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BOUNDARY POLARIZATION OF THE RATIONAL SIX-VERTEX MODEL ON A SEMI-INFINITE LATTICE

Abstract. We consider the six-vertex model on a finite square lattice with the so-called partial domain wall boundary conditions. For the case of the rational Boltzmann weights, we compute the polarization on the free boundary of the lattice. For the finite lattice the result is given in terms of a ratio of determinants. In the limit where the side of the lattice with the free boundary tends to infinity (the limit of a semi-infinite lattice), they simplify and can be evaluated in a closed form.

§1. Introduction

Boundary correlation functions play an important role in the study of dimer and vertex models both in their relation with combinatorics and phase separation phenomena [1–5]. A famous example is provided by the six-vertex model with domain wall boundary conditions (DWBC) and their modifications, such as the six-vertex model on an L-shaped domain [6–8].

In [9], it was shown numerically, that rather interesting limit shape phenomena take place in the six-vertex model with a particular variation of the domain wall boundary conditions, the so-called partial domain wall boundary conditions (pDWBC). The partition function of this model was studied in [10, 11], where determinant representations, generalizing those for the domain wall case (Izergin-Korepin partition function) were given under certain restrictions on the Boltzmann weights. The partition function was also recently discussed in [12], where a Pfaffian formula was proposed for the partition function in the limit of semi-infinite lattice.

In the present paper, we address the problem of calculation of the simplest one-point boundary correlation function, describing polarization on the free boundary. We assume that the weights are symmetric (invariant

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under the reversal of arrows) and correspond to the rational parameterization in the underlying Yang-Baxter algebra. We show that the boundary polarization of this model can be given in terms of a ratio of determinants. Furthermore, in the limit of a semi-infinite lattice, these determinants simplify and can be evaluated explicitly.

The paper is organized as follows. In the next section we give definition of the six-vertex model with pDWBC and recall the result of [10] for the partition function of the model with the rational weights. In section 3, we introduce the boundary polarization and calculate it using the quantum inverse scattering method (QISM) [13]. In section 4, we study the obtained expression in the limit of a semi-infinite lattice, where we derive a closed expression in terms of a sum over Jacobi polynomials.

§2. The six-vertex model with pDWBC

In this section we define the model and recall a known result for the partition function.

2.1. The model. Configurations of the six-vertex model are described in terms of arrows placed on edges of a square lattice, which obey the "ice" rule, that is, there are two arrows pointing inward and two arrows pointing outward each lattice vertex. An equivalent way to represent configurations is in terms of solid lines "flowing" through the lattice, namely, the edge contains a solid line if an arrow points down or left, otherwise it is empty. The six vertex configurations, in the standard order (see, e.g., [14]) are shown on Fig. 1.

The partition function is defined as

$$Z = \sum_{\mathcal{C}} \prod_{i=1,\dots,6} w_i^{\#(i;\mathcal{C})},$$

where the sum is over configurations, $\#(i;\mathcal{C})$ denotes the number of vertices of type i in the configuration $\mathcal{C}$, and w_i is the corresponding Boltzmann weight, $i = 1, \dots, 6$.

In this paper, we consider the six-vertex model on a square $s \times N$ lattice (the lattice obtained by intersection of s horizontal and N vertical lines). The states on the three boundaries are fixed, and on the remaining one a summation over all possible arrow configurations is performed. Namely, the arrows on the left and right boundaries are outgoing, and at the bottom one are incoming, see Fig. 2. Note that, in the special case $s = N$, the only

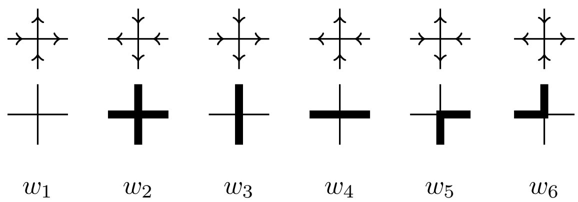

Figure 1. Vertex configurations in terms of arrows (first row), solid lines (second row), and their Boltzmann weights (third row).

possible configuration on the top boundary is with all incoming arrows, that corresponds to the DWBC [10, 15].

We assume that Boltzmann weights are invariant under reversal of all arrows, $w_1 = w_2 =: a$, $w_3 = w_4 =: b$, $w_5 = w_6 =: c$. To apply the quantum inverse scattering method (QISM) we consider the inhomogeneous version of the model where the Boltzmann weights a, b, and c are site-dependent. Namely, we introduce two sets of parameters, each parameter associated with a line of the lattice. Parameters $\lambda_1, \dots, \lambda_s$ correspond to horizontal lines enumerated from top to bottom and $\nu_1, \dots, \nu_N$ correspond to vertical lines enumerated from right to left. The weights of the vertex being at the intersection of j-th horizontal and k-th vertical lines are

$$a_{jk} = a(\lambda_j, \nu_k), \qquad b_{jk} = b(\lambda_j, \nu_k), \qquad c_{jk} = c(\lambda_j, \nu_k).$$

Specifically, we consider here only the case where these functions satisfy the relation:

$$a(\lambda, \nu) = b(\lambda, \nu) + c(\lambda, \nu),$$

and given by

$$a(\lambda, \nu) = 1, \qquad b(\lambda, \nu) = \frac{\lambda - \nu - \frac{1}{2}}{\lambda - \nu + \frac{1}{2}}, \qquad c(\lambda, \nu) = \frac{1}{\lambda - \nu + \frac{1}{2}}.$$

The case where all the parameters are equal to each other within each set, $\lambda_1 = \ldots = \lambda_s = \lambda$ and $\nu_1 = \ldots = \nu_N = \nu$ corresponds to the homogeneous model; we call this case as homogeneous limit.

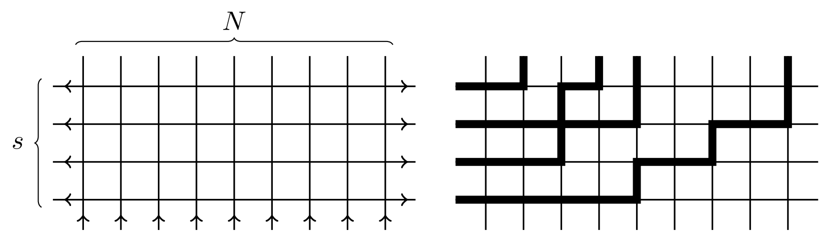

Figure 2. An $s \times N$ lattice with pDWBC, the summation over all possible arrow orientations on empty edges is implied (left), and one of possible configurations in terms of solid lines (right).

2.2. QISM formulation. To formulate the model in the framework of QISM, we first introduce the spin-up and spin-down states:

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We assume that the projection to the spin-up state corresponds to an empty edge (in the "line" language), and the projection to the spin-down state corresponds to an edge with a solid line.

To the vertex being at the intersection of kth column and jth row we associate an operator $L_{jk}(\lambda_j, \nu_k)$ which acts non-trivially in the direct tensor product of two vector spaces $\mathbb{C}^2$: the "horizontal" or "quantum" space $\mathcal{H}_j = \mathbb{C}^2$ (associated with the j-th row) and the "vertical" or "auxiliary" space $\mathcal{V}_k = \mathbb{C}^2$ (associated with the kth column). Graphically, the L-operator acts from top to bottom and from right to left.

For the six-vertex model the L-operator is nothing but the matrix of the Boltzmann weights in the spin-up and spin-down basis. It is convenient to think of L-operator as an operator, acting in $\mathcal{V}_k \otimes \mathcal{H}$, where $\mathcal{H} = \otimes_{j=1}^{s} \mathcal{H}_j$, which acts nontrivially in jth horizontal space only,

$$L_{jk}(\lambda_j, \nu_k) = \begin{pmatrix} \frac{1+\sigma_j^z}{2} a_{jk} + \frac{1-\sigma_j^z}{2} b_{jk} & \sigma_j^- c_{jk} \\ \sigma_j^+ c_{jk} & \frac{1+\sigma_j^z}{2} b_{jk} + \frac{1-\sigma_j^z}{2} a_{jk} \end{pmatrix}_{[\mathcal{V}_k]}.$$

Here, σ_j^z and $\sigma_j^\pm = \frac{1}{2}\left(\sigma_j^x \pm \sigma_j^y\right)$ are quantum operators in $\mathcal{H}$ acting as Pauli matrices in $\mathcal{H}_j$, and the subscript of the matrix emphasizes that this is a matrix in the space $\mathcal{V}_k$.

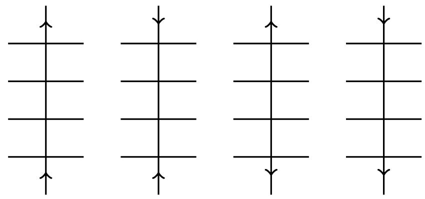

Figure 3. Graphical interpretation of the operators—elements of the quantum monodromy matrix $T(\nu)$.

An ordered product of L-operators along a vertical line is the quantum monodromy matrix:

$$T_k(\nu_k) = L_{sk}(\lambda_s, \nu_k) \cdots L_{2k}(\lambda_2, \nu_k) L_{1k}(\lambda_1, \nu_k) = \begin{pmatrix} A(\nu_k) & B(\nu_k) \\ C(\nu_k) & D(\nu_k) \end{pmatrix}_{[\mathcal{V}_k]}.$$

Fig. 3 shows graphical interpretation of the operators $A(\nu)$, $B(\nu)$, $C(\nu)$ and $D(\nu)$, associated with a column of the lattice in the arrow language.

The basic role for QISM plays the intertwining relation for the L-operators

$$R_{kl}(\nu_k, \nu_l)\big(L_{jk}(\lambda_j, \nu_k) \otimes L_{jl}(\lambda_j, \nu_l)\big) = \big(L_{jl}(\lambda_j, \nu_l) \otimes L_{jk}(\lambda_j, \nu_k)\big) R_{kl}(\nu_k, \nu_l). \quad (2.1)$$

This relation is written as an operator equation in the direct product of spaces $\mathcal{V}_k \otimes \mathcal{V}_l \otimes \mathcal{H}_j$. The matrix R_{kl} acts nontrivially in the direct product of $\mathcal{V}_k \otimes \mathcal{V}_l$ and has the following form

$$R_{kl}(\nu, \mu) = \begin{pmatrix} f(\mu, \nu) & 0 & 0 & 0 \\ 0 & 1 & g(\mu, \nu) & 0 \\ 0 & g(\mu, \nu) & 1 & 0 \\ 0 & 0 & 0 & f(\mu, \nu) \end{pmatrix}_{[\mathcal{V}_k \otimes \mathcal{V}_l]}$$

where functions $f(\mu, \nu)$ and $g(\mu, \nu)$ are

$$f(\mu, \nu) = 1 + \frac{1}{\nu - \mu}, \qquad g(\mu, \nu) = \frac{1}{\nu - \mu}.$$

This R-matrix is called rational R-matrix and it is related to the Heisenberg XXX spin chain.

The relation (2.1) implies a similar relation for the monodromy matrices:

$$R_{kl}(\nu_k,\nu_l)\big(T_k(\nu_k)\otimes T_l(\nu_l)\big) = \big(T_l(\nu_l)\otimes T_k(\nu_k)\big)R_{kl}(\nu_k,\nu_l). \tag{2.2}$$

Relation (2.2) encodes the commutation relations between operators $A(\nu)$, $B(\nu)$, $C(\nu)$ and $D(\nu)$, the so-called Yang-Baxter algebra. The relations which are important for our purposes below are

$$\big[A(\nu),A(\mu)\big] = \big[B(\nu),B(\mu)\big] = 0$$

and

$$A(\nu)B(\mu) = f(\nu,\mu)B(\mu)A(\nu) + g(\mu,\nu)B(\nu)A(\mu).$$

In the case of the rational R-matrix, the Yang-Baxter algebra is simplified due to the fact that

$$f(\nu,\mu) - g(\nu,\mu) = 1. \tag{2.3}$$

In particular, the following commutation relations are valid

$$\big[A(\nu)+B(\nu),A(\mu)+B(\mu)\big] = 0 \tag{2.4}$$

and

$$\begin{aligned} A(\nu)\big(A(\mu)+B(\mu)\big) = f(\nu,\mu)\left(A(\mu)+B(\mu)\right)A(\nu) \\ + g(\mu,\nu)\left(A(\nu)+B(\nu)\right)A(\mu). \end{aligned} \tag{2.5}$$

These relations follow from (2.3).

2.3. The partition function. The partition function of the six-vertex model on a $s\times N$ square lattice with pDWBC can be written as the matrix element

$$Z = \langle\Downarrow_s|\prod_{k=1}^{N}\left(A(\nu_k)+B(\nu_k)\right)|\Uparrow_s\rangle.$$

Here,

$$|\Uparrow_s\rangle = \otimes_{j=1}^{s}|\uparrow_j\rangle, \qquad |\Downarrow_s\rangle = \otimes_{j=1}^{s}|\downarrow_j\rangle,$$

where $|\uparrow\rangle_j$ and $|\downarrow_j\rangle$ are basic vectors in $\mathcal{H}_j$.

Note, that the vectors $|\Uparrow_s\rangle$ and $|\Downarrow_s\rangle$ are the eigenvectors of the operators $A(\nu)$ and $D(\nu)$, in particular, we have

$$A(\nu)|\Uparrow_s\rangle = \prod_{j=1}^{s} a(\lambda_j,\nu)|\Uparrow_s\rangle = |\Uparrow_s\rangle. \tag{2.6}$$

An explicit formula for the partition function was derived in [10], where it was given in terms of a determinant. In our parameterization, the result of [10] reads

$$Z = \frac{\prod_{j=1}^{s}\prod_{k=1}^{N}\left(\lambda_j - \nu_k - \frac{1}{2}\right)}{\prod_{1\leqslant j<k\leqslant s}(\lambda_k-\lambda_j)\prod_{1\leqslant j<k\leqslant N}(\nu_j-\nu_k)}\det\nolimits_N \mathcal{Z}, \tag{2.7}$$

where $\mathcal{Z}$ is the following $N\times N$ matrix:

$$\mathcal{Z} = \begin{pmatrix} \varphi(\lambda_1,\nu_1) & \dots & \varphi(\lambda_1,\nu_N) \\ \dots & \dots & \dots \\ \varphi(\lambda_s,\nu_1) & \dots & \varphi(\lambda_s,\nu_N) \\ \nu_1^{N-s-1} & \dots & \nu_N^{N-s-1} \\ \dots & \dots & \dots \\ \nu_1^0 & \dots & \nu_N^0 \end{pmatrix}.$$

Here,

$$\varphi(\lambda,\nu) = \frac{1}{\left(\lambda-\nu+\frac{1}{2}\right)\left(\lambda-\nu-\frac{1}{2}\right)}.$$

The formula (2.7) provides a generalization for the partition function of the model with domain wall boundary conditions on an $s\times s$ square lattice [15–17] to the case of partial domain wall ones on an $s\times N$ lattice, valid for the rational weights [10].

§3. Boundary polarization

In this section we introduce and calculate the boundary polarization of the six-vertex model on the $s\times N$ lattice with pDWBC.

3.1. Definition of the polarization. We are interested in computing the one-point correlation function $G_\downarrow(m)$, which can be defined as the probability that the external edge of the mth vertical line from the left has the arrow pointing down, see Fig. 4. In the "line" language this correlation function describes configurations such that one of the s solid lines passes though this edge. Note that, since there are s solid lines in total,

$$\sum_{m=1}^{N} G_\downarrow(m) = s.$$

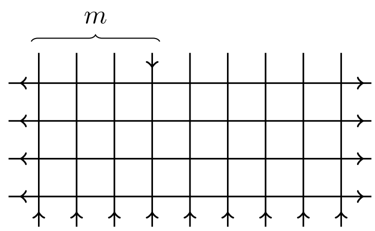

Figure 4. Definition of the boundary polarization $G_{\downarrow}(m)$.

To calculate this correlation function, we use that equivalently we can consider the probability of having an up arrow on the indicated edge,

$$G_{\uparrow}(m) = 1 - G_{\downarrow}(m).$$

It turns out that $G_{\uparrow}(m)$ can be computed in rather straightforward manner in the framework of QISM. At the final stage of calculations, we turn back to $G_{\downarrow}(m)$.

Furthermore, we introduce the distance from the right boundary

$$M = N - m + 1.$$

In terms of operators of the Yang-Baxter algebra, $G_{\uparrow}(m)$ can written as

$$\begin{aligned} G_{\uparrow}(m) &= Z^{-1} \\ &\times \langle \Downarrow_s | \prod_{k=M+1}^{N} \big(A(\nu_k) + B(\nu_k)\big) A(\nu_M) \prod_{k=1}^{M-1} \big(A(\nu_k) + B(\nu_k)\big) | \Uparrow_s \rangle. \end{aligned} \tag{3.1}$$

Note that, it is the presence of the operator $A(\nu_M)$ (rather than $B(\nu_M)$, in the case of $G_{\downarrow}(m)$) that makes it possible to use directly relations (2.4), (2.5) and (2.6) to compute the matrix element in (3.1).

3.2. Calculation of the polarization. Indeed, using the commutation relation (2.5) we commute operator $A(\nu)$ to the right and act with it on the all-spins-up eigenstate:

$$A(\nu_M)\prod_{k=1}^{M-1}\big(A(\nu_k)+B(\nu_k)\big)|\Uparrow_s\rangle = \sum_{j=1}^{M}\frac{f(\nu_j,\nu_M)}{g(\nu_j,\nu_M)}\prod_{\substack{k=1\\k\neq j}}^{M} f(\nu_j,\nu_k)\prod_{\substack{k=1\\k\neq j}}^{M}\big(A(\nu_k)+B(\nu_k)\big)|\Uparrow_s\rangle.$$

This formula follows in a completely standard manner for QISM. To prove it, one has to consider the term $j=M$ which contains the operators $A(\nu_k)+B(\nu_k)$ for all values of k, except $k=M$. To get this term one has to use the first term in the RHS of the commutation relation (2.5). The remaining terms originate from the symmetry with respect to permutations of $\nu_1,\dots,\nu_{M-1}$, see (2.4).

This relation allows us to rewrite the correlation function as the following sum

$$G_{\uparrow}(m) = Z^{-1}\sum_{j=1}^{M}\frac{f(\nu_j,\nu_M)}{g(\nu_j,\nu_M)}\prod_{\substack{k=1\\k\neq j}}^{M} f(\nu_j,\nu_k)\langle\Downarrow_s|\prod_{\substack{k=1\\k\neq j}}^{N}\big(A(\nu_k)+B(\nu_k)\big)|\Uparrow_s\rangle.$$

The last factor is nothing but the partition function of the model on a $s\times(N-1)$ lattice with a set of parameters $\{\nu_k\}_{k=1,k\neq j}^{N}$. Using the determinant representation of the partition function (2.7) with $N\mapsto N-1$, we get the following expression

$$G_{\uparrow}(m) = \sum_{j=1}^{M}(-1)^{M-j}\frac{\prod_{k=1}^{M-1}(\nu_k-\nu_j+1)\prod_{k=M+1}^{N}(\nu_j-\nu_k)}{\prod_{k=1}^{s}\big(\lambda_k-\nu_j-\frac{1}{2}\big)} \times\frac{\det_{N-1}\mathcal{Z}(\backslash\nu_j)}{\det_N\mathcal{Z}}.$$

Here, $\mathcal{Z}(\backslash\nu_j)$ denotes the matrix obtained from $\mathcal{Z}$ by removing jth column and $(s+1)$th row.

We note that the expression above is a determinant of some $N\times N$ matrix whose first M entries in the column are not equal to zero developed along the $(s+1)$th row. Finally, we end up with the following determinant representation of the correlation function

$$G_{\uparrow}(m) = (-1)^{s+M}\frac{\det_N\mathcal{G}}{\det_N\mathcal{Z}}, \tag{3.2}$$

where the matrix $\mathcal{G}$ reads

$$\mathcal{G}=\begin{pmatrix}\varphi(\lambda_1,\nu_1) & \dots & \varphi(\lambda_1,\nu_N)\\ \dots & \dots & \dots\\ \varphi(\lambda_s,\nu_1) & \dots & \varphi(\lambda_s,\nu_N)\\ h(\nu_1) & \dots & h(\nu_N)\\ \nu_1^{N-s-2} & \dots & \nu_N^{N-s-2}\\ \dots & \dots & \dots\\ \nu_1^0 & \dots & \nu_N^0\end{pmatrix}.$$

Here, the function $h(\nu)$ is given by

$$h(\nu)=\frac{\prod_{k=1}^{M-1}(\nu-\nu_k+1)\prod_{k=M+1}^{N}(\nu_k-\nu)}{\prod_{k=1}^{s}\left(\lambda_k-\nu-\frac{1}{2}\right)}.$$

Note that $h(\nu)$ has zeros at the points $\nu=\nu_{M+1},\dots,\nu_N$.

3.3. Homogeneous limit. Let us now derive an expression for the correlation function in the homogeneous limit, that is as $\lambda_1,\dots,\lambda_s\to\lambda$ and $\nu_1,\dots,\nu_N\to\nu$. Note that, since the weights of the homogeneous model depends only on the difference $\lambda-\nu$, we can also put $\nu=0$ without loss of generality.

To derive the limit, we multiply both numerator and denominator of (3.2) by the factor

$$\prod_{1\leqslant j<k\leqslant N}\frac{1}{\nu_j-\nu_k}\prod_{1\leqslant j<k\leqslant s}\frac{1}{\lambda_k-\lambda_j}$$

and consider the limit separately in the denominator and numerator. The key relation here is that if $\vec{f}(\nu)$ is an N-component column vector, with the components of at least $N-1$ times differentiable with respect to ν, then

$$\lim_{\nu_1,\dots,\nu_N\to\nu}\prod_{1\leqslant j<k\leqslant N}\frac{1}{\nu_k-\nu_j}\left|\vec{f}(\nu_1)\quad \vec{f}(\nu_2)\quad \dots\quad \vec{f}(\nu_N)\right|$$
$$=\prod_{k=1}^{N-1}\frac{1}{k!}\left|\vec{f}(\nu)\quad \partial_\nu\vec{f}(\nu)\quad \dots\quad \partial_\nu^{N-1}\vec{f}(\nu)\right|.$$

Consider first the denominator and perform the limit in ν's keeping λ's arbitrary. Using that $\partial_\nu\varphi(\lambda,\nu)=-\partial_\lambda\varphi(\lambda,\nu)$, we get

$$\lim_{\nu_1,\dots,\nu_N\to 0}\prod_{1\leqslant j<k\leqslant N}\frac{1}{\nu_j-\nu_k}\det{}_N\mathcal{Z}=\prod_{k=1}^{N-1}\frac{1}{k!}$$

$$\times\begin{vmatrix}\varphi(\lambda_1) & \varphi'(\lambda_1) & \dots & \varphi^{(N-s-1)}(\lambda_1) & \dots & \varphi^{(N-1)}(\lambda_1)\\ \dots & \dots & \dots & \dots & \dots & \dots\\ \varphi(\lambda_s) & \varphi'(\lambda_s) & \dots & \varphi^{(N-s-1)}(\lambda_s) & \dots & \varphi^{(N-1)}(\lambda_s)\\ 0 & 0 & \dots & (-1)^{N-s-1}(N-s-1)! & \dots & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & -1! & \dots & 0 & \dots & 0\\ 0! & 0 & \dots & 0 & \dots & 0\end{vmatrix}$$

$$=(-1)^{s(N-s)}\prod_{k=1}^{s}\frac{1}{(N-k)!}\begin{vmatrix}\varphi^{(N-s)}(\lambda_1) & \dots & \varphi^{(N-1)}(\lambda_1)\\ \dots & \dots & \dots\\ \varphi^{(N-s)}(\lambda_s) & \dots & \varphi^{(N-1)}(\lambda_s)\end{vmatrix}.$$

Here, the last determinant is of size $s\times s$, and $\varphi(\lambda)\equiv\varphi(\lambda,0)$. Taking the limit in λ's, for the denominator we finally obtain

$$\lim_{\substack{\nu_1,\dots,\nu_N\to 0\\ \lambda_1,\dots,\lambda_s\to\lambda}}\prod_{1\leqslant j<k\leqslant s}\frac{1}{\lambda_k-\lambda_j}\prod_{1\leqslant j<k\leqslant N}\frac{1}{\nu_j-\nu_k}\det{}_N\mathcal{Z}$$

$$=\frac{(-1)^{s(N-s)}}{\prod_{k=1}^{s}(N-k)!\prod_{k=1}^{s-1}k!}\det{}_s\mathcal{Z}_{\mathrm{hom}},$$

where $\mathcal{Z}_{\mathrm{hom}}$ is an $s\times s$ matrix with entries

$$(\mathcal{Z}_{\mathrm{hom}})_{ij}=\varphi^{(N-s+i+j-2)}(\lambda),\qquad i,j=1,\dots,s.$$

Essentially similarly, for the numerator we obtain

$$\lim_{\substack{\nu_1,\dots,\nu_N\to 0\\ \lambda_1,\dots,\lambda_s\to\lambda}}\prod_{1\leqslant j<k\leqslant N}\frac{1}{\nu_j-\nu_k}\prod_{1\leqslant j<k\leqslant s}\frac{1}{\lambda_k-\lambda_j}\det{}_N\mathcal{G}$$

$$=\frac{(-1)^{N-M+1}}{\prod_{k=1}^{s+1}(N-k)!\prod_{k=1}^{s-1}k!}\det{}_{s+1}\mathcal{G}_{\mathrm{hom}},$$

where entries of the $(s+1)\times(s+1)$ matrix $\mathcal{G}_{\mathrm{hom}}$ are

$$(\mathcal{G}_{\mathrm{hom}})_{ij}=\begin{cases}\varphi^{(N-s-3+i+j)}(\lambda) & i=1,\dots,s\\ (-1)^{s-j+1}H^{(N-s+j-2)}(0) & i=s+1.\end{cases}$$

Here, the function $H(\varepsilon)$ is

$$H(\nu) = (-1)^{N-M} \lim_{\substack{\nu_1,\dots,\nu_N \to 0\\ \lambda_1,\dots,\lambda_s \to \lambda}} h(\nu) = \frac{(\nu+1)^{M-1}\nu^{N-M}}{\left(\lambda - \nu - \frac{1}{2}\right)^s}.$$

As a result, we obtain the following expression for the boundary polarization of the homogeneous model:

$$G_{\uparrow}(m) = \frac{(-1)^{N-s-1}}{(N-s-1)!} \frac{\det_{s+1}\mathcal{G}_{\text{hom}}}{\det_s \mathcal{Z}_{\text{hom}}}. \tag{3.3}$$

§4. Semi-infinite lattice

In this section we consider the model in the limit where N tends to infinity, and derive a closed formula for the boundary polarization.

4.1. Reparameterization of the determinants. We begin with introducing a new variable

$$t = b(\lambda, 0) = \frac{\lambda - \frac{1}{2}}{\lambda + \frac{1}{2}}$$

as the main parameter of the model. Recall that the function $\varphi(\lambda)$ is given by

$$\varphi(\lambda) = \frac{1}{\lambda - \frac{1}{2}} - \frac{1}{\lambda + \frac{1}{2}} = \frac{(1-t)^2}{t}$$

and the n-th derivative with respect to λ in terms of t reads

$$\varphi^{(n)}(\lambda) = (-1)^n n! \left(\frac{1-t}{t}\right)^{n+1} (1 - t^{n+1}).$$

Taking into account this expression, we now rewrite the result (3.3) for the correlation function introducing renormalized matrices

$$G_{\uparrow}(m) = \frac{1}{(\alpha-1)!} \frac{\det_{s+1}\mathcal{G}_{\text{ren}}}{\det_s \mathcal{Z}_{\text{ren}}},$$

where $\alpha \equiv N - s$. The entries of the matrix $\mathcal{Z}_{\text{ren}}$ are

$$(\mathcal{Z}_{\text{ren}})_{ij} = (\alpha + i + j - 2)! \left(1 - t^{\alpha+i+j-1}\right), \qquad i, j = 1, \dots, s, \tag{4.1}$$

and those of the matrix $\mathcal{G}_{\text{ren}}$ are

$$(\mathcal{G}_{\text{ren}})_{ij} = \begin{cases} (\alpha + i + j - 3)! \left(1 - t^{\alpha+i+j-2}\right) & i = 1, \dots, s \\ \left(\frac{t}{1-t}\right)^{j-1} H^{(\alpha-2+j)}(0) & i = s+1. \end{cases}$$

The function $H(\nu)$ now reads

$$H(\nu)=\frac{(\nu+1)^{N-m}\nu^{m-1}}{\left(\frac{t}{1-t}-\nu\right)^{s}}.$$

Using Cauchy contour integral formula, one can write the entries of the last row of the matrix $\mathcal{G}_{\mathrm{ren}}$ as follows

$$\frac{1}{(\alpha-2+j)!}H^{(\alpha-2+j)}(0)=(-1)^{s}\delta_{1j}+(1-t)^{s}t^{m-N-j+1}I_j. \tag{4.2}$$

Here,

$$I_j=\frac{1}{2\pi\mathrm{i}}\oint_{C_1}\frac{z^{N-m}(z-t)^{j-2}}{(z-1)^{s}}\mathrm{d}z,$$

where c_1 denotes an infinitesimal contour with positive orientation encircling the point $z=1$. The details are given in appendix A.

Apparently, the first term in the RHS of (4.2) is relevant only for the $(s+1,1)$ entry of the matrix $\mathcal{G}_{\mathrm{ren}}$. Picking up the contribution coming from this term, we get

$$G_{\uparrow}(m)=1-\frac{t^{m-N}}{(\alpha-1)!}\frac{\det_{s+1}\tilde{\mathcal{G}}_{\mathrm{ren}}}{\det_{s}\mathcal{Z}_{\mathrm{ren}}}. \tag{4.3}$$

Here, the entries of the $(s+1)\times(s+1)$ matrix $\tilde{\mathcal{G}}_{\mathrm{ren}}$ are

$$\left(\tilde{\mathcal{G}}_{\mathrm{ren}}\right)_{ij}=\begin{cases}(\alpha-3+i+j)!\left(1-t^{\alpha-2+i+j}\right) & i=1,\dots,s\\ (\alpha-2+j)!\left(\frac{1}{1-t}\right)^{j-s-1}I_j & i=s+1.\end{cases}$$

Recall, that the entries of $\mathcal{Z}_{\mathrm{ren}}$ are given by (4.1).

The second term in (4.3) is in fact nothing but the correlation function $G_{\downarrow}(m)$ which gives the probability of finding the down arrow on the top boundary on the mth column from the left, that is

$$G_{\downarrow}(m)=\frac{t^{m-N}}{(\alpha-1)!}\frac{\det_{s+1}\tilde{\mathcal{G}}_{\mathrm{ren}}}{\det_{s}\mathcal{Z}_{\mathrm{ren}}}. \tag{4.4}$$

In what follows, we focus on the representation (4.4).

4.2. Large N limit. As indicated in [12], the limit $N\to\infty$ means that the right boundary goes to infinity, so that the lattice is semi-infinite, with s rows. The boundary conditions on the right boundary become effectively vanishing, that is guaranteed by fact that the a-weight is normalized to one and that the b-weight, which is now the parameter t, satisfies $0 \leqslant t < 1$. The partition function in the limit is

$$\lim_{N\to\infty} Z = 1. \tag{4.5}$$

This property reflects stochasticity of the six-vertex model with the weights satisfying the relation $a = b + c$, see, e.g., [18].

Indeed, the partition function of the model on the $s \times N$ lattice reads

$$Z = \frac{1}{\prod_{k=1}^{s}(N-k)!\prod_{k=1}^{s-1} k!}\det{}_s\mathcal{Z}_{\mathrm{ren}}.$$

Since $t^N \to 0$ as $N \to \infty$, for the entries of $\mathcal{Z}_{\mathrm{ren}}$ we have,

$$(\mathcal{Z}_{\mathrm{ren}})_{ij} = (\alpha + i + j - 2)! + O(t^N).$$

The leading term is nothing but the moment of the orthogonality measure of the Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$,

$$(\alpha + i + j - 2)! = \int_0^\infty x^{i+j-2}\mathrm{e}^{-x}x^\alpha \mathrm{d}x.$$

The standard technique of evaluation of the determinant of a Hankel matrix yields

$$\det{}_s\mathcal{Z}_{\mathrm{ren}} = \prod_{j=0}^{s-1}(\alpha + j)!j! + O(t^N),$$

that gives (4.5).

In what follows we address the problem of obtaining an expression for the boundary polarization in the limit where the number of vertical lines tends to infinity, $N \to \infty$. Correspondingly, in the limit, the distance to the right boundary, M, must also be large, but the distance to the left boundary, $m = N - M + 1$, remains finite.

To study the correlation function (4.4) for large N, but finite s and m, we note that the presence of the prefactor t^{m-N} in (4.4) suggests that the determinant of matrix $\tilde{\mathcal{G}}_{\mathrm{ren}}$ is of $O(t^N)$, since the polarization should remain finite as $N \to \infty$. This means that the leading term of the determinant vanishes, contrary the case of $\det_s\mathcal{Z}_{\mathrm{ren}}$. In what follows our aim is to derive the first nonzero contribution in the large N limit (the remaining terms are exponentially small as $N \to \infty$).

4.3. Evaluation of the polarization. To derive an expression for the boundary polarization in the $N \to \infty$ limit, we represent the matrix $\tilde{\mathcal{G}}_{\mathrm{ren}}$ in the form

$$\tilde{\mathcal{G}}_{\mathrm{ren}} = A_0 - A_1 t^N.$$

The entries of the $(s+1) \times (s+1)$ matrices A_0 and A_1 are

$$(A_0)_{ij} = \begin{cases} (\alpha+i+j-3)! & i = 1, \dots, s \\ (\alpha+j-2)!\left(\frac{1}{1-t}\right)^{j-s-1} I_j & i = s+1 \end{cases}$$

and

$$(A_1)_{ij} = \begin{cases} (\alpha-3+i+j)!\, t^{-s-2+i+j} & i = 1, \dots, s \\ 0 & i = s+1. \end{cases}$$

respectively.

Consider the matrix A_0. Its last row appears to be a linear combination of all other rows,

$$(\alpha-2+j)! \left(\frac{1}{1-t}\right)^{j-s-1} I_j = \sum_{i=1}^{s} C_i \cdot (\alpha-3+i+j)!, \tag{4.6}$$

where

$$C_i = \frac{(-1)^{s+i}(N-1)!}{(s-i)!(\alpha+i-1)!(i-1)!}\, {}_2F_1\left(\begin{matrix} i-s,\, m-N \\ 1-N \end{matrix} \middle| 1-t \right). \tag{4.7}$$

Hence, $\det_{s+1} A_0 = 0$.

To derive the first nonzero contribution to the polarization, consider the matrix

$$A(i,j) = A_0 - e_{ij}(A_1)_{ij},$$

where $i = 1, \dots, s$ and $j = 1, \dots, s+1$ and e_{ij} is a matrix whose only nonzero element is 1 at the (i,j) entry.

Using the identity (4.6) we subtract from the last rows other rows with the coefficients C_i. Denoting the new matrix by $\tilde{A}(i,j)$ we find that the entries of this matrix at the last row are all zeros, except the jth, which is equal to

$$\left(\tilde{A}(i,j)\right)_{s+1,j} = t^{\alpha-2+i+j}(\alpha+j-3)!\, C_i.$$

Developing the determinant along the last row, we find

$$\det_{s+1} A(i,j) = (-1)^{s+j} \left(\tilde{A}(i,j)\right)_{s+1,j} M(j),$$

where $M(j)$ is the determinant of the matrix A_0 with the $(s+1)$th row and the jth column removed.

We sum over all possible i and j and end up with the following double sum representation

$$\det_{s+1}\tilde{\mathcal{G}}_{\text{ren}} = (-1)^s t^\alpha \sum_{i=1}^{s}\sum_{j=1}^{s+1}(-1)^j(\alpha+i+j-3)!t^{i+j-2}C_i M(j) + O(t^N).$$

It can be shown that (see, e.g., calculations in [19])

$$M(j) = \frac{1}{(j-1)!\ (\alpha+j-2)!(s-j+1)!}\prod_{k=0}^{s}(\alpha+k-1)!k!.$$

As a result, we obtain that, as $N\to\infty$,

$$\begin{aligned} G_\downarrow(m) &= (-1)^s s! t^{m-s} \\ &\times \sum_{i=1}^{s}\sum_{j=1}^{s+1}\frac{(-1)^j(\alpha-3+i+j)!}{(\alpha-2+j)!(s+j-1)!(j-1)!}C_i t^{i+j-2} + O(t^N). \end{aligned} \tag{4.8}$$

Recall, that the coefficients C_i, given by (4.7), are some polynomials in t. The peculiarity of the expression in (4.8) is that the sum over i in (4.8) is independent of N (recall that $\alpha = N - s$) and it is equal to

$$\begin{aligned} &\sum_{i=1}^{s}\frac{(n-1)!(\alpha+i+j-3)!}{(s-i)!(\alpha+i-1)!}\frac{(-t)^{i-1}}{(i-1)!}\,{}_2F_1\left(\begin{matrix} i-s,\ m-N \\ 1-N\end{matrix}\middle| 1-t\right) \\ &\qquad = (-1)^s\frac{(m-s+1)_{s-1}}{(s-1)!}\,{}_2F_1\left(\begin{matrix} 1-s,\ m-s+j \\ m-s+1\end{matrix}\middle| t\right). \end{aligned}$$

Here $(a)_n$ is a Pochhammer symbol. The identity above is proved in the appendix (see (B.1)) using the following Jacobi polynomials representation for the hypergeometric function

$${}_2F_1\left(\begin{matrix} -a,\ b \\ c\end{matrix}\middle| t\right) = \frac{a!}{(c)_a}P_a^{(c-1,b-a-c)}(1-2t).$$

Finally, we end up with the following expressions for the correlation function in the $N\to\infty$ limit in terms of the hypergeometric function:

$$\begin{aligned} \lim_{N\to\infty} G_\downarrow(m) &= t^{m-s}\frac{(m-s+1)_{s-1}}{(s-1)!} \\ &\qquad \times \sum_{j=0}^{s}(-t)^j\binom{s}{j}\,{}_2F_1\left(\begin{matrix} 1-s,\ m-s+j \\ m-s+1\end{matrix}\middle| t\right). \end{aligned}$$

Or, in terms of Jacobi polynomials,

$$\lim_{N\to\infty} G_{\downarrow}(m) = t^{m-s} \sum_{j=0}^{s} (-t)^j \binom{s}{j} P_{s-1}^{(m-s,j-s)}(1-2t).$$

Appendix §A. Contour integral representation

Here we consider contour integral representation of the elements on the last row of the matrix $\mathcal{G}_{\mathrm{ren}}$.

We rewrite the derivative of the function $H(\varepsilon)$ via contour integral representation

$$\frac{1}{(\alpha+j-2)!} H^{(\alpha+j-2)}(0) = \frac{1}{2\pi \mathrm{i}} \oint_{C_0} \frac{(\varepsilon+1)^{N-m}\varepsilon^{m-1}}{\left(\frac{t}{1-t}-\varepsilon\right)^s \varepsilon^{\alpha-1+j}} \mathrm{d}\varepsilon,$$

where C_0 is an infinitesimal contour with positive orientation encircling the point $z=0$ Making the change $\varepsilon \mapsto 1/z$, transform the contour integral

$$\frac{1}{2\pi \mathrm{i}} \oint_{C_0} \frac{(\varepsilon+1)^{N-m}\varepsilon^{m-1}}{\left(\frac{t}{1-t}-\varepsilon\right)^s \varepsilon^{\alpha-1+j}} \mathrm{d}\varepsilon = \frac{(1-t)^s}{2\pi i} \oint_{C_\infty} \frac{(z+1)^{N-m} z^{j-2}}{(tz-1+t)^s} \mathrm{d}z.$$

If $j=1$, then there is a pole at the point $z=0$,

$$\frac{(1-t)^s}{2\pi \mathrm{i}} \oint_{C_0} \frac{(z+1)^{N-m} z^{j-2}}{(tz-1+t)^s} \mathrm{d}z = (-1)^s \delta_{1j}.$$

There is also a pole at the point $z=(1-t)/t$, such that

$$\oint_{C_{\frac{1-t}{t}}} \frac{(z+1)^{N-m} z^{j-2}}{(tz-1+t)^s} \mathrm{d}z = t^{m-N+1-j} \oint_{C_1} \frac{z^{N-m}(z-t)^{j-2}}{(z-1)^s} \mathrm{d}z.$$

Summarizing, we conclude that

$$\frac{1}{(\alpha-2+j)!} H^{(\alpha+j-2)}(0) = (-1)^s \delta_{1j} + (1-t)^s t^{m-N-j+1} I_j,$$

where I_j is the contour integral

$$I_j = \frac{1}{2\pi \mathrm{i}} \oint_{C_1} \frac{z^{N-m}(z-t)^{j-2}}{(z-1)^s} \mathrm{d}z.$$

In terms of Jacobi polynomials,

$$I_j = (-1)^{s-1}(1-t)^{j-1-s} P_{s-1}^{(m-N+1-j,j-s-1)}(1-2t).$$

Appendix §B. An identity for Jacobi polynomials

The following identity for Jacobi polynomials holds

$$P_{s-1}^{(m-s,j-s-1)}(1-2t) = \sum_{k=0}^{s-1} \frac{\Gamma(N-s-1+j+k)}{\Gamma(N-s-1+j)} \frac{(-t)^k}{k!} P_{s-k-1}^{(m-s+k,-N)}(1-2t). \quad \text{(B.1)}$$

To prove this identity we use the following representation for Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)} \sum_{i=0}^{n} \binom{n}{i} \frac{\Gamma(\alpha+\beta+n+i+1)}{\Gamma(\alpha+i+1)} \left(\frac{z-1}{2}\right)^i.$$

The LHS of identity (B.1) is

$$P_{s-1}^{(m-s,j-s-1)}(1-2t) = \frac{\Gamma(m)}{(s-1)!\Gamma(m+j-s-1)} \times \sum_{i=0}^{s-1} \binom{s-1}{i} \frac{\Gamma(m+j-s+i-1)}{\Gamma(m-s+i+1)} (-t)^i.$$

The RHS is

$$P_{s-k}^{(m-s+k,-N)}(1-2t) = \frac{\Gamma(m)}{(s-k-1)!\Gamma(m-N)} \times \sum_{l=0}^{s-k-1} \binom{s-k-1}{l} \frac{\Gamma(m-N+l)}{\Gamma(m-s+k+l-1)} (-t)^l.$$

Consider the coefficient of the term t^i in both sides. The identity then reads:

$$\frac{1}{(s-1)!\Gamma(m+j-s-1)} \binom{s-1}{i} \frac{\Gamma(m+j-s+i-1)}{\Gamma(m-s+i+1)} = \sum_{k=0}^{i} \frac{\Gamma(N-s-1+j+k)}{\Gamma(N-s-1+j)} \frac{(-1)^k}{k!} \frac{1}{(s-k-1)!\Gamma(m-N)} \times \binom{s-k-1}{i-k} \frac{\Gamma(m-N+i-k)}{\Gamma(m-s+i+1)} (-1)^{i-k}.$$

Indeed, to show that this relation is valid, it is suffice to prove that

$$\frac{(m+j-s-1)_i}{(m-N)_i} = \sum_{k=0}^{i} \frac{(N+j-s-1)_k(-i)_k}{(N-m-i+1)_k}\frac{1}{k!}. \tag{B.2}$$

This is indeed true, due to the Chu-Vandermonde identity

$${}_2F_1\left(\begin{matrix}-i,\, b\\ c\end{matrix}\middle|\,1\right) = \frac{(c-b)_i}{(c)_i},$$

with $b = N-s-1+j$ and $c = N-m-i+1$, for the RHS of (B.2) written in terms of a hypergeometric function.

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