k+1}]. \tag{3.8}$$

We have

$$[\![\widehat{v} - v_\circ]\!]^2_{1,0,Q_T} = \sum_{k=0}^{m} \int_{t_k}^{t_{k+1}} \|\nabla(\widehat{v} - v_\circ)\|^2_\Omega dt$$

$$\leqslant \mathbb{C}^2_\Omega \sum_{k=0}^{m} \int_{t_k}^{t_{k+1}} \|\mathrm{div}\widehat{v}\|^2_\Omega dt = \mathbb{C}^2_\Omega \|\mathrm{div}\widehat{v}\|^2_{Q_T}.$$

Hence for any $\widehat{v} \in V_0^{(m)}(Q_T,\mathbb{R}^d)$

$$\mathrm{d}(\widehat{v}, \overset{\circ}{\mathrm{S}}{}^{1,0}(Q_T,\mathbb{R}^d)) \leqslant \mathbb{C}_\Omega \|\mathrm{div}\widehat{v}\|_{Q_T}.$$

Let $\widetilde{v}$ be a smooth function vanishing on S_T. For any $\epsilon > 0$, we can find sufficiently large m and the corresponding $\widehat{v} \in V_0^{(m)}(Q_T,\mathbb{R}^d)$ such that $[\![\nabla(\widetilde{v} - \widehat{v})]\!]_{1,0,Q_T} \leqslant \epsilon$. Therefore,

$$[\![\widetilde{v} - v_\circ]\!]_{1,0,Q_T} \leqslant [\![\widehat{v} - v_\circ]\!]_{1,0,Q_T} + \epsilon \leqslant \mathbb{C}_\Omega \|\mathrm{div}\widehat{v}\|_{Q_T} + \epsilon \leqslant \mathbb{C}_\Omega \|\mathrm{div}\widetilde{v}\|_{Q_T} + 2\epsilon,$$

and the distance estimate also holds for $\widetilde{v}$. Since smooth functions are dense in $\overset{\circ}{W}{}^{1,1}_2(Q_T,\mathbb{R}^d)$, we extend the estimate to this class of functions.

Consider the norm $[\![\widehat{v} - v_\circ]\!]_{1,1,Q_T}$, where $\widehat{v} \in \overset{\circ}{V}{}^{1,1}_2(Q_T,\mathbb{R}^d)$. We have

$$(\widehat{v} - v_\circ)_t = \frac{1}{\delta_k}\left(\widehat{v}_{k+1} - \widehat{v}_k - v_{\circ,k+1} + v_{\circ,k}\right) \quad \text{for} \quad t \in [t_k, t_{k+1}]. \tag{3.9}$$

Notice that

$$\|\widehat{v}_{k+1} - \widehat{v}_k - v_{\circ,k+1} + v_{\circ,k}\|_\Omega \leqslant C_F(\Omega)\|\nabla(\widehat{v}_{k+1} - \widehat{v}_k - v_{\circ,k+1} + v_{\circ,k})\|_\Omega$$

and $\mathbb{P}(\widehat{v}_{k+1} - \widehat{v}_k) = v_{\circ,k+1} - v_{\circ,k}$. We conclude that

$$\|\widehat{v}_{k+1} - \widehat{v}_k - \widehat{v}_{\circ,k+1} + \widehat{v}_{\circ,k}\|_\Omega \;\leqslant\; C_F(\Omega)\mathbb{C}_\Omega \|\mathrm{div}(\widehat{v}_{k+1} - \widehat{v}_k)\|. \tag{3.10}$$

On each interval $[t_k, t_{k+1}]$

$$\mathrm{div}\widehat{v}_{k+1} - \mathrm{div}\widehat{v}_k = \delta_k (\mathrm{div}\widehat{v})_t = \delta_k \mathrm{div}\widehat{v}_t.$$

Using (3.9) and (3.10), we obtain

$$\|(\widehat{v} - v_\circ)_t\|^2_{Q_T} \leqslant C_F^2 \mathbb{C}_\Omega^2 \sum_{k=0}^{m} \int\limits_{t_k}^{t_{k+1}} \|\mathrm{div}\widehat{v}_t\|^2 dt = C_F^2 \mathbb{C}_\Omega^2 \|\mathrm{div}\widehat{v}_t\|^2_{Q_T}. \tag{3.11}$$

Since

$$|\!|\!|\widehat{v} - v_\circ|\!|\!|^2_{1,1,Q_T} = |\!|\!|\widehat{v} - v_\circ|\!|\!|^2_{1,0,Q_T} + \|(v - v_\circ)_t\|^2_{Q_T},$$

we obtain (3.3) for any $\widehat{v} \in\in V_0^{(m)}(Q_T, \mathbb{R}^d)$. By analogous arguments based on density of smooth functions in $\overset{\circ}{V}{}_2^{1,1}(Q_T, \mathbb{R}^d)$, this estimate is extended to this class of functions.

Finally, consider the quantity

$$\|\!|\, \widehat{v} - v_\circ \,|\!\|^2_{Q_T} := \|\nabla(\widehat{v} - v_\circ)\|^2_{Q_T} + \frac{1}{2}\|(\widehat{v} - v_\circ)(T)\|^2_\Omega$$

for $\widehat{\in} \overset{\circ}{W}{}_2^{1,1}(Q_T, \mathbb{R}^d)$. Here the first term is estimated by (3.8) and for the second one we have

$$\begin{aligned}\|(\widehat{v} - v_\circ)(T)\|_\Omega = \|\widehat{v}_m - v_{\circ,m}\|_\Omega \leqslant C_F(\Omega)\|\nabla(\widehat{v}_m - v_{\circ,m})\|_\Omega \\ \leqslant \mathbb{C}_\Omega C_F(\Omega)\|\mathrm{div}\widehat{v}_m\|_\Omega = \mathbb{C}_\Omega C_F(\Omega)\|\mathrm{div}\widehat{v}(T)\|_\Omega.\end{aligned}$$

Using density of smooth fields in $\overset{\circ}{V}{}_2^{1,1}(Q_T, \mathbb{R}^d)$ and properties of their traces on the faces of Q_T related to $t = 0$ and $t = T$, this estimate is extended to the whole class of functions. □

Remark 3. From (3.2), it follows that for any $f(x,t) \in L_2(Q_T)$ such that $f = \mathrm{div}\widehat{v}$ for some vector valued function $\widehat{v} \in \overset{\circ}{W}{}_2^{1,1}(Q_T, \mathbb{R}^d)$, there exists $\widehat{w} \in \overset{\circ}{W}{}_2^{1,1}(Q_T, \mathbb{R}^d)$ satisfying the conditions

$$\begin{aligned}&\mathrm{div}\widehat{w} = f \qquad \text{for a.e. } t \in (0,T),\\ &|\!|\!|\widehat{w}|\!|\!|_{1,0,Q_T} \leqslant \mathbb{C}_\Omega \|f\|_{Q_T}.\end{aligned}$$

Analogous estimates for the norms $|\!|\!|\widehat{w}|\!|\!|_{1,1,Q_T}$ and $\|\!|\, \widehat{w} \,|\!\|$ follow from (3.3) and (3.4).

§4. Estimates for non–solenoidal approximations

Now we extend the estimates derived in Sect. 2 to functions, which may not satisfy the divergence free condition. In the vast majority of cases, the function $\widehat{v}$ is obtained by an incremental type computational method and, therefore, belongs to $V_0^{(m)}(Q_T,\mathbb{R}^d) \subset \overset{\circ}{V}{}_2^{1,1}(Q_T,\mathbb{R}^d)$, so that we can use Lemma 1.

Let us estimate the quantity

$$\| e \|^2_{Q_T} := \nu \int\limits_0^T \|\varepsilon(e)\|^2_\Omega dt + \frac{1}{2}\|e(T)\|^2_\Omega,$$

where $e = u - \widehat{v}$ and for simplicity, we assume that $\widehat{v}(x,0) = \phi(x)$.

Using (3.7) we define $v_\circ \in \overset{\circ}{\mathrm{S}}{}^{1,1}(Q_T,\mathbb{R}^d)$. Then, the estimates (3.3)– (3.4) hold. Hence

$$\| e \|^2_{Q_T} = \| u - \widehat{v} \|^2_{Q_T} \leqslant \alpha \| u - v_\circ \|^2_{Q_T} + \alpha' \| \widehat{v} - v_\circ \|^2_{Q_T}, \tag{4.1}$$

where α and α' are arbitrary positive numbers satisfying $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$. The first norm is estimated as follows (see Theorem 1):

$$\| u - v_\circ \|^2_{Q_T} \leqslant \frac{C_K^2}{\nu}\int\limits_0^T (\|\tau - \nu\varepsilon(v_\circ) + q\mathbb{I}\|_\Omega + \|f - v_{\circ t} + \mathrm{Div}\tau - (\mathrm{a}\cdot\nabla)v_\circ\|_\Omega)^2 dt.$$

In view of (3.8), we have for $t \in [0,T]$

$$\|\tau - \nu\varepsilon(v_\circ) + q\mathbb{I}\|_\Omega \leqslant \widehat{\mathcal{R}}_1(t) + \nu\|\varepsilon(\widehat{v} - v_\circ)\|_\Omega \leqslant \widehat{\mathcal{R}}_1(t) + \nu\mathbb{C}_\Omega\|\mathrm{div}\widehat{v}\|_\Omega$$

and (cf. (3.11))

$$\begin{aligned}\|f - v_{\circ t} + \mathrm{Div}\tau - (\mathrm{a}\cdot\nabla)v_\circ\|_\Omega &\leqslant \widehat{\mathcal{R}}_2(t) + \|(\mathrm{a}\cdot\nabla)(\widehat{v} - v_\circ)\|_\Omega + \|\widehat{v}_t - v_{\circ t}\|_\Omega \\ &\leqslant \widehat{\mathcal{R}}_2(t) + \mathbb{C}_\Omega\left(\|\mathrm{a}\|_\infty\|\mathrm{div}\widehat{v}\|_\Omega + C_F(\Omega)\|\mathrm{div}\widehat{v}_t\|_\Omega\right),\end{aligned}$$

where

$$\begin{aligned}\widehat{\mathcal{R}}_1(t) &:= \|\tau - \nu\varepsilon(\widehat{v}) + q\mathbb{I}\|_\Omega,\\ \widehat{\mathcal{R}}_2(t) &= \|f - \widehat{v}_t + \mathrm{Div}\tau - (\mathrm{a}\cdot\nabla)\widehat{v}\|_\Omega.\end{aligned}$$

Hence we find that

$$\begin{aligned} \| \widehat{v} - v_\circ \|^2_{Q_T} \leqslant \frac{C_K^2}{\nu} \int\limits_0^T \beta \left(\widehat{\mathcal{R}}_1(t) + \widehat{\mathcal{R}}_2(t) \right)^2 dt \\ + \beta' \mathbb{C}^2_\Omega \int\limits_0^T \left((\nu + \|a\|_\infty) \|\mathrm{div} \widehat{v}\|_\Omega + C_F(\Omega) \|\mathrm{div} \widehat{v}_t\|_\Omega \right)^2 dt, \end{aligned} \tag{4.2}$$

where β and β' are positive conjugate numbers.

The second term in (4.1) is estimated with the help of Lemma 1:

$$\| \widehat{v} - v_\circ \|^2_{Q_T} \leqslant \mathbb{C}^2_\Omega (\nu \|\mathrm{div} \widehat{v}\|^2_{Q_T} + \frac{1}{2} C_F^2(\Omega) \|\mathrm{div} \widehat{v}(T)\|^2_\Omega). \tag{4.3}$$

Using (4.1)–(4.2) and making simple calculations, we obtain the following generalization of Theorem 1.

Theorem 2. *For any* $\widehat{v} \in V_0^{(m)}(Q_T, \mathbb{R}^d)$, $\tau \in \mathrm{H}_{\mathrm{Div}}(Q_T)$, *and* $q \in L_2(Q_T)$, *it holds*

$$\| e \|^2_{Q_T} \leqslant \frac{C_K^2}{\nu} \int\limits_0^T \alpha\beta \left(\widehat{\mathcal{R}}_1(t) + \widehat{\mathcal{R}}_2(t) \right)^2 dt + \mathbb{C}^2_\Omega \Phi(\widehat{v}), \tag{4.4}$$

where

$$\Phi(\widehat{v}) = c_1 \|\mathrm{div} \widehat{v}\|^2_{Q_T} + c_2 \|\mathrm{div} \widehat{v}_t\|^2_{Q_T} + c_3 \|\mathrm{div} \widehat{v}(T)\|^2_\Omega,$$

$$c_1 = \alpha\beta'\gamma \left((\nu + \|\mathrm{a}\|_\infty)^2 + \alpha'\nu \right), c_2 = \alpha\beta'\gamma' \, C_F^2(\Omega), c_3 = \frac{\alpha'}{2} C_F^2(\Omega),$$

and α, β, *and* γ *are arbitrary numbers greater than* 1, α', β', *and* γ' *are the corresponding conjugate numbers. The right hand side of* (4.4) *vanishes if and only if* $\widehat{v} = u$, *and* $\tau = \nu \nabla u - p\mathbb{I}$, *and* $q = p$.

Remark 4. The term $\Phi(\widehat{v})$ vanishes if $\widehat{v}$ is a solenoidal function. Therefore, this term can be viewed as a penalty for the violation of the divergence–free condition. In general, the estimate has the same structure as analogous estimates derived in [7, 8, 11] for the stationary Stokes and generalized Oseen problems. However, the penalty term in (4.4) has a more complicated structure and together with norm of $\|\mathrm{div} \widehat{v}\|_{Q_T}$ includes two other terms generated by the evolutionary nature of the problem (1.1)–(1.4).

The constants α, β, and γ are in our disposal. They should be selected to minimize the right hand side of (4.4) with $\widehat{v}$, τ, and q found in a numerical experiment.

Also, it is worth noting that the functionals forming right hand sides of the estimates (2.6) and (4.4) generate variational functionals associated with the problem (1.1)–(1.4), minimization of which with respect to v, τ, and q results in the exact velocity, stress, and pressure, which therefore can be found by direct minimization. The corresponding values of the functionals serve as reliable measures of the distance to the exact solution.

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