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THE ASYMPTOTICS OF PLANE PARTITIONS WITH FIXED VOLUMES OF DIAGONAL PARTS

Abstract. Determinantal representation for the generating function of plane partitions with fixed volumes of diagonal parts is investigated in limiting cases.

The enumeration of plane partitions of different shape and subjected to various restrictions being a classical problem of enumerative combinatorics [1–7] naturally appears in the studies of integrable quantum systems [8–10]. The investigation of the behaviour of the correlation functions [11, 12] and of the partition functions at the large values of parameters leads to the analysis of asymptotics of the generating function of plane partitions [13, 14].

In the paper [10] we have calculated the partition function of the four vertex model in the external linearly growing inhomogeneous field and established its connection with the boxed plane partitions with the fixed values of its diagonal parts. The aim of this paper is to calculate the asymptotics of the correspondent generating function in the case when the height of the box is infinite and its sides are large but finite.

A plane partition $\boldsymbol{\pi}$ is an array

$$\boldsymbol{\pi} = \begin{pmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1j} & \cdots \\ \pi_{21} & \pi_{22} & \cdots & \pi_{2j} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \pi_{i1} & \pi_{i2} & \cdots & \pi_{ij} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

of non-negative integers π_{ij} that are non-increasing as functions both of i and j [15]. The entries π_{ij} are the parts of the plane partition, and its norm (volume) is $|\boldsymbol{\pi}| = \sum_{i,j} \pi_{ij}$. Each plane partition is represented by a three-dimensional Young diagram consisting of unit cubes arranged into stacks so that the stack with coordinates (i, j) is of height π_{ij}. It is said that the plane partition is contained in a box with side lengths L, N, P if

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$i \leqslant L$, $j \leqslant N$ and $\pi_{ij} \leqslant P$ for all cubes of the Young diagram (Fig. 1). Let us introduce the notation $\mathrm{tr}_s\,\boldsymbol{\pi}$ for the sum of the entries of $\boldsymbol{\pi}$ along

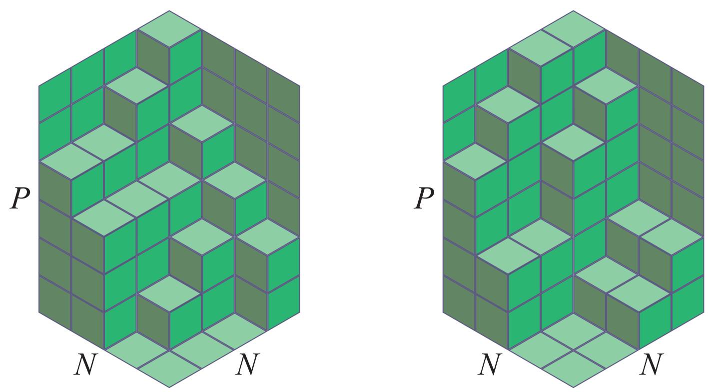

Figure 1. A plane partitions in a box with a symmetrical base.

non-principal diagonals counted from the left down corner. The arrays

$$\boldsymbol{\pi}_1 = \begin{pmatrix} 6 & 4 & 3 & 2 \\ 5 & 3 & 2 & 0 \\ 4 & 3 & 1 & 0 \\ 4 & 3 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\pi}_2 = \begin{pmatrix} 6 & 5 & 2 & 2 \\ 6 & 4 & 1 & 1 \\ 5 & 2 & 0 & 0 \\ 4 & 2 & 0 & 0 \end{pmatrix} \tag{1}$$

correspond to Fig. 1. The alternative image of plane partitions is presented in Fig. 2: integers in rhombi correspond to the heights of their columns.

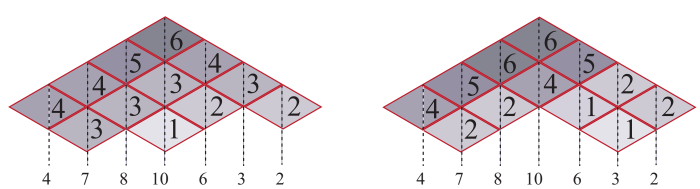

Figure 2. Two different plane partitions with the same traces of their diagonals.

The generating function of plane partitions in $N \times N \times P$ box with the fixed volumes of their diagonal parts has the polynomial form [6, 10]:

$$G(N, N, P|q, \mathbf{a}) \equiv \sum_{\{\boldsymbol{\pi}\}} q^{|\boldsymbol{\pi}|} \prod_{s=1}^{2N-1} a_s^{\mathrm{tr}_s \boldsymbol{\pi}}, \tag{2}$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_{2N-1})$ is $(2N-1)$-tuple, and summation goes over all plane partitions in the box. It is possible to formulate the following

Proposition. *The determinantal representation is valid for the polynomial* (2):

$$\begin{aligned}\sum_{\{\boldsymbol{\pi}\}} q^{|\boldsymbol{\pi}|} \prod_{s=1}^{2N-1} a_s^{\mathrm{tr}_s \boldsymbol{\pi}} &= \prod_{1 \leqslant m < k \leqslant N} \left(q^k \prod_{j=k}^{N} a_j - q^m \prod_{j=m}^{N} a_j \right)^{-1} \\ &\times \prod_{1 \leqslant m < k \leqslant N} \left(q^{k-1} \prod_{j=1}^{k-1} a_{N+j} - q^{m-1} \prod_{j=1}^{m-1} a_{N+j} \right)^{-1} \\ \times \det &\left[\frac{1 - \left(q^{k+m-1} \prod_{j=1}^{k-1} a_{N+j} \prod_{j=m}^{N} a_j \right)^{P+N}}{1 - q^{k+m-1} \prod_{j=1}^{k-1} a_{N+j} \prod_{j=m}^{N} a_j} \right]_{1 \leqslant k, m \leqslant N}.\end{aligned} \tag{3}$$

Proof. The identity (3) results from the calculation in [10] of the partition function of the integrable four-vertex model in external inhomogeneous field by Quantum Inverse Scattering Method [16, 17]. □

In a box of unbounded height $P \to \infty$ and finite N, the polynomial (2) turns into a series, the determinant in (3) turns into the Cauchy determinant which is calculated:

$$\begin{aligned}\mathcal{G}_N(q, \mathbf{a}) &\equiv \lim_{P \to \infty} G(N, N, P|q, \mathbf{a}) \\ &= \prod_{k=1}^{N} \prod_{m=1}^{N} \frac{1}{1 - q^{k+m-1} \prod_{j=1}^{k-1} a_{N+j} \prod_{j=m}^{N} a_j}.\end{aligned} \tag{4}$$

Let us consider the inhomogeneous $(2N-1)$-tuple $\mathbf{a}$ specified as follows:

$$\mathbf{a} = (\underbrace{1, 1, \ldots, 1}_{s-1 \text{ times}}, a, \underbrace{1, 1, \ldots, 1}_{2N-s-1 \text{ times}}) \equiv (\mathbf{1}_{s-1}, a, \mathbf{1}_{2N-s-1}) \tag{5}$$

where $1 \leqslant s \leqslant N$. In this case Eq. (2) is written as:

$$\mathcal{G}_N\big(q, (\mathbf{1}_{s-1}, a, \mathbf{1}_{2N-s-1})\big) = \sum_{\{\boldsymbol{\pi}\}} q^{|\boldsymbol{\pi}|}\, a^{\operatorname{tr}_s \boldsymbol{\pi}}\,, \tag{6}$$

while (4) takes the form:

$$\mathcal{G}_N\big(q, (\mathbf{1}_{s-1}, a, \mathbf{1}_{2N-s-1})\big) = \prod_{k=1}^{N} \Bigg\{ \prod_{m=1}^{s} \frac{1}{1 - aq^{k+m-1}} \times \prod_{m=s+1}^{N} \frac{1}{1 - q^{k+m-1}} \Bigg\}. \tag{7}$$

Equation (7) is the generating function of plane partitions with fixed trace $\operatorname{tr}_s \boldsymbol{\pi}$, $1 \leqslant s \leqslant N$. Equation (7) at $s = N$ reduces to the norm-trace generating function obtained by Stanley in [4]. It is interesting to mention that in this case the series (6) was considered by [18] in connection with the five-dimensional supersymmetric Yang-Mills theories.

Let us introduce the q-deformed Barnes G-function:

$$G_q(n+1) \equiv \prod_{k=1}^{n} \Gamma_q(k)\,, \tag{8}$$

where $\Gamma_q(n)$ is q-Gamma function [20]:

$$\Gamma_q(n) \equiv [1]\,[2]\,\cdots\,[n-1]\,, \qquad [k] \equiv \frac{1-q^k}{1-q}\,, \quad k \in \mathbb{Z}^+\,. \tag{9}$$

With the help of (8) and (9) the generating function (7) may be brought into the form

$$\mathcal{G}_N\big(q, (\mathbf{1}_{s-1}, a, \mathbf{1}_{2N-s-1})\big) = (1-q)^{N(s-N)}\, \mathcal{G}^{(1)}_{N,s}(q)\, \mathcal{G}^{(2)}_N(q,a)\,, \tag{10}$$

where

$$\mathcal{G}^{(1)}_{N,s}(q) \equiv \frac{G_q(N+1) G_q(N+s+1)}{G_q(s+1) G_q(2N+1)}, \quad \mathcal{G}^{(2)}_N(q,a) \equiv \prod_{k=1}^{N} \frac{(a,q)_k}{(a,q)_{k+s}}\,, \tag{11}$$

and $(a,q)_n$ is the shifted q-factorial:

$$(a,q)_n \equiv (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), \quad (a,q)_0 = 1. \tag{12}$$

The function $\mathcal{G}^{(1)}_{N,s}(q)$ (11) is the unity for $s = N$, and Eq. (10) is simplified:

$$\mathcal{G}_N\big(q, (\mathbf{1}_{N-1}, a, \mathbf{1}_{N-1})\big) = \prod_{k=1}^{N} \frac{(a,q)_k}{(a,q)_{k+N}}. \tag{13}$$

Let us consider the asymptotics of $\mathcal{G}^{(1)}_{N,s}(q)$ (11) at $1 \ll s < N \ll P$. To this end, the asymptotics of the q-Barnes function (8) is required, which is obtainable from the properties [19] of the logarithm of q-Gamma function $\Gamma_q(z)$. For instance, the asymptotics of logarithm of conventional Barnes G-function $G(N+1)$ at $N \gg 1$, [8], results (provided that the leading logarithmic approximation $\log N, \log s \gg 1$ is considered) in the following estimate:

$$\begin{aligned}\log\frac{G(N+1)}{G(s+1)} &= \sum_{k=s+1}^{N} \log\Gamma(k) \\ &= \frac{N^2}{2}\log N - \frac{s^2}{2}\log s + \mathcal{O}(N^2, s^2)\,. \end{aligned} \tag{14}$$

The q-Gamma function at $0 < q \leqslant 1$ and $|z| \to \infty$ behaves [19]:

$$\begin{aligned}\log\Gamma_q(z) \simeq \Big(z - \frac{1}{2}\Big)\log\frac{q^z-1}{q-1} + \frac{1}{\log q}\int_{-\log q}^{-z\log q} \frac{u\,du}{e^u-1} + C_q \\ + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\Big(\frac{\log q}{q^z-1}\Big)^{2k-1} q^z\, P_{2k-3}(q^z)\,, \end{aligned} \tag{15}$$

where B_{2k} are the Bernoulli numbers, C_q is some constant, [19], and $P_n(z)$ is a polynomial of n^{th} order respecting the relation

$$P_n(z) = (z - z^2)P'_{n-1}(z) + (nz+1)P_{n-1}(z)\,, \qquad P_0 = 1\,.$$

Assuming that q is close to unity, $q = 1-\epsilon$, $\epsilon \ll 1$, and taking into account only the first contribution in right-hand side of (15) we obtain at $N, s \gg 1$:

$$\sum_{k=s+1}^{N} \log\Gamma_q(k) \approx \sum_{k=s}^{N} k\log k\,. \tag{16}$$

Applying the Euler-Maclaurin formula [21] to (16), we get the leading term of (14).

Let us turn to $\mathcal{G}^{(1)}_{N,s}(q)$ (11). The estimate $\log(1-q^k) \simeq -q^k$ is valid provided that $0 < q < 1$ and k is large enough. Therefore, restricting with the first contribution in (15), one obtains:

$$\mathcal{G}^{(1)}_{N,s}(q) \simeq (1-q)^{N(N-s)} \times \exp\Big((q^N-1)\sum_{k=s}^{N} kq^k + Nq^N\sum_{k=s}^{N} q^k\Big)\,. \tag{17}$$

The sums in right-hand side of (17) are calculated:

$$(1-q)^{N(s-N)}\mathcal{G}^{(1)}_{N,s}(q) \simeq \exp\Big((Nq^N - sq^s)\frac{q^N-1}{q-1} + Nq^N\frac{q^N-q^s}{q-1}\Big)\,. \tag{18}$$

The expansion of the exponential gives:

$$(1-q)^{N(s-N)}\mathcal{G}^{(1)}_{N,s}(q) \simeq 1 + (Nq^N - sq^s)(1+q+\ldots+q^r)\,, \tag{19}$$

where $1 < r \ll s$.

The function $\mathcal{G}^{(2)}_{N}(q,a)$ (11) can be studied by means of the formulas [4]:

$$\frac{1}{(1-a)(1-aq)\cdots(1-aq^{n-1})} = \sum_{k=0}^{\infty} a^k \begin{bmatrix} k+n-1 \\ n-1 \end{bmatrix}_q\,, \tag{20}$$

$$(1-a)(1-aq)\cdots(1-aq^{l-1}) = \sum_{k=0}^{l}(-a)^k q^{k(k-1)/2}\begin{bmatrix} l \\ k \end{bmatrix}_q\,, \tag{21}$$

where $\begin{bmatrix} l \\ k \end{bmatrix}_q$ are the q-binomial coefficients [20]. The q-binomial coefficients are the polynomials in powers of q,

$$\begin{bmatrix} k+n \\ n \end{bmatrix}_q = \sum_{j=0}^{kn} N_j(k,n)\, q^j\,. \tag{22}$$

They may be considered as the generating functions of lattice paths, and the numbers $N_j(k,n)$ have been studied in [7]. Left-hand side of (22) at $k=1$ is the q-number $[n+1]$, and the corresponding numbers $N_j(1,n)=1$. Using (20) and (21) we obtain for $\mathcal{G}^{(2)}_{N}(q,a)$ in the first order in a:

$$\mathcal{G}^{(2)}_{N}(q,a) = \prod_{k=1}^{N}\frac{1}{(1-aq^k)(1-aq^{k+1})\cdots(1-aq^{k+s-1})} \tag{23}$$

$$= 1 + aq[N][s] + \mathcal{O}(a^2)\,, \tag{24}$$

where, by straightforward computation,

$$
\begin{aligned}
[N][s] = 1 &+ 2q + 3q^2 + \ldots + (s-1)q^{s-2} \\
&+ s(q^{s-1} + q^s + \ldots + q^{N-1}) \\
&+ (s-1)q^N + (s-2)q^{N+1} + \ldots + 1 \cdot q^{N+s-2}\,. \qquad (25)
\end{aligned}
$$

Straightforward application of (20) and (21) to expansion of $\mathcal{G}_N^{(2)}(q,a)$ (23) in powers of a is combersome even in the second order a^2. A more tractable representation for $\mathcal{G}_N^{(2)}(q,a)$ (23) arises provided we put:

$$
\log \frac{1}{\mathcal{G}_N^{(2)}(q,a)} = \sum_{k=1}^{N} \Big(\log(1-aq^k) + \log(1-aq^{k+1}) + \ldots + \log(1-aq^{k+s-1}) \Big)\,. \qquad (26)
$$

The logarithms in right-hand side of (26) are expanded in power series (valid for $0 < a, q < 1$). We meet in right-hand side of (26) the polynomials:

$$
\sum_{i=0}^{s-1} \frac{a^n}{n} q^{ni} \sum_{k=1}^{N} q^{nk} = \sum_{i=0}^{s-1} \frac{a^n}{n} q^{n(i+1)} [N]_n = \frac{a^n q^n}{n} [N]_n [s]_n\,,
\qquad [N]_n \equiv \frac{q^{nN}-1}{q^n - 1}. \qquad (27)
$$

Equation (26) is formally re-expressed:

$$
\mathcal{G}_N^{(2)}(q,a) = \exp\Big(\sum_{n=1}^{\infty} \frac{a^n q^n}{n} [N]_n [s]_n \Big) = \prod_{n \geqslant 1} \exp\Big(\frac{a^n q^n}{n} [N]_n [s]_n \Big)\,, \qquad (28)
$$

where

$$
\begin{aligned}
[N]_n[s]_n = 1 &+ 2q^n + 3q^{2n} + \ldots + (s-1)q^{(s-2)n} \\
&+ s\big(q^{(s-1)n} + q^{sn} + \ldots + q^{(N-1)n}\big) \\
&+ (s-1)q^{Nn} + (s-2)q^{(N+1)n} + \ldots + 1 \cdot q^{(N+s-2)n}\,. \qquad (29)
\end{aligned}
$$

The representation (28) is the generating function of a formal expansion in powers of a:

$$\mathcal{G}_N^{(2)}(q,a) = \sum_{k=0}^{\infty} a^n A_n\,, \tag{30}$$

$$A_1 = \frac{q[N]_1[s]_1}{1!}, \quad A_2 = \frac{(q[N]_1[s]_1)^2}{2!} + \frac{q^2[N]_2[s]_2}{2},$$

$$A_3 = \frac{(q[N]_1[s]_1)^3}{3!} + \frac{q^2[N]_2[s]_2}{2}\frac{q[N]_1[s]_1}{1} + \frac{q^3[N]_3[s]_3}{3}.$$

The term $\sim a$ in (30) coincides with that in (24), and the corresponding coefficient $[N]_1[s]_1 \equiv [N][s]$ is given by (25). In the leading order (neglecting $q^s, q^N \ll 1$ and smaller terms) we get from (30):

$$\mathcal{G}_N^{(2)}(q,a) \simeq 1 + a\big(q + 2q^2 + 3q^3 + \ldots\big) + a^2\big(q^2 + 2q^3 + 6q^4 + \ldots\big). \tag{31}$$

Generally, the coefficients at a^p are given by

$$A_p \;=\; \frac{1}{p!}\,\frac{d^p}{da^p}\,\mathcal{G}_N^{(2)}(q,a)\Big|_{a=0} = \frac{1}{p!}\sum_{k_1+k_2+\ldots+k_p=p} \binom{p}{k_1,k_2,\ldots,k_p}$$

$$\times \prod_{l=1}^{p} \frac{d^{k_l}}{da^{k_l}}\,\exp\Big(\frac{a^l q^l}{l}[N]_l[s]_l\Big)\Big|_{a=0}, \tag{32}$$

where $\binom{p}{k_1,k_2,\ldots,k_p}$ are the multinomial coefficients.

The representation (28) admits another re-arrangement:

$$\mathcal{G}_N^{(2)}(q,a) = \prod_{n\geqslant 1} e^{q^n a_n}\,, \tag{33}$$

where

$$a_1 = a, \quad a_2 = 2a + \frac{a^2}{2}, \quad a_3 = 3a + \frac{a^3}{3}, \quad a_4 = 4a + a^2 + \frac{a^4}{4}. \tag{34}$$

From (33) it follows that

$$\mathcal{G}_N^{(2)}(q,a) = \sum_{k=0}^{\infty} q^n Q_n\,, \tag{35}$$

$$Q_1 = a_1 = a\,, \quad Q_2 = \frac{a_1^2}{2!} + a_2 = a^2 + 2a\,,$$

$$Q_3 = \frac{a_1^3}{3!} + a_1 a_2 + a_3 = a^3 + 2a^2 + 3a\,. \tag{36}$$

The numbers Q_n (35) are given by the formula analogous to (32). The relations (36) appeared in [22] in connection with re-arrangment of (33) into (35) in framework of a combinatorial approach to Mayer's theory of cluster expansion.

The choice of inhomogeneous $(2N-1)$-tuple

$$\mathbf{a} = (\underbrace{1, 1, \ldots, 1}_{N-l \text{ times}}, \underbrace{a, a, \ldots, a}_{l \text{ times}}, \underbrace{1, 1, \ldots, 1}_{N-1 \text{ times}}) \equiv (\mathbf{1}_{N-l}, \mathbf{a}_l, \mathbf{1}_{N-1}), \qquad (37)$$

leads to the generating function of plane partitions with the fixed value of the sum $\sum_{s=N-l+1}^{N} \mathrm{tr}_s\, \boldsymbol{\pi}$, i.e., to an l-trace type formula of Gansner [5]:

$$\mathcal{G}_N\big(q, (\mathbf{1}_{N-l}, \mathbf{a}_l, \mathbf{1}_{N-1})\big) = \prod_{k=1}^{N} \frac{(a^{N-k+1}, q)_k}{(a^{N-k+1}, q)_{k+N}}$$
$$\times \prod_{m=1}^{N-l+1} \left(\frac{(a^{N-m+1}, q)_{m+N}}{(a^{N-m+1}, q)_m} \frac{(a^l, q)_m}{(a^l, q)_{m+N}} \right). \qquad (38)$$

is reduced to (13) at $l = 1$. The approach developed may be used in the study of the asymptotics of Eq. (38).

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