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NOTES ON FUNCTIONAL INTEGRATION

Abstract. This work is devoted to the construction of an "integral" on an infinite-dimensional space which combines the approaches proposed earlier and at the same time is the simpliest one. We give a new definition of the construction and study its properties on a special class of functionals. We also consider the introduction of a quasi-scalar product, an orthonormal system, and applications in physics (path integral, loop space, functional derivative). In addition the paper contains a discussion of generalized functionals.

§1. Introduction

The term "functional integral" has a long history and was first proposed by Daniell [1] in 1919 and since then it has been playing an important role in theoretical (see [2,3]) and mathematical physics (see [4]). There are many ways to understand this construction, each of which contains deep mathematical problems. To begin the description of the ideas of the paper it is necessary to mention the major ways of the construction developments, to point out some problems, and also to emphasize necessary properties which functional integral should satisfies.

The first way to define the construction has a probabilistic nature and was proposed and developed by many authors (for example see [4–6]). It is related to the study of Gaussian measures in the infinite-dimensional spaces. It is known that Wick's theorem holds in this case but there is a big problem with Kolmogorov's zero-one law, which leads to divergences. It is also not clear how to define a measure on the chosen infinite-dimensional subsets.

The second way was proposed by Feynman [7] and it is more physical in nature. The main idea is to use a limit of piecewise constant functions. It means that we consider a transition from $N\sum\limits_{k=1}^{N}(x_{k+1}-x_k)^2$ to $\int\limits_0^1 ds\,\dot{v}^2(s)$,

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when $N \to \infty$ and where v is a limit function. There are several challenges. The main ones are the lack of information about the function v (smoothness, number of break points) and the difference of countable and uncountable sets. Of course, the approach keeps Wick's theorem and linearity of the path integral, but at the same time it does not allow you to make estimates.

There are many other ways, proposed by Wiener [8], Kac [9], Smolyanov and Shavgulidze [10], and many other authors. The main aim of this work is to introduce an object, which would combine the features of the first two approaches. It means that we should have such properties as Wick's theorem, a complex linearity, and an "integration of inequality", and at the same time we want to have an explicit way to do integration. Also we are going to give a physical meaning of the "integral", not by functions from a domain, but by means of a spectral parameter which corresponds to a Sturm–Liouville problem.

§2. The main construction

Let $\mathcal{V} = \mathbb{R}^\infty$ be a Cartesian product of infinite number of $\mathbb{R}$ with the Tikhonov topology. Elements of the space $\mathcal{V}$ are sequences, so if $x \in \mathcal{V}$ then $x = \{x_k\}_{k=1}^\infty$. A scalar product is the real one and defined by the formula $(x, y) = \sum\limits_{j=1}^{\infty} x_j y_j$. By symbol x^α, where x and α are from $\mathcal{V}$, we will denote the sequence $\{\alpha_k x_k\}_{k=1}^\infty$ as alement from $\mathcal{V}$. Then we introduce two sets which will be used in the next sections:

$$\mathcal{A} = \{x \in \mathcal{V} : x_k \in \mathbb{N} \cup \{0\} \,\text{for all}\, k\}\,, \ \mathcal{B} = \{x \in \mathcal{V} : x_k > 0 \,\text{for all}\, k\}\,. \quad (1)$$

2.1. Definition. First of all we need to introduce some extra notations. Let x be from $\mathcal{V}$ then by $\mathbb{X}_n$ we will denote an element from $\mathcal{V}$ according to the formula: $(\mathbb{X}_n)_k = x_k$ for $k \in (1, \ldots, n)$, and $(\mathbb{X}_n)_k = 0$ for $k > n$. For instance, under the conditions described above, we have $(\mathbb{X}_n^\alpha, \mathbb{X}_n) = \sum\limits_{k=1}^{n} \alpha_k x_k^2$ for all α from $\mathcal{V}$. If $y \in \mathcal{V}$ and $\beta \in \mathcal{B}$, then we can give a definition of a functional in the following way

$$\Phi_\beta^y(F) = \lim_{n\to\infty} \int\limits_{\mathbb{R}^n + i\mathbb{Y}_n} d^n\mathbb{X}_n \, e^{-\frac{1}{2}(\mathbb{X}_n^\beta, \mathbb{X}_n)} F(\mathbb{X}_n) \prod_{k=1}^{n} \left(\frac{\beta_k}{2\pi}\right)^{\frac{1}{2}}, \quad (2)$$

where $\mathbb{R}^n + i\mathbb{Y}_n = \{z \in \mathbb{C}^n : \mathrm{Re}(z) \in \mathbb{R}^n \text{ and } \mathrm{Im}(z)_k = y_k \text{ for } k \in (1, \ldots, n)\}$, and F is a functional on $\mathcal{V}$.

2.2. Basic properties.

Lemma 1. *Let $y \in \mathcal{V}$ and $c \in \mathbb{C}$. Let also F and G be functionals on $\mathcal{V}$ such that $\Phi^y_\beta(F)$ and $\Phi^y_\beta(G)$ are finite. Then $\Phi^y_\beta(F+G)$ and $\Phi^y_\beta(cF)$ are finite and we have the following equalities*

$$\Phi^y_\beta(F+G) = \Phi^y_\beta(F) + \Phi^y_\beta(G), \quad \Phi^y_\beta(cF) = c\,\Phi^y_\beta(F). \tag{3}$$

Lemma 2. *Let $y \in \mathcal{V}$. Let also F and G be functionals on $\mathcal{V}$ such that $\Phi^y_\beta(F\overline{G})$ is finite. Then, $\Phi^{-y}_\beta(\overline{F}G)$ is also finite and we have*

$$\Phi^y_\beta(F\overline{G}) = \overline{\Phi^{-y}_\beta(\overline{F}G)}. \tag{4}$$

Lemma 3. *Let F be a functional on $\mathcal{V}$ such that for all $x \in \mathcal{V}$ we have the inequality $F[x] \geqslant 0$. Then, we have*

$$\Phi^0_\beta(F) \geqslant 0. \tag{5}$$

From the last properties it is easy to see that we can introduce an operation according to the formula $\langle F, G\rangle_\beta = \Phi^0_\beta(F\overline{G})$ on a set of functionals. Indeed, from Lemma 1 we have linearity by the first argument and the Hermitian symmetry. From Lemma 3 the positive definiteness follows, but there is no the property: $\langle F, F\rangle_\beta = 0$ if and only if $F = \mathbb{O}$. So, we will name it quasi-scalar product.

2.3. Examples. Let $\beta \in \mathcal{B}$ be fixed. By symbol $\tilde{\beta}$ we denote an element from $\mathcal{V}$, which is defined by the formula $\tilde{\beta} = \{\beta_k^{-1}\}_{k=1}^\infty$. Further we are going to work with such functionals F on $\mathcal{V}$, which satisfy the conditions of view: there is such $C \in \mathbb{R}_+$, $n \in \mathbb{N}$, $\alpha \in \mathcal{B}$, and $z \in \mathcal{V}$, that for all $x \in \mathcal{V}$ we have

$$\left|\frac{F(x)e^{-(z,x)}}{1+(x^\alpha, x)^n}\right| \leqslant C, \quad (z^{\tilde{\beta}}, z) < +\infty, \quad \{\alpha_k \tilde{\beta}_k\}_{k=1}^\infty \in l^1. \tag{6}$$

The functionals which obey the last condition we will name as functionals from a valid class, denoted by $\mathcal{S}_\beta$. In fact it is a monoid with respect to the standard product. The set $\mathcal{S}_\beta$ is quite natural in theoretical physics, because it contains the polynomials and exponentials. Let us give some examples of the functionals from the set $\mathcal{S}_\beta$ and there norms, which will be useful in further reasoning.

1) If $F(x) = x_j^k$, where $x \in \mathcal{V}$ and $k, j \in \mathbb{N}$, then we can write

$$\langle F, F\rangle_\beta = \int_{\mathbb{R}} dx_j\, e^{-\frac{1}{2}\beta_j x_j^2} x_j^{2k} \left(\frac{\beta_j}{2\pi}\right)^{\frac{1}{2}} = \beta_j^{-k}(2k-1)!!. \tag{7}$$

2) Let us introduce the Hermite polynomials. It is known that they are defined by the formula

$$H_n(x) = (-1)^n e^{x^2/2} \partial_x^n e^{-x^2/2},\ n \geqslant 0. \tag{8}$$

If we take a functional like $H_{n,j}[x] = (n!)^{-1/2} H_n(\beta_j^{\frac{1}{2}} x_j)$, where $j > 0$ and $n \geqslant 0$, then

$$\langle H_{n,j}, H_{n,j}\rangle_\beta = \int_{\mathbb{R}} dx_j\, e^{-\frac{1}{2}\beta_j x_j^2} (n!)^{-1} H_n^2(\beta_j^{\frac{1}{2}} x_j) \left(\frac{\beta_j}{2\pi}\right)^{\frac{1}{2}} = 1. \tag{9}$$

3) If $G^{\pm iy}[x] = \exp(\pm i(x, y))$, where $x, y \in \mathcal{V}$, then $\langle G^{\pm iy}, G^{\pm iy}\rangle_\beta = 1$.

4) Let us take $G^{\pm\frac{1}{2}y}[x]$, where $x, y \in \mathcal{V}$. We need to do the change of variables in such form $x_j \to x_j \pm y_j \beta_j^{-1}$ for $j > 0$, then we obtain

$$\langle G^{\pm\frac{1}{2}y}, G^{\pm\frac{1}{2}y}\rangle_\beta = \lim_{n\to\infty} \exp\left(\frac{1}{2}(\mathbb{Y}_n^{\tilde{\beta}}, \mathbb{Y}_n)\right). \tag{10}$$

From the last formula one can see that the answer depends on $y \in \mathcal{V}$ and $\beta \in \mathcal{B}$ and can have the values from the closed interval $[1, +\infty]$. If $(y^{\tilde{\beta}}, y) < +\infty$, then $G^{\pm\frac{1}{2}y} \in \mathcal{S}_\beta$.

5) If x is from $\mathcal{V}$, α is from $\mathcal{B}$, and $F(x) = \exp\left(-\frac{1}{4}(x^\alpha, x)\right)$, then

$$\langle F, F\rangle_\beta = \lim_{n\to\infty} \prod_{k=1}^{n} \left(\frac{\beta_k}{\alpha_k + \beta_k}\right)^{\frac{1}{2}}. \tag{11}$$

The last answer depends on a choice of β and α. If $\{\alpha_k \tilde{\beta}_k\}_{k=1}^\infty \in l^1$ (see [11]) then $0 < \langle F, F\rangle_\beta \leqslant 1$, in other cases we have $\langle F, F\rangle_\beta = 0$.

2.4. Orthonormal system. In the priveous section we considered the normalization properties of the Hermite polynomials. It was seen that we have functionals $H_{n,j}$ for $n \geqslant 0$ and $j > 0$. After using the orthogonality property for the Hermite polynomials we can write for $n, k \geqslant 0$ and $i, j > 0$ the equality

$$\langle H_{k,i}, H_{n,j}\rangle_\beta = \delta_{kn}\delta_{ij} + \delta_{k0}\delta_{n0}(1 - \delta_{i,j}). \tag{12}$$

The next step is connected with studying of the products of such functionals when all second indices are different. We are going to give the most general definition and then consider the special cases. Let $\rho = \{\rho_k\}_{k=1}^{\infty}$ be from $\mathcal{A}$, and $x \in \mathcal{V}$, then we define

$$\mathbb{H}_\rho[x] = \prod_{k=1}^{\infty} H_{\rho_k,k}[x] : \forall\, n > 0 \text{ we have } \mathbb{H}_\rho[\mathbb{X}_n] = \prod_{k=1}^{n} H_{\rho_k,k}[\mathbb{X}_n]. \tag{13}$$

To understand the definition it is convenient to consider a few examples:

1) If $\rho_i \geqslant 0$ for fixed $i > 0$ and $\rho_l = 0$ for $l \neq i$, then $\mathbb{H}_\rho[x] = H_{\rho_i,i}[x]$, because of $H_{0,k}[x] = 1$;

2) If $\rho_i, \rho_j \geqslant 0$ for some $i \neq j$, and $\rho_l = 0$ for other l, then $\mathbb{H}_\rho[x] = H_{\rho_i,i}[x]H_{\rho_j,j}[x]$. Therefore, using the Hermite polynomial properties we can write the equality

$$\langle \mathbb{H}_\rho[x], \mathbb{H}_{\tilde\rho}[x]\rangle_\beta = \delta_{\rho_i\tilde\rho_i}\delta_{\rho_j\tilde\rho_j}, \tag{14}$$

where we assume that i and j are fixed, and $\tilde\rho_i, \tilde\rho_j \geqslant 0$.

Lemma 4. *Let $\rho = \{\rho_k\}_{k=1}^{\infty}$ and $\tilde\rho = \{\tilde\rho_k\}_{k=1}^{\infty}$ be from $\mathcal{A}$ and there is such $N > 0$ that $\rho_j = \tilde\rho_j = 0$ for all $j > N$. Then, under the conditions described above, we have the equality*

$$\langle \mathbb{H}_\rho[x], \mathbb{H}_{\tilde\rho}[x]\rangle_\beta = \prod_{k=1}^{N} \delta_{\rho_k\tilde\rho_k}. \tag{15}$$

Proof follows from the orthogonality property for the Hermite polynomials. It should be noted that $\mathbb{H}_\rho$, $\mathbb{H}_{\tilde\rho}$ and their product are from $\in \mathcal{S}_\beta$, because $\#\rho < \infty$ and $\#\tilde\rho < \infty$, where $\#\rho$ denotes a number of nonzero elements. The last Lemma means that we have orthonormal system. In this case a set of orthonormal elements is $\{\mathbb{H}_\rho : \rho \in \mathcal{A}, \#\rho < \infty\}$. It is clear that the set is countable. The completeness property is a delicate matter and it is not discussed here.

2.5. Further properties. Let us formulate properties which play a crucial role in physical applications. For this purpose we need to introduce a derivative in the form of sequence: if $x \in \mathcal{V}$ then $\partial_x = \{\partial_{x_k}\}_{k=1}^{\infty}$. We can extend notations as it was done earlier. It means that if $x, \alpha \in \mathcal{V}$ and $n \in \mathbb{N}$, then $\partial_{\mathbb{X}_n^\alpha}$ equals to $\alpha_k^{-1}\partial_{x_k}$ for $k \in (1, \ldots, n)$, and zero for $k > n$.

Lemma 5. *Let F be a functional from $\mathcal{S}_\beta$, which can be represented by a series in powers of x. In this case we have*

$$\Phi_\beta^0(F) = \lim_{n\to\infty} F[-\partial_{\mathbb{Y}_n}] \exp\left(\frac{1}{2}(\mathbb{Y}_n^{\tilde\beta}, \mathbb{Y}_n)\right)\bigg|_{y=0}. \tag{16}$$

Proof. First of all we are going to take the functional $G^{-y}[x] = \exp(-(x,y))$ and substitute one into the integrand. Thereby we can write the equation $\Phi^0_\beta(F) = \Phi^0_\beta(FG^{-y})|_{y=0}$. Then, after using the equality $F[x]G^{-y}[x] = F[-\partial_y]G^{-y}[x]$, we need to find only $\Phi^0_\beta(G^{-y})$, which was calculated in the Section 2.3. □

Lemma 6. *Let $y \in \mathcal{V}$ and $\beta \in \mathcal{B}$, then we have $\Phi^y_\beta(\mathbb{I}) = 1$.*

Proof. Let $\tilde{\beta}$ and the functional $G^{-iy^\beta}[x] = \exp(-i(y^\beta, x)$ be from the Section 2.3. Then, using the transformation

$$e^{-\frac{1}{2}(\mathbb{X}^\beta_n,\mathbb{X}_n)} \to e^{-\frac{1}{2}(\mathbb{X}^\beta_n,\mathbb{X}_n)}e^{-i(\mathbb{X}^\beta_n,\mathbb{Y}_n)}e^{\frac{1}{2}(\mathbb{Y}^\beta_n,\mathbb{Y}_n)} \tag{17}$$

after a shift in the following form $\mathbb{X}_n \to \mathbb{X}_n + i\mathbb{Y}_n$ and Lemma 5, we have the equality

$$\Phi^y_\beta(\mathbb{I}) = \lim_{n\to\infty} e^{\frac{1}{2}(\mathbb{Y}^\beta_n,\mathbb{Y}_n)}G^{-iy^\beta}[-\partial_{\mathbb{Z}_n}]e^{\frac{1}{2}(\mathbb{Z}^{\tilde{\beta}}_n,\mathbb{Z}_n)}\Big|_{z=0}. \tag{18}$$

At the same time we know that

$$\begin{aligned}&\exp(-i(\mathbb{Y}^\beta_n, \partial_{\mathbb{Z}_n}))e^{\frac{1}{2}(\mathbb{Z}^{\tilde{\beta}}_n,\mathbb{Z}_n)}\Big|_{z=0}\\ &= e^{\frac{1}{2}(\partial_{\mathbb{Z}^{\beta}_n},\partial_{\mathbb{Z}_n})}\exp(-i(\mathbb{Y}^\beta_n,\mathbb{Z}_n))\Big|_{z=0} = e^{-\frac{1}{2}(\mathbb{Y}^\beta_n,\mathbb{Y}_n)}.\end{aligned} \tag{19}$$

And after substitution the last equality into previous one the statement follows. □

Actually, the performed proof is very instructive and contains important equalities which are used in various applications. But it should be noted that the last Lemma can be proven by using complex analisys. Indeed, we can use the Cauchy's theorem and move the contour in the complex plane.

Lemma 7. *If $F \in \mathcal{S}_\beta$, then $|\Phi^0_\beta(F)| < +\infty$.*

Proof. The requirement $F \in \mathcal{S}_\beta$ means there are such $C \in \mathbb{R}_+$, $n \in \mathbb{N}$, $\alpha \in \mathcal{B}$, and $z \in \mathcal{V}$, that for all $x \in \mathcal{V}$ the conditions (6) hold. Let us define two functionals by the formulas

$$G(x) = C(1 + (x^\alpha, x)), \quad G^z(x) = e^{(z,x)}. \tag{20}$$

Then, by using Lemma 3, we have the following inequalities

$$|\Phi^0_\beta(F)| \leqslant \Phi^0_\beta(|F|) \leqslant \Phi^0_\beta(GG^z). \tag{21}$$

At the same time from the Section 2.3 we know that $\Phi^0_\beta(G^z)=e^{\frac{1}{2}(z^{\tilde\beta},z)}$. Thus, from Lemma 5 we obtain

$$|\Phi^0_\beta(F)|\leqslant G\,(\partial_z)\,e^{\frac{1}{2}(z^{\tilde\beta},z)}<+\infty, \tag{22}$$

because of the sums $\sum\limits_{n=1}^{+\infty} z_n^2\tilde\beta_n^k\alpha_n^{k-1}$ and $\sum\limits_{n=1}^{+\infty}\tilde\beta_n^k\alpha_n^k$ are finite for all $k\geqslant 1$. $\square$

2.6. Delta-functional. In the construction of the classical theory (see [12]) after the introduction of a set of test functions the generalized functions (or distributions) are defined. They have such features as linearity and continuity. In the infinite-dimensional case instead of the set of test functions we have a set of functionals on $\mathcal{V}$. It is obvious that any element of the set $\mathcal{S}_\beta$ is also linear generalized functional on the $\mathcal{S}_\beta$. Let us give an example of a special type. If x and y are from $\mathcal{V}$, then a δ-functional $\mathbb{D}(y,x)$ is defined by the formula

$$\mathbb{D}^\beta(y,\mathbb{X}_n)=\prod_{k=1}^{n}\left(\frac{2\pi}{\beta_k}\right)^{\frac{1}{2}}e^{\frac{1}{2}\beta_k x_k^2}\,\delta(y_k-x_k),\ \forall\, n>0. \tag{23}$$

The last definition consistent with the construction described above. It can be verified by substitution that the equality $\langle\mathbb{D}[y,\cdot\,],F\rangle_\beta=F[y]$ holds for all $F\in\mathcal{S}_\beta$, for which the limit $\lim\limits_{n\to+\infty}F(\mathbb{Y}_n)$ exists and equals to $F[y]$.

§3. Applications

3.1. Path integral. As it was noted above we are going to connect the physical meaning of the integral with the parameter $\beta\in\mathcal{B}$, which plays a role of spectrum for some problem. Of course, it follows from the definition of the set $\mathcal{B}$ that the spectrum should be discrete, infinite, and positive. The transition to the finite spectrum can be done by using projection of $\mathcal{V}$ on a finite-dimensional subspace.

To see it and to give sufficient definitions we need to introduce some new notations. First of all we enter an infinite-dimensional basis $\psi=\{\psi_k\}_{k=1}^\infty$ and a scalar product $\langle\cdot,\cdot\rangle$, such that $\langle\psi_i,\psi_j\rangle=\delta_{ij}$. Then by symbol $\mathcal{V}$ we denote (in the sence of generalized functions) a set

$$\{v=(x,\psi):\ \{x_k\}_{k=1}^\infty\in\mathcal{V}\}, \tag{24}$$

where ψ is understood as a function with value in the $\mathcal{V}$. In the same way we introduce a set

$$\mathcal{H} = \{v = (x, \psi) : (x^\beta, x) < \infty\}. \tag{25}$$

Let F be a functional on $\mathcal{V}$. It is clear that we can proceed to consideration of a functionals $\mathcal{F}$ on $\mathcal{V}$ by using a substitution $x_j = \langle v, \psi_j \rangle$ for all j. So we have a transition $F(x) \to \mathcal{F}(v)$, where $x \in \mathcal{V}$ and $v = (x, \psi) \in \mathcal{V}$.

Now we have the ability to introduce an integral. Under the conditions described above the path integral of the functional $\mathcal{F}$ over a set $\mathcal{H} + iu$ is defined by the formula

$$\int_{\mathcal{H}+iu} \mathcal{D}v\, e^{-\mathcal{S}[v]} \mathcal{F}(v) = \Phi_\beta^y(F), \tag{26}$$

where $\mathcal{S}[v] = \frac{1}{2}\langle v^\beta, v\rangle$, $v^\beta = (x^\beta, \psi)$, and $u = (y, \psi)$.

It is very easy to see that the last construction inherits all properties of Φ-functional. Thus we can in the same way introduce quasi-scalar product and orthonormal system. Also we need to note that the left hand side of the formula (26) contains a little more information, because we have the basis and the scalar product. Actually, it does not matter because any two infinite-dimensional separable Hilbert spaces are isomorphic to each other.

3.2. Loop space. Let us introduce more concrete β from $\mathcal{B}$, a basis, and a scalar product, which are useful in theoretical physics. The first one is given by the formula $\beta = \{\pi^2 k^2\}_{k=1}^\infty$. The second one we define as $\psi_k(s) = \sqrt{2}\sin(\pi k s)$ for all $k \in \mathbb{N}$, where $s \in [0, 1]$. The scalar product is a real and given by the integral over the interval $[0, 1]$. One can verify that $\int_0^1 ds\, \psi_i(s)\psi_j(s) = \delta_{ij}$.

Indeed, the spectrum β and the basis ψ correspond to the solution of the Sturm–Liouville problem with the equation $-u'' = \lambda u$ and the homogenious boundary conditions $u(0) = u(1) = 0$. Taking into account the definition of the quadratic form $\mathcal{S}$ we have

$$\mathcal{H} = H_0^1(0, 1) = \{u \in L_2(0, 1) : \int_0^1 ds\, \dot{u}^2(s) < \infty,\, u(0) = u(1) = 0\}. \tag{27}$$

It is very well known that any function from $\mathcal{H}$ is the continuous one. Due to the boundary conditions we deal with loops. So in this particular case the path integral is called the loop integral and plays a crucial role in physics. For example, at studying of the heat kernel expansions (see [13]).

3.3. Functional derivative. Under the conditions stated in the previous section let $u(s)$, where $s \in [0,1]$, be some function from the set $\mathcal{V}$. Then a functional derivative is based on the equality

$$\frac{\delta u(t)}{\delta u(s)} = \delta(t-s), \tag{28}$$

where $t \in [0,1]$. It is a very important object which allows us to work with Feynman diagram technique through the path integral. To connect the last formula with the definitions described above we need an extra lemma.

Lemma 8. *Let $x = \{x_k\}_{k=1}^{\infty} \in \mathcal{V}$, $\psi = \{\psi_k\}_{k=1}^{\infty}$ is the basis from the Section 3.2, and $u = (\psi, x) \in \mathcal{V}$. Then we have the equality*

$$\frac{\delta}{\delta u(t)} = (\psi(t), \partial_x),\ \ t \in [0,1]. \tag{29}$$

The statement follows from the completness of the basis functions $(\psi(t), \psi(s)) = \delta(t-s)$. Using the last Lemma it is easy to obtain some new properties of the integral. For example, due to the equality

$$\int_{\mathcal{H}+iu} \mathcal{D}v\, \frac{\delta}{\delta v(t)} e^{-\mathcal{S}[v]}\mathcal{F}(v) = 0,\ \ \forall t \in [0,1], \tag{30}$$

we can use an integration by parts in the case of the functional integration, or, due to the equality

$$\int_{\mathcal{H}} \mathcal{D}v\, e^{-\mathcal{S}[v]}\mathcal{F}(v) = \mathcal{F}[-\frac{\delta}{\delta u}] \exp\left(\frac{1}{2}(u^{\tilde{\beta}}, u)\right)\bigg|_{u=0}, \tag{31}$$

we have the ability to use the standard theory of perturbative expansion.

§4. Conclusion

The main purpose of this work was to define the "integral" construction and to give as many examples and physical applications as possible. We also focused on the possible ways of development as well as the problems that arose. Let us mention some of them.

1) At the beginning of the Section 2 we pointed out the set $\mathcal{B}$ elements of which play the role of the spectrum. The transition to a finite-dimensional or positive continuous one is quite easy, but adding a negative or zero

spectrum brings difficulties associated with divergences.

2) In the Section 2.4 we considered the orthonormal system. It is clear that we have the completness on the polynomial classes. It is very interesting to find a class of functionals on which completeness is performed. It is also not completely clear whether it is necessary to supplement the basis with generalized functionals $\mathbb{H}_\rho$, where $\#\rho = \infty$.

3) In the Section 2.6 we have outlined an analogy of functional sets with the classical theory of generalized functions. At the same time new problems, related to functional continuity or to the convergence of the functional sequence, appear.

4) Also in the Section 2.6 we proposed definitions for δ-functional. In the same way it is possible to introduce a δ-sequence $\mathbb{D}^\beta_\tau(y,x)$, which is a set according to the formula

$$\mathbb{D}^\beta_\tau(y,\mathbb{X}_n) = \prod_{k=1}^{n} \frac{1}{\sqrt{2\tau}} e^{\frac{1}{2}\beta_k x_k^2} e^{-\frac{\beta_k}{4\tau}(y_k - x_k)^2}, \ \forall\, n > 0,\ \tau > 0. \tag{32}$$

In fact the last one is a heat kernel for infinite-dimensional initial value problem, because it satisfies the equation

$$\big(\partial_\tau - (\partial_{y^\beta}, \partial_y)\big)\, \mathbb{D}^\beta_\tau(y,x) = 0 \tag{33}$$

and the initial condition $\mathbb{D}^\beta_0(y,x) = \mathbb{D}^\beta(y,x)$. Using the transition to the loop case from the Section 3, we can rewrite the last problem in the following form

$$\begin{aligned} &\Big(\partial_\tau - \int_0^1\!\!\int_0^1 ds dt\, g(s,t) \frac{\delta}{\delta u(s)} \frac{\delta}{\delta u(t)}\Big) \mathfrak{D}^\beta_\tau(u,v) = 0, \\ &\mathfrak{D}^\beta_0(u,v) = \mathfrak{D}^\beta(u,v), \end{aligned} \tag{34}$$

where $g(s,t) = (\psi^{\tilde\beta}(s), \psi(t))$. It is possible to develop the theory of the Laplace operator with functional derivative (class of potentials, spectral theory, a decomposition into a direct integral). Earlier attempts were made in [6, 14, 15].

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