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## QUANTUM EQUATION OF MOTION AND TWO-LOOP CUTOFF RENORMALIZATION FOR $\phi^{3}$ MODEL


#### Abstract

We present two-loop renormalization of $\phi^{3}$ model effective action by using the background field method and cutoff momentum regularization. In this paper we also study a derivation of the quantum equation of motion and its application to the renormalization procedure.


## §1. Introduction

Renormalization theory (see [1]) plays a crucial role in the quantum field theory and largely depends on the regularization. This work is devoted to the cutoff momentum one, which has its pros and cons. On the one hand it breaks invariance and adds the non-logarithmic divergences, but on the other hand it is more physical and retains the dimension. As a rule, for studying the properties of regularization and renormalization we often choose the simplest theory (not necessarily physical one), which clearly shows the process. We are going to work with a scalar $\phi^{3}$ model, which was used in the study of dimensional regularization in the four- (see [2]) and six-dimensional (see [3]) cases as well as for more intricate versions of the theory [4-10].

In the paper we present two-loop renormalization of the scalar $\phi^{3}$ theory with cutoff momentum regularization in $3,4,5$ dimensions (super-renormalizable cases), and in six dimensions (renormalizable case). We are going to use the background field method (see [11-14]), obtain a quantum equation of motion, and explain its applications to the renormalization theory.

First of all we need to introduce a Lagrangian density of the euclidian $\phi^{3}$ model

$$
\begin{equation*}
\mathcal{L}[\phi](x)=\frac{1}{2} \partial_{\mu} \phi(x) \partial^{\mu} \phi(x)+\frac{1}{2} m^{2} \phi^{2}(x)-\frac{g}{6} \phi^{3}(x), x \in \mathbb{R}^{n} \tag{1}
\end{equation*}
$$

[^0]where $m$ is a mass parameter, $g$ is a coupling constant, and $n$ is a dimension. Then we can define an action of the theory as $S[\phi]=\int_{\mathbb{R}^{n}} d^{n} x \mathcal{L}[\phi](x)$. Next we assume that the scalar field $\phi$ has a decreasing at infinity so one can integrate by parts and obtain a crucial property
\[

$$
\begin{equation*}
S[\phi+B]=S[B]+(M, \phi)+\frac{1}{2}(N \phi, \phi)-\frac{g}{6} \int_{\mathbb{R}^{n}} d^{n} x \phi^{3}(x) \tag{2}
\end{equation*}
$$

\]

where an operator $N$ and a field $M$ in the point $x \in \mathbb{R}^{n}$ are defined by the formulas

$$
\begin{align*}
& N(x)=-\partial_{\mu} \partial^{\mu}+m^{2}-g B(x) \\
& M(x)=-\partial_{\mu} \partial^{\mu} B(x)+m^{2} B(x)-\frac{g}{2} B^{2}(x) \tag{3}
\end{align*}
$$

and where $B$ is a background field, which will be defined below (see Sec. 5).

## §2. GREEN FUNCTION AND HEAT KERNEL

Let us introduce some extra definitions related to the operator $N(x)$. By $G(x, y)$ and $K(x, y ; \tau)$ we denote a Green function and a heat kernel respectively which satisfy the problems

$$
N(x) G(x, y)=\delta(x, y), \quad\left\{\begin{array}{l}
\left(\frac{\partial}{\partial \tau}+N(x)\right) K(x, y ; \tau)=0  \tag{4}\\
K(x, y ; 0)=\delta(x-y)
\end{array}\right.
$$

for all $x, y \in \mathbb{R}^{n}$ and $\tau \in \mathbb{R}_{+}$. Under the conditions described above, we have

$$
\begin{align*}
& \frac{\delta}{\delta B(z)} G(x, y)=g G(x, z) G(z, y) \\
& \frac{\delta}{\delta B(z)} K(x, y ; \tau)=g \int_{0}^{\tau} d s K(x, z ; \tau-s) K(z, y ; s) . \tag{5}
\end{align*}
$$

To prove the last formulas we need to apply the functional derivative, which satisfies the equality

$$
\begin{equation*}
\frac{\delta B(y)}{\delta B(x)}=\delta(x-y) \tag{6}
\end{equation*}
$$

to the problems (4) for the Green function and the heat kernel. Then we introduce a logarithm of determinant of the oparetor $N$ as the following
integral (see [15])

$$
\begin{equation*}
\ln \operatorname{det}\left(N /\left.N\right|_{B=0}\right)=-\int_{\mathbb{R}^{n}} d^{n} x \int_{0}^{\infty} \frac{d \tau}{\tau}\left[K(x, x ; \tau)-e^{-m^{2} \tau}\right] . \tag{7}
\end{equation*}
$$

Therefore, using the equality for the heat kernel

$$
\begin{equation*}
\int_{\mathbb{R}^{n}} d^{n} x K(z, x ; \tau) K(x, y ; s)=K(z, y ; \tau+s), \tag{8}
\end{equation*}
$$

one can find the first variation

$$
\begin{equation*}
\frac{\delta}{\delta B(x)} \ln \operatorname{det}(N)=-g G(x, x) . \tag{9}
\end{equation*}
$$

This equality makes sense for the regularized objects. Additional properties one can find in the Appendix A.

## §3. Diagram technique

For clarity it is convenient to introduce a diagram technique. We will denote the Green function $G(x, y)$ by a line with two indices $x$ and $y$, and the integration - by a dot. Let us give some examples of using the technique.

1) Let a functional $\rho(g, B)$ equals to one-particle irreducible (1PI) diagrams and their products from

$$
\begin{equation*}
\left.e^{\frac{g}{\mathbb{R}^{n}} \int^{n} d^{n} x\left(\frac{\delta}{\delta \eta(x)}\right)^{3}} e^{\frac{1}{2}(G \eta, \eta)}\right|_{\eta=0} \tag{10}
\end{equation*}
$$

It is just a sum of 1PI vacuum diagrams (and their products). On the Figure 1 one can see the first two terms of the expansion in powers of the coupling constant $g$. The next correction is multiplied by $g^{4}$.


Figure 1. The main terms of the $\rho(g, B)$.
2) Let us define an extended Green function $\mathcal{G}(x, y)$ as a sum of all 1PI contributions to the functional

$$
\begin{equation*}
\left.\frac{\delta}{\delta \eta(x)} \frac{\delta}{\delta \eta(y)} e^{\frac{g}{6} \int_{\mathbb{R}^{n}} d^{n} x\left(\frac{\delta}{\delta \eta(x)}\right)^{3}} e^{\frac{1}{2}(G \eta, \eta)}\right|_{\eta=0} \tag{11}
\end{equation*}
$$

then it takes the following view


Figure 2. The extended Green function with the first correction.

Lemma 1. Under the conditions described above $\rho(g, B) \mathcal{G}(x, y)$ equals to 1PI diagrams and their products from the functional (11).

This statement can be proved using combinatorial methods and binomial coefficients.

## §4. BACKGROUND FIELD METHOD

Primarily we need to enter an effective action $W$ as the path integral

$$
\begin{equation*}
e^{-W}=\int_{H} \mathcal{D} \phi e^{-S[\phi]} \tag{12}
\end{equation*}
$$

where $H$ is a functional set, which is determined by using physical reasons. Actually the effective action is a function of the set $H$. Then, according to the background field method, we do a shift $\phi \rightarrow \phi+B$. So, using the formula (2), we get

$$
\begin{equation*}
e^{-W[B]}=e^{-S[B]} \int_{H_{0}} \mathcal{D} \phi e^{-(M, \phi)-\frac{1}{2}(N \phi, \phi)+\frac{g}{6} \int_{\mathbb{R}^{n}} d^{n} x \phi^{3}(x)} \tag{13}
\end{equation*}
$$

where $H_{0}=\{\phi-B: \phi \in H\}$ is a new set of integration after the shift $H \rightarrow$ $H_{0}$. We suppose that the dependence of $W=W[B]$ on $H$ is dictated by the background field $B$, which will be defined below by using the quantum equation of motion. Then we do one more shift $\phi \rightarrow \phi+G \eta$, where $G$ is an
integration operator with the kernel $G(x, y)$ and $\eta$ is a smooth auxiliary field. In this case we have

$$
\begin{align*}
& \int_{H_{0}} \mathcal{D} \phi e^{-(M, \phi)-\frac{1}{2}(N \phi, \phi)+\frac{g}{6} \int_{\mathbb{R}^{n}} d^{n} x \phi^{3}(x)} \\
& \quad=\left.\operatorname{det}(N)^{-1 / 2} e^{-\left(M, \frac{\delta}{\delta \eta}\right)+\frac{g}{6} \int_{\mathbb{R}^{n}} d^{n} x\left(\frac{\delta}{\delta \eta(x)}\right)^{3}} e^{\frac{1}{2}(G \eta, \eta)}\right|_{\eta=0} \tag{14}
\end{align*}
$$

where we fixed the normalization property of the measure from the formula (12) by using the following condition

$$
\begin{equation*}
\int_{H_{0}} \mathcal{D} \phi e^{-\frac{1}{2}(N \phi, \phi)}=[\operatorname{det}(N)]^{-1 / 2} \tag{15}
\end{equation*}
$$

## §5. QuANTUM EQUATION OF MOTION

Let us obtain the equation of motion. For this purpose we need to find two kinds of contributions to the effective action $W[B]$.
Lemma 2. The coefficient for $(G M, M)$ in $W[B]$, consisting of 1 PI diagrams and their products, equals to $\frac{1}{2} \rho(g, B)$.
Lemma 3. The effective action $W[B]$ contains terms, which have one $M$ vertex and can be represented as a product of 1 PI diagrams. The sum of all such contributions equals to $-\frac{g}{2} \rho(g, B) \int_{\mathbb{R}^{n}} d^{n} x G M(x) \mathcal{G}(x, x)$.

Proof. To find the contribution we need to consider the chain of equalities. The first one is

$$
\begin{align*}
- & \left.\left(M, \frac{\delta}{\delta \eta}\right) e^{\frac{g}{6} \int \mathbb{R}^{n} d^{n} x\left(\frac{\delta}{\delta \eta(x)}\right)^{3}} e^{\frac{1}{2}(G \eta, \eta)}\right|_{\eta=0} \\
& =-\left.e^{\frac{g}{6} \int_{\mathbb{R}^{n}} d^{n} x\left(\frac{\delta}{\delta \eta(x)}\right)^{3}}(G M, \eta) e^{\frac{1}{2}(G \eta, \eta)}\right|_{\eta=0} \tag{16}
\end{align*}
$$

Then we need to use properties of the functional derivative in the form

$$
\begin{equation*}
\left[e^{\frac{g}{6} \int_{\mathbb{R}^{n}} d^{n} x\left(\frac{\delta}{\delta \eta(x)}\right)^{3}},(G M, \eta)\right]=\frac{g}{2} e^{\frac{g}{6} \int_{\mathbb{R}^{n}} d^{n} x\left(\frac{\delta}{\delta \eta(x)}\right)^{3}}\left(G M, \frac{\delta^{2}}{\delta \eta^{2}}\right) \tag{17}
\end{equation*}
$$

Finally, the statement follows from Lemma 1.

Therefore we can give a definition of the quantum equation of motion. Using the results of Lemmas 2 and 3, one can write down the equation in the following form

$$
\begin{equation*}
M(x)=\frac{g}{2} \mathcal{G}(x, x) \tag{18}
\end{equation*}
$$

where $x \in \mathbb{R}^{n}$. Of course, it contains the divergencies, so we should understand it in the regularization sence. It is very easy to see that the equation is nonlinear with respect to the background field. In particular case, after regularization we can express a trace part of the Green function

$$
\begin{equation*}
G(x, x)=\frac{2}{g} M(x)+O\left(g^{2}\right) \tag{19}
\end{equation*}
$$

Now we can define the background field $B$ as a solution of the problem which consists of the quantum equation of motion (18) and asymptotic behaviour at infinity. The last condition is taken from the definition of $H$.

Theorem 1. Under the conditions described above for all $x$ from $\mathbb{R}^{n}$ we have in the regularization sence

$$
\begin{equation*}
\frac{\delta}{\delta B(x)} W[B]=M(x)-\frac{g}{2} \mathcal{G}(x, x) \tag{20}
\end{equation*}
$$

The last expression follows from the formulas (3) and (9), and definition of the function $\mathcal{G}(x, y)$. From the equalities (13) and (14) one can express the effective action, which after using Theorem 1 has the following form


Figure 3. The effective action with the 1PI corrections.

In particular, it means that diagrams such as "glasses" are cancelled.

## §6. REGULARIZATION

There are a lot of ways to do regularization (dimensional one, Pauli Villars one, and other). We are going to use the cutoff momentum regularization in a special form. It should be noted that we are going to find infrared divergencies in the coordinate representation. It means one
should regularize the Green function expansion when $x \sim y$. The rules are following:
(1) The factor $r^{-k}$ with $k \in \mathbb{N}$ goes to $\chi_{r \Lambda>1} r^{-k}$;
(2) The factor $\ln r$ goes to $\chi_{r \Lambda>1} \ln r-\chi_{r \Lambda \leqslant 1} \ln \Lambda$,
where $\chi_{(a, b)}$ is a characteristic function of $(a, b)$, and $\Lambda$ is a parameter of regularization. It means that $G^{\Lambda} \rightarrow G$ in the sence of generalized functions when $\Lambda \rightarrow+\infty$. In this case one can write down the trace parts of Green function for $n=3,4,5,6$ dimensional cases:

$$
\begin{align*}
& G_{3}^{\Lambda}(y, y)=P S_{3}(y, y)  \tag{21}\\
& G_{4}^{\Lambda}(y, y)=\frac{L}{8 \pi^{2}} a_{1}(y, y)+P S_{4}(y, y)  \tag{22}\\
& G_{5}^{\Lambda}(y, y)=P S_{5}(y, y)  \tag{23}\\
& G_{6}^{\Lambda}(y, y)=\frac{L}{32 \pi^{3}} a_{2}(y, y)+P S_{6}(y, y), \tag{24}
\end{align*}
$$

where the bottom index corresponds to the dimension of the space and $L=$ $\ln (\Lambda / \mu)$. The last equalities do not violate the limit transition for Green function $G^{\Lambda}(x, y)$, they just redefine the value on the diagonal $x=y$. Of course, the Green function after the cutoff regularization has logarithmic $L$ and powers $\Lambda$ singularities. The second kind of ones has a different nature (see for example [16,17]).

## §7. Renormalization

The process of renormalization is based on redefining of model parameters $m^{2}, \phi$, and $g$. We are going to consider renormalizable case, when $n=6$, and then super-renormalizable cases, when $n=3,4,5$. For the convenience one introduces some extra types of sign " $=$ ". The letters IR $(\stackrel{\text { IR }}{=})$ mean that both sides of an equality contain the same infrared singular contributions without consideration of parts which are proportional to the zero or the first degree of the background field $B$. Let us also note that we will use the logic and notations proposed in the work [18].
7.1. $\mathbf{n}=\mathbf{6}$ dimensional case. In the renormalizable case we have an infinite number of divergencies, and thus we need to find the renormalization constants $Z, Z_{0}$, and $Z_{m}$. Using the fact that the process of renormalization is equivalent to the transitions

$$
\begin{equation*}
\phi \rightarrow \sqrt{Z} \phi, \quad g \rightarrow Z_{0} Z^{-\frac{3}{2}} g, \quad m^{2} \rightarrow Z_{m} Z^{-1} m^{2}, \tag{25}
\end{equation*}
$$

which cancel the singularities, we plan to consider two-loop renormalization. Using the Lagrangian density (1) one can conclude that only finite number of the coefficients should be found:

$$
\begin{align*}
Z_{0}(g) & =1-a_{12} g^{2} L-a_{14} g^{4} L-a_{24} g^{4} L^{2}+o\left(g^{4}\right)  \tag{26}\\
Z_{m}(g) & =1-b_{12} g^{2} L-b_{14} g^{4} L-b_{24} g^{4} L^{2}+o\left(g^{4}\right)  \tag{27}\\
Z(g) & =1-c_{12} g^{2} L-c_{14} g^{4} L-c_{24} g^{4} L^{2}+o\left(g^{4}\right) \tag{28}
\end{align*}
$$

Firstly we find the coefficients proportional to $g^{2} L$. For this purpose we need to consider a singularity from the one-loop correction. It follows from the formulas (9) and (54), that the singilar logarithmic part has the following from

$$
\begin{equation*}
\ln \operatorname{det}(N) \stackrel{\mathrm{IR}}{=}-\frac{L}{32 \pi^{3}} \int_{\mathbb{R}^{6}} d^{6} x a_{3}(x, x) \tag{29}
\end{equation*}
$$

Thereby the contribution to the effective action has a view

$$
\begin{equation*}
\frac{1}{2} \ln \operatorname{det}(N) \stackrel{\mathrm{IR}}{=} \frac{g^{2} L}{6(4 \pi)^{3}} \frac{\left(\partial_{\mu} B, \partial_{\mu} B\right)}{2}+\frac{m^{2} g^{2} L}{(4 \pi)^{3}} \frac{(B, B)}{2}-\frac{g^{3} L}{(4 \pi)^{3}} \frac{\left(B^{2}, B\right)}{6} \tag{30}
\end{equation*}
$$

and the coefficients are

$$
\begin{equation*}
c_{12}=\frac{1}{6(4 \pi)^{3}}, \quad b_{12}=\frac{1}{(4 \pi)^{3}}, \quad a_{12}=\frac{1}{(4 \pi)^{3}} \tag{31}
\end{equation*}
$$

Let us find the coefficients proportional to $g^{4} L$. They appear from the two-loop correction. Summig up all terms from the formulas (71)-(72) and (74)-(75), and using the equalities (58), (13), and (14), one can obtain the contribution to the effective action as

$$
\begin{equation*}
-\frac{11 g^{4} L}{36(4 \pi)^{6}} \frac{\left(\partial_{\mu} B, \partial_{\mu} B\right)}{2}+\frac{m^{2} g^{4} L}{6(4 \pi)^{6}} \frac{(B, B)}{2}-\frac{g^{5} L}{6(4 \pi)^{6}} \frac{\left(B^{2}, B\right)}{6} \tag{32}
\end{equation*}
$$

It means that the coefficients are

$$
\begin{equation*}
c_{14}=-\frac{11}{36(4 \pi)^{6}}, \quad b_{14}=\frac{1}{6(4 \pi)^{6}}, \quad a_{14}=\frac{1}{6(4 \pi)^{6}} . \tag{33}
\end{equation*}
$$

In the same way, using the formulas (77)-(79), a contribution proportional to $g^{4} L^{2}$ is the following

$$
\begin{equation*}
\frac{5 g^{4} L^{2}}{36(4 \pi)^{6}} \frac{\left(\partial_{\mu} B, \partial_{\mu} B\right)}{2}+\frac{5 m^{2} g^{4} L^{2}}{4(4 \pi)^{6}} \frac{(B, B)}{2}-\frac{5 g^{5} L^{2}}{4(4 \pi)^{6}} \frac{\left(B^{2}, B\right)}{6} \tag{34}
\end{equation*}
$$

so we have

$$
\begin{equation*}
c_{24}=\frac{5}{36(4 \pi)^{6}}, \quad b_{24}=\frac{5}{4(4 \pi)^{6}}, \quad a_{24}=\frac{5}{4(4 \pi)^{6}} . \tag{35}
\end{equation*}
$$

The coefficients, obtained above, are in full agreement with the results obtained earlier (see [3]) in the case of dimensional regularization. We deliberately did not take into account the contributions of the type (73). A sum of all such terms equals to $-\frac{5}{6} \frac{g^{2} L}{2(4 \pi)^{3}} \int d^{6} x v(x) P S_{6}(x, x)$ and will be considered in the Remark 1 in the Section 7.3. Also it should be noted that the two-loop correction contains a term of view (see formulas (69) and (76))

$$
\begin{equation*}
\frac{g \Lambda^{2}}{2(4 \pi)^{3}} \int_{\mathbb{R}^{6}} d^{6} x \frac{\delta}{\delta B(x)} \ln \operatorname{det}(N)+\frac{g^{2} \Lambda^{2}}{2(4 \pi)^{6}} \int_{\mathbb{R}^{6}} d^{6} x a_{2}(x, x) . \tag{36}
\end{equation*}
$$

It seems that the first term contains a high degree of the field $B$, but it is not so. One can use the expansion of the quantum equation of motion in the form (19). Therefore

$$
\begin{equation*}
\frac{\delta}{\delta B(x)} \ln \operatorname{det}(N) \sim-g B^{2}(x)+O\left(g^{3}\right), \tag{37}
\end{equation*}
$$

where the terms proportional to $B^{1}$ and $B^{0}$ are not taken into account. Further using the formula (56), we can rewrite the contribution as

$$
\begin{equation*}
-\frac{g^{2} \Lambda^{2}}{(4 \pi)^{3}}\left(1-\frac{g^{2}}{2(4 \pi)^{3}}\right) \frac{(B, B)}{2} . \tag{38}
\end{equation*}
$$

Actually the singularity $\Lambda^{2}$ has a different nature and can be eliminated by redefining a regularized trace part of Green function, or by renormalization of the mass parameter.
7.2. $\mathrm{n}=5$ dimensional case. In the five-dimensional case we have only finite number of divergencies. From the formula (23) it follows that the one-loop correction does not have singularities. Thereby from the equalities (62)-(66) we obtain a contribution to the effective action:

$$
\begin{equation*}
-\frac{g^{4} L}{12(4 \pi)^{4}} \frac{(B, B)}{2}+\frac{g \Lambda}{6(4 \pi)^{2}} \int_{\mathbb{R}^{5}} d^{5} x \frac{\delta}{\delta B(x)} \ln \operatorname{det}(N), \tag{39}
\end{equation*}
$$

where the formulas $S^{4}=\frac{8}{3} \pi^{2}$ and (9) were used. The second term in the last formula also can be considered by using the quantum equation of
motion in the form (37). Therefore to do the renormalization we need to shift only the mass parameter in the following way

$$
\begin{equation*}
m^{2} \longrightarrow m^{2}+\frac{g^{4}}{12(4 \pi)^{4}} L \tag{40}
\end{equation*}
$$

7.3. $n=4$ dimensional case. The divergencies in the effective action in the four-dimensional case follow from the equalities (52) and (22), and formulas (59) and (60). So the contributions from first two loops have a form

$$
\begin{equation*}
-\frac{g^{2} L}{(4 \pi)^{2}} \frac{(B, B)}{2}-\frac{g^{2} L}{2(4 \pi)^{2}} \int_{\mathbb{R}^{4}} d^{4} x P S_{4}(x, x) \tag{41}
\end{equation*}
$$

In this case we have only logarithmic divergencies. To renormalize the effective action only the mass parameter should be shifted as follows

$$
\begin{equation*}
m^{2} \longrightarrow m^{2}+\frac{g^{2}}{(4 \pi)^{2}} L \tag{42}
\end{equation*}
$$

The four-dimensional case is the super-renormalizable one, therefore let us see how the second singularity in the formula (41) is cancelled. Let $\sigma$ be a finite part of the $\ln \operatorname{det}(N)$ such that

$$
\begin{equation*}
g \frac{\delta \sigma}{\delta v(x)}=\frac{\delta \sigma}{\delta B(x)}=-g P S_{4}(x, x) \tag{43}
\end{equation*}
$$

where $v(x)=-m^{2}+g B(x)$. Hence using the shift (42) the effective action $W[B]$ after one-loop renormalization contains the term

$$
\begin{equation*}
\left.\frac{1}{2}\left(\sigma+\frac{g^{2} L}{(4 \pi)^{2}} \int_{\mathbb{R}^{4}} d^{4} x \frac{\delta \sigma}{\delta v(x)}\right)\right|_{m^{2} \rightarrow m^{2}+\frac{g^{2}}{(4 \pi)^{2}} L} \tag{44}
\end{equation*}
$$

However all objects are constructed by using the Green function. It means that they are functionals of the field $v(x)=-m^{2}+g B(x)$. At the same time the operator

$$
\begin{equation*}
\exp \left(\frac{g^{2} L}{(4 \pi)^{2}} \int_{\mathbb{R}^{4}} d^{4} x \frac{\delta}{\delta v(x)}\right) \tag{45}
\end{equation*}
$$

does a shift in the following form

$$
\begin{equation*}
v(x) \longrightarrow v(x)+\frac{g^{2} L}{(4 \pi)^{2}} . \tag{46}
\end{equation*}
$$

So one can obtain that formula (44) equals to $\frac{1}{2} \sigma$ plus term which is cancelled by the next high loop corrections. It is supposed that the same calculations can be done for a finite part of the two-loop correction, using the high loop contributions.
Remark 1. Let us get back to the case $n=6$, where we noted that the term

$$
\begin{equation*}
-\frac{5}{6} \frac{g^{2} L}{2(4 \pi)^{3}} \int d^{6} x v(x) P S_{6}(x, x) \tag{47}
\end{equation*}
$$

exists. By $\sigma$ we denote such part of the $\ln \operatorname{det}(N)$ that $\frac{\delta \sigma}{\delta v(x)}=-P S_{6}(x, x)$. Drawing an analogy with the $n=4$ case we see that the term (47) is a part of exponential operator which transforms the potential in the $\sigma$ from the value $v$ to the $v+\frac{5}{6} \frac{g^{2} L}{(4 \pi)^{3}} v$. At the same time after one-loop renormalization we have the shift

$$
\begin{align*}
v(x) & =-m^{2}+g B(x) \rightarrow-Z_{m} Z^{-1} m^{2}+Z_{0} Z^{-1} g B(x) \\
& =v-\frac{5}{6} \frac{g^{2} L}{(4 \pi)^{3}} v+\ldots \tag{48}
\end{align*}
$$

It means that the shifts cancel each other. A similar procedure should work in the high loops.

## §8. Appendix A

It is very well known (see [19]) that we can represent the heat kernel $K(x, y ; \tau)$ as a series in powers of a proper time $\tau$. The coefficients $a_{k}(x, y)$, $k \in \mathbb{N}$, of the expansion satisfy the problem

$$
\left\{\begin{array}{l}
a_{0}(x, y)=1  \tag{49}\\
\left(k+(x-y)^{\mu} \partial_{\mu}\right) a_{k}(x, y)=\left(\partial_{\mu} \partial^{\mu}+v(x)\right) a_{k-1}(x, y), k>0
\end{array}\right.
$$

and are called Seeley-DeWitt coefficients. They play an important role in physics. In particular case, they give an asymptotic expansion for the Green function $G_{n}(x, y)$ when $x \sim y$. Let us introduce some notations:

$$
\begin{equation*}
(x-y)^{\mu_{1} \ldots \mu_{k}}=(x-y)^{\mu_{1}} \ldots(x-y)^{\mu_{k}}, \quad \partial_{\mu_{1} \ldots \mu_{k}}=\partial_{\mu_{1}} \ldots \partial_{\mu_{k}} \tag{50}
\end{equation*}
$$

where $k \in \mathbb{N}$ and $\mu_{i} \in\{1, \ldots, n\}$. So we can write down the expansions for $n=3,4,5,6$ :

$$
\begin{equation*}
G_{3}(x, y)=\frac{1}{4 \pi r}-\frac{r}{8 \pi} a_{1}(x, y)+P S_{3}(x, y)+o(r) \tag{51}
\end{equation*}
$$

$$
\begin{align*}
G_{4}(x, y) & =\frac{1}{4 \pi^{2} r^{2}}-\frac{\ln (r \mu)^{2}}{16 \pi^{2}} a_{1}(x, y)  \tag{52}\\
& +\frac{r^{2} \ln (r \mu)^{2}}{64 \pi^{2}} a_{2}(x, y)+P S_{4}(x, y)+o\left(r^{2} \ln r^{2}\right) ; \\
G_{5}(x, y)=\frac{1}{8 \pi^{2} r^{3}} & +\frac{1}{16 \pi^{2} r} a_{1}(x, y)-\frac{r}{32 \pi^{2}} a_{2}(x, y)+P S_{5}(x, y)+o(r) ;  \tag{53}\\
G_{6}(x, y) & =\frac{1}{4 \pi^{3} r^{4}}+\frac{1}{16 \pi^{3} r^{2}} a_{1}(x, y)-\frac{\ln (r \mu)^{2}}{64 \pi^{3}} a_{2}(x, y)  \tag{54}\\
& +\frac{r^{2} \ln (r \mu)^{2}}{256 \pi^{3}} a_{3}(x, y)+P S_{6}(x, y)+o\left(r^{2} \ln r^{2}\right),
\end{align*}
$$

where $r=|x-y|, P S_{k}(x, y)$ for $k=3,4,5,6$ are regular parts and depend on $\mu$, although $G(x, y)$ does not (see [20]). The first three coefficients have the forms (from [21-23]):

$$
\begin{align*}
& a_{1}(x, y)=v(y)+\frac{1}{2}(x-y)^{\mu} \partial_{\mu} v(y)+\frac{1}{6}(x-y)^{\mu \nu} \partial_{\mu \nu} v(y) \\
&+\frac{1}{24}(x-y)^{\mu \nu \rho} \partial_{\mu \nu \rho} v(y)+\frac{1}{120}(x-y)^{\mu \nu \rho \sigma} \partial_{\mu \nu \rho \sigma} v(y)+o\left(r^{4}\right)  \tag{55}\\
& a_{2}(x, y)=\frac{1}{6} \partial_{\mu \mu} v(y)+\frac{1}{2} v^{2}(y)+\frac{1}{12}(x-y)^{\mu} \partial_{\mu \nu \nu} v(y) \\
&+\frac{1}{2} v(y)(x-y)^{\mu} \partial_{\mu} v(y)+\frac{1}{40}(x-y)^{\nu \rho} \partial_{\nu \rho \mu \mu} v(y)  \tag{56}\\
&+\frac{1}{8}\left((x-y)^{\mu} \partial_{\mu} v(y)\right)^{2} \\
&+\frac{1}{6} v(y)(x-y)^{\mu \nu} \partial_{\mu \nu} v(y)+o\left(r^{2}\right) \\
& a_{3}(y, y)= \frac{1}{60} \partial_{\mu \mu \nu \nu} v(y)+\frac{1}{6} v^{3}(y)+\frac{1}{12} \partial_{\mu} v(y) \partial_{\mu} v(y)+\frac{1}{6} v(y) \partial_{\mu \mu} v(y) \tag{57}
\end{align*}
$$

At the same time after applying the operator $N(x)$ to the equality (54) and using the Green function definition we have the following equality for $n=6$

$$
\begin{equation*}
-\frac{a_{3}(y, y)}{16 \pi^{3}}-v(y) P S_{6}(y, y)-\left.\partial_{\mu} \partial^{\mu} P S_{6}(x, y)\right|_{x=y}=0 \tag{58}
\end{equation*}
$$

## §9. Appendix B

9.1. $\mathbf{n}=4$ : In the four-dimensional case the two-loop diagram contains only two singularities, which can be obtained by using formulas (52)
and (55):

$$
\begin{gather*}
3 \int d^{4} x d^{4} y\left(\frac{1}{4 \pi^{2} r^{2}}\right)^{2}\left(-\frac{\ln (r \mu)^{2}}{16 \pi^{2}} v(y)\right) \stackrel{\mathrm{IR}}{=}-\frac{3 S^{3}}{2^{8} \pi^{6}} \int d^{4} y v(y) L^{2}  \tag{59}\\
3 \int d^{4} x d^{4} y\left(\frac{1}{4 \pi^{2} r^{2}}\right)^{2} P S_{4}(x, y) \stackrel{\mathrm{IR}}{=} \frac{3 S^{3}}{2^{4} \pi^{4}} \int d^{4} y P S_{4}(y, y) L \tag{60}
\end{gather*}
$$

9.2. $\mathbf{n}=5$ : In the five-dimensional case we have five terms with singularities, among which there are not only logarithmic. So, to obtain them, we need to use the expressions (53), (55), (56), and the equality of view

$$
\begin{equation*}
\int d^{n} x\left(x_{j}-y_{j}\right)^{2} f(r)=\frac{1}{n} \int d^{n} x r^{2} f(r), \quad j \in\{1, \ldots, n\} . \tag{61}
\end{equation*}
$$

So we have the contributions, which are proportional to $\Lambda^{2}, \Lambda$, and $L$ :

$$
\begin{array}{r}
3 \int d^{5} x d^{5} y\left(\frac{1}{8 \pi^{2} r^{3}}\right)^{2} \frac{v(y)}{16 \pi^{2} r} \stackrel{\mathrm{IR}}{=} \frac{3 S^{4}}{2^{11} \pi^{6}} \int d^{5} y v(y) \Lambda^{2} ; \\
3 \int d^{5} x d^{5} y\left(\frac{1}{8 \pi^{2} r^{3}}\right)^{2} P S_{5}(x, y) \stackrel{\mathrm{IR}}{=} \frac{3 S^{4}}{2^{6} \pi^{4}} \int d^{5} y P S_{5}(y, y) \Lambda ; \\
3 \int d^{5} x d^{5} y \frac{1}{8 \pi^{2} r^{3}}\left(\frac{v(y)}{16 \pi^{2} r}\right)^{2} \stackrel{\mathrm{IR}}{=} \frac{3 S^{4}}{2^{11} \pi^{6}} \int d^{5} y v^{2}(y) L ; \\
3 \int d^{5} x d^{5} y\left(\frac{1}{8 \pi^{2} r^{3}}\right)^{2} \frac{(x-y)^{\mu \nu} \partial_{\mu \nu} v(y)}{6 \cdot 16 \pi^{2} r} \stackrel{\mathrm{IR}}{=} \frac{S^{4}}{2^{11} 5 \pi^{6}} \int d^{5} y \partial_{\mu \mu} v(y) L ; \\
3 \int d^{5} x d^{5} y\left(\frac{1}{8 \pi^{2} r^{3}}\right)^{2}\left(-\frac{r}{32 \pi^{2}} a_{2}(x, y)\right) \\
\stackrel{\mathrm{IR}}{=}-\frac{3 S^{4}}{2^{12} \pi^{6}} \int d^{5} y\left(\frac{1}{3} \partial_{\mu \mu} v(y)+v^{2}(y)\right) L . \tag{66}
\end{array}
$$

9.3. $\mathbf{n}=\mathbf{6}$ : In the six-dimensional case we have 13 contributions with singularities, among which there are not only logarithmic. To calculate the divergencies, we are going to use the expressions (54), (55), (56), (57), and (61). Then we have:

$$
\begin{gather*}
3 \int d^{6} x d^{6} y\left(\frac{1}{4 \pi^{3} r^{4}}\right)^{2} \frac{v(y)}{16 \pi^{3} r^{2}} \stackrel{\text { IR }}{=} \frac{3 S^{5}}{2^{10} \pi^{9}} \int d^{6} y v(y) \Lambda^{4} ;  \tag{67}\\
3 \int d^{6} x d^{6} y\left(\frac{1}{4 \pi^{3} r^{4}}\right)^{2} \frac{(x-y)^{\mu \nu} \partial_{\mu \nu} v(y)}{6 \cdot 16 \pi^{3} r^{2}} \stackrel{\mathrm{IR}}{=} \frac{S^{5}}{2^{11} 3 \pi^{9}} \int d^{6} y \partial_{\mu \mu} v(y) \Lambda^{2} ; \tag{68}
\end{gather*}
$$

$$
\begin{align*}
& 3 \int d^{6} x d^{6} y\left(\frac{1}{4 \pi^{3} r^{4}}\right)^{2} P S_{6}(y, y) \stackrel{\mathrm{IR}}{=} \frac{3 S^{5}}{2^{5} \pi^{6}} \int d^{6} y P S_{6}(y, y) \Lambda^{2} ;  \tag{69}\\
& 3 \int d^{6} x d^{6} y\left(\frac{1}{4 \pi^{3} r^{4}}\right)^{2} \frac{(x-y)^{\mu \nu \rho \sigma} \partial_{\mu \nu \rho \sigma} v(y)}{120 \cdot 16 \pi^{3} r^{2}} \\
& \stackrel{\mathrm{IR}}{=} \frac{1}{2^{15} 5 \pi^{6}} \int d^{6} y\left(\sum_{\mu, \nu=1}^{6} \partial_{\mu}^{2} \partial_{\nu}^{2} v(y)\right) L  \tag{70}\\
& \int d^{6} x d^{6} y\left(\frac{v(y)}{16 \pi^{3} r^{2}}\right)^{3} \stackrel{\mathrm{IR}}{=} \frac{S^{5}}{2^{12} \pi^{9}} \int d^{6} y v^{3}(y) L ;  \tag{71}\\
& 6 \int d^{6} x d^{6} y \frac{1}{4 \pi^{3} r^{4}} \frac{v(y)}{16 \pi^{3} r^{2}} \frac{(x-y)^{\mu \nu} \partial_{\mu \nu} v(y)}{6 \cdot 16 \pi^{3} r^{2}} \\
& \stackrel{\mathrm{IR}}{=} \frac{S^{5}}{2^{11} 3 \pi^{9}} \int d^{6} y v(y) \partial_{\mu \mu} v(y) L ;  \tag{72}\\
& 6 \int d^{6} x d^{6} y \frac{1}{4 \pi^{3} r^{4}} \frac{v(y)}{16 \pi^{3} r^{2}} P S_{6}(x, y) \stackrel{\mathrm{IR}}{=} \frac{3 S^{5}}{2^{5} \pi^{6}} \int d^{6} y v(y) P S_{6}(y, y) L ;  \tag{73}\\
& 3 \int d^{6} x d^{6} y\left(\frac{1}{4 \pi^{3} r^{4}}\right)^{2} \frac{1}{2}(x-y)^{\mu \nu} \partial_{\mu \nu} P S_{6}(x, y) \\
& \left.\stackrel{\mathrm{IR}}{=} \frac{S^{5}}{2^{6} \pi^{6}} \int d^{6} y \partial_{\mu \mu} P S_{6}(x, y)\right|_{x=y} L ;  \tag{74}\\
& 3 \int d^{6} x d^{6} y \frac{1}{4 \pi^{3} r^{4}}\left(\frac{(x-y)^{\mu} \partial_{\mu} v(y)}{2 \cdot 16 \pi^{3} r^{2}}\right)^{2} \stackrel{\mathrm{IR}}{=} \frac{S^{5}}{2^{13} \pi^{9}} \int d^{6} y \partial_{\mu} v(y) \partial_{\mu} v(y) L ;  \tag{75}\\
& 3 \int d^{6} x d^{6} y\left(\frac{1}{4 \pi^{3} r^{4}}\right)^{2}\left(-\frac{\ln (r \mu)^{2}}{64 \pi^{3}} a_{2}(y, y)\right) \\
& \stackrel{\mathrm{IR}}{=}-\frac{3 S^{5}}{2^{9} \pi^{9}} \int d^{6} y a_{2}(y, y)\left(\frac{1}{4} \Lambda^{2}-\frac{1}{2} \Lambda^{2} L\right) ;  \tag{76}\\
& 3 \int d^{6} x d^{6} y\left(\frac{1}{4 \pi^{3} r^{4}}\right)^{2}\left(-\frac{\ln (r \mu)^{2}}{64 \pi^{3}} \frac{1}{2}(x-y)^{\nu \rho} \partial_{\nu \rho} a_{2}(x, y)\right) \\
& \stackrel{\mathrm{IR}}{=} \frac{S^{5}}{2^{11} \pi^{9}} \int d^{6} y\left(\left.\frac{1}{2} \partial_{\mu \mu} a_{2}(x, y)\right|_{x=y}\right) L^{2} ; \tag{77}
\end{align*}
$$

$$
\begin{array}{r}
3 \int d^{6} x d^{6} y\left(\frac{1}{4 \pi^{3} r^{4}}\right)^{2}\left(\frac{r^{2} \ln (r \mu)^{2}}{256 \pi^{3}} a_{3}(y, y)\right) \\
\stackrel{\mathrm{IR}}{=}-\frac{3 S^{5}}{2^{12} \pi^{9}} \int d^{6} y a_{3}(y, y) L^{2} \\
6 \int d^{6} x d^{6} y \frac{1}{4 \pi^{3} r^{4}} \frac{v(y)}{16 \pi^{3} r^{2}}\left(-\frac{\ln (r \mu)^{2}}{64 \pi^{3}} a_{2}(y, y)\right) \\
\stackrel{\mathrm{IR}}{=} \frac{3 S^{5}}{2^{11} \pi^{9}} \int d^{6} y v(y) a_{2}(y, y) L^{2} \tag{79}
\end{array}
$$

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