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ON EXTENSIONS OF CANONICAL SYMPLECTIC STRUCTURE FROM COADJOINT ORBIT OF COMPLEX GENERAL LINEAR GROUP

Abstract. The problem of the extensions of the canonical Lee–Poisson–Kirillov–Kostant symplectic structure of the coadjoint orbit of the complex general linear group is considered. The introduced method uses the concept of the flag coordinates and does not depend on the Jordan structure of matrices forming the orbit. The principal bundle associated with the fibration of the orbit over the Grassmanian of flags is constructed.

§1. Introduction

1.1. Extension by D. Korotkin and M. Bertolla. A natural way to extend the Lee–Poisson–Kirillov–Kostant (LPKK) structure in the case of the coadjoint orbit of general position was introduced in [7]. The approach is based on the interpretation of the coadjoint orbit as a quotient manifold of the right-invariant section of $\mathrm{T}^*\mathrm{GL}(N,\mathbb{C})$ with respect to the subgroup of the left shifts that keeps this section invariant.

To present it in more detail let us consider $\mathrm{GL}(N,\mathbb{C})$ and its cotangent bundle $\mathrm{T}^*\mathrm{GL}(N,\mathbb{C}) \ni (g,B)$. We identify $\mathrm{gl}(N,\mathbb{C})$ and $\mathrm{gl}^*(N,\mathbb{C})$ using Killing product

$$B \in \mathrm{gl}(N,\mathbb{C}) \leftrightarrow B \in \mathrm{gl}^*(N,\mathbb{C}) : \{A \to \mathrm{tr}\, AB\}.$$

The right-invariant section that is equal to some $\Lambda \in \mathrm{gl}^*(N,\mathbb{C})$ at the unit is

$$\bigcup_{g\in \mathrm{GL}(N,\mathbb{C})} (g, g^{-1}\Lambda) \subset \mathrm{T}^*\mathrm{GL}(N,\mathbb{C}).$$

It carries the two-form that is the restriction of the standard form $\omega_{T^*} = \mathrm{tr}\, dB \wedge dg$ on $\mathrm{T}^*\mathrm{GL}(N,\mathbb{C})$.

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The quotient manifold with respect to the subgroup of matrices that commute with Λ can be identified with the coadjoint orbit

$$\bigcup_{g\in \mathrm{GL}(N,\mathbb{C})} g^{-1}\Lambda g \subset \mathrm{gl}^*(N,\mathbb{C}).$$

It can be proved that the contraction of ω_{T^*} on the section is well defined on this quotient manifold and defines a symplectic structure there, see [8].

Let $\Delta \subset \mathrm{gl}^*(N,\mathbb{C})$ be a locus of the diagonal matrices with the distinct eigenvalues. It is N-dimentional manifold. Let M_Δ be a $N+N^2$-dimensional submanifold of the cotangent bundle swept by the right shifts of the elements of $(\mathbf{e},\Delta) \in \mathrm{T}^*_{\mathbf{e}}\mathrm{GL}(N,\mathbb{C})$:

$$M_\Delta := \bigcup_{g\in \mathrm{GL}(N,\mathbb{C}),\Lambda\in\Delta} (g, g^{-1}\Lambda) \subset \mathrm{T}^*\mathrm{GL}(N,\mathbb{C}).$$

The restriction on M_Δ of the standard form ω_{T^*} is

$$\omega_{T^*} = \operatorname{tr} d(g^{-1}\Lambda) \wedge dg.$$

The form is evidently closed as the contraction of the closed form. The non-degeneration is equivalent to the absence of the non-trivial solutions $(X, \dot{\Lambda})$ of the equations

$$\dot{\Lambda} + [X,\Lambda] = 0, \text{ and } \operatorname{tr} \Lambda' X = 0 \ \forall \Lambda' \in \Delta,$$

i.e., for all Λ' tangent to the set of values of Λ's. The first equation is equivalent to vanishing both $\dot{\Lambda}$ and $[X,\Lambda]$. The vanishing of $\operatorname{tr}\Lambda' X$ set some restrictions on the diagonal part of X only. That is why this method of the extension of LPKK structure works in the case of the distinct eigenvalues of Λ only. It is the case $[X,\Lambda] = 0$ is equivalent to the diagonality of X and $\operatorname{tr}\Lambda' X = 0 \ \forall \Lambda'$ implies $X = 0$, see [7].

So M_Δ is a symplectic manifold that can be considered as an embedding of the extension of the LPKK-symplectic structure of the coadjoint orbit constructed at [7].

1.2. Flag extension. Let us introduce another representation of the coadjoint orbit. In the works [1, 2, 3] the concept of flag coordinates was introduced. A natural domain of definition of these coordinates contains the coadjoint orbit as an open subset. The flag coordinates can be constructed for the matrices of *any* Jordan structure uniformly.

The set of local isomorphisms between $\mathcal{O}$ and the linear symplectic spaces was built. The elements of the set are numerated by the permutations of the roots of the minimal polynomial.

Main objects of the present investigation are fiber bundles over the Grassman manifold of flags. A flag in question is $\{\mathscr{F}\} : F_{k-1} \subset F_k$, $F_k :=$ $\ker \prod_{i=1}^{k} (A - \lambda_i \mathrm{I})$. It is defined for any ordered set of the eigenvalues λ_i of A.

We construct a fiber bundle over the manifold of these flags. The fiber is a set of the spaces of the linear maps between $\mathbb{C}^N/F_k$ and F_k/F_{k-1}. We extend just this fiber bundle.

Let us consider the extension itself. Each point $A = g^{-1}\Lambda g$ of the orbit $\mathcal{O}$ is the result of the factorization. The factorization is made with respect to the subgroup of matrices commuting with Λ:

$$g \simeq g_\Lambda g \text{ if } g_\Lambda \Lambda = \Lambda g_\Lambda, \quad g^{-1}\Lambda g = g^{-1} g_\Lambda^{-1} \Lambda g_\Lambda g = A.$$

The representation of $\mathcal{O}$ as the fiberbundle over the flag Grassmanian "remember" this factorization. There is an isomorphism between the set of matrices over the point A and a manifold $\otimes_k \mathrm{GL}(F_k/F_{k-1})$ associated with the flag. The cotangent bundle of this manifold is the symplectic space that extends the orbits fibered over this flag Grassmanian.

Each orbit can be projected symplectically to this extended space as a submanifold which added coordinates are equal to

$$\{(id, \lambda_1 \mathrm{I}), \ldots, (id, \lambda_M \mathrm{I})\} = const \in \otimes_{k=1}^{M} \mathrm{T}^* \mathrm{GL}(F_k/F_{k-1}).$$

§2. Stratification of $\mathrm{gl}(N, \mathbb{C})$ and Young tableaux

A diagram technique for the investigation of the set of matrices with the complicated Jordan structure was developed in [5].

Each matrix defines the Young tableaux $\mathscr{Y}$. If the set of roots of the minimal polynomial of the matrix is ordered, the Young tableaux can be marked by these roots that are the eigenvalues of the matrix. The marked Young tableaux $\overleftarrow{\mathscr{Y}}$ uniquely defines the flag $\mathscr{F}$ that was defined in the previous section. The conjugated class of the matrix is defined by tableaux the marked by the eigenvalues.

The unmarked tableaux defines the submanifold $\{\cup\mathcal{O}\}_{\mathscr{Y}}$ in the space of matrices. The matrices in the submanifold have the same integer-valued functions

$$k \to \dim \ker \prod_{i=1}^{k} (A - \lambda_i \mathrm{I})$$

for some orderings of the roots λ_i of their minimal polynomials. It can be treated as some special factorization of the characteristic polynomial, where all the eigenvalues λ_i substituted by one formal variable, say $\bar{\lambda}$:

$$\prod_{k=1}^{M}(\lambda-\bar{\lambda})^{n_k}.$$

Let us fix an unmarked diagram $\mathscr{Y}$. It defines the decomposition of N on M summands n_k that are the values of jumps of the dimensions of $\ker\prod_k(A-\lambda_k)$, the dimensions of the subspaces forming the flag. The same numbers n_k are the dimensions of the subspaces $[F_k] := F_k/F_{k-1}$ of the factor-spaces $V_{k-1} = \mathbb{C}^N/F_{k-1} \supset [F_k]$, for some ordering of λ_k's. The different orderings of λ_k imply just the reordering of the same set of n_k. The action of matrix A on the auxiliary $V \simeq \mathbb{C}^N$ can be contracted on the factor-spaces $V_k = V/F_k$. These contractions define the sequence of the transformations $\mathcal{A}_k, k = 1,\dots,M$. The contraction of $\mathcal{A}_{k-1} \in \operatorname{End} V_{k-1}$, on $[F_k] \subset V_{k-1}$ is the homothetic dilatation of $[F_k]$ with coefficient λ_k.

§3. Projection of coanjoint orbit of $\mathrm{GL}(N,\mathbb{C})$ to flag manifold. Flag coordinates.

Consider any $A \in \mathrm{gl}(N,\mathbb{C}) \simeq \mathrm{gl}^*(N,\mathbb{C})$. We treat A as a matrix of the linear transformation $\mathcal{A} \in \operatorname{End}(V)$ of some auxiliary $V \simeq \mathbb{C}^N$ in a fixed basis.

Let us consider minimal polynomial of A and order the set of its roots that are the eigenvalues λ_i of A. Let us consider flag $\{\mathscr{F}\}$:

$$F_k := \ker\prod_{i=1}^{k}(A-\lambda_i \mathrm{I}), \quad F_0 := 0,$$

where the product is taken over such values "i" that $\dim F_k \neq \dim F_{k-1}$. The jumps of the dimension of F_k we denote by n_k, the number of the jumps we denote by M:

$$\dim F_k - \dim F_{k-1} =: n_k, \quad \sum_{k=1}^{M} n_k = N.$$

The value of M is the order of the minimal polynomial.

The index k takes no more than N values excluding $k = 0$. The flag is complete in the case of one-dimensional eigenspaces, it is the case when each eigenvalue has just one Jordan box. It follows from the concept (existence) of the minimal polynomial that for some $k = M \leqslant N$ the corresponding subspace F_M becomes the whole V.

Let us consider a flag $\{\mathscr{V}\}$ that is complementary to this one in some sense:

$$V_k := V/\ker\prod_{i=1}^{k}(A-\lambda_i \mathrm{I}), \quad V_0 := V, \quad V_N = 0.$$

Linear spaces V_k of the flag $\{\mathscr{V}\}$ can be embedded into V, where F_i lives, as fiberations of V by the linear submanifolds parallel to F_k. The linear structure is defined by the projection on any transversal subspace of the complementary dimension.

It is evident, that $\mathcal{A}$ induces some linear transformation on each of the spaces F_k and V_k. It is $\mathcal{A}_{F_k} := \mathcal{A}|_{F_k} \in \mathrm{End}(F_k)$, and $\mathcal{A}_k \in \mathrm{End}(V_k)$ that can be defined on representatives. The image of any point of a fiber is contained in the same fiber because $\mathcal{A}F_k \subset F_k$. The factorization gives one point of V_k.

We can see that all the objects, namely linear transformation $\mathcal{A}_k \in$ End V_k, the flag $\ker \prod\limits_{i=k+1}^{s} (\mathcal{A}_k - \lambda_i \mathrm{I})$, and the complementary flag of the quotients with respect to these subspaces are correctly defined. So we can consider the sequence of the nested flags and transformations $\mathcal{A}_k \in V_k$ defined on them.

Let us denote a Grassman manifold of all flags at V with the set $\{n_k\}_{k=1}^{M}$ of the jumps of the dimensions by $\mathrm{Gr}^{\Phi}_{\{n_k\}} \ni \{\mathscr{F}\}$.

We define a map from the coadjoint orbit of $\Lambda \in \mathrm{gl}(N,\mathbb{C})$ to the flag manifold $\mathrm{Gr}^{\Phi}_{\{n_k\}} \ni \{\mathscr{F}\}$: $\mathcal{O}(\Lambda) \to \{\mathscr{F}\}$.

Let us consider a level of value of this map. We “forget” the positions of images $\mathcal{A}_{k-1}(v)$, $v \in V_k$ on the fibers $w \in V_k$ during each factorization. The result of this “forgetting” is $\mathcal{A}_k \in$ End V_k. We see that we are forgetting the positions of $\dim V_k$ points on the fibers on each step. Each fiber is isomorphic to F_k/F_{k-1}. It can be proved that the image is algebraically open subset in the fiber, see [3].

Finally, we see that the level-set of the map $\mathcal{O}(\Lambda) \to \{\mathscr{F}\}$ is isomorphic to the open subset of the manifold of the linear maps

$$V/F_k \to [F_k],$$

where $[F_k] := \ker(\mathcal{A}_{k-1} - \lambda_k \mathrm{I}) = F_k/F_{k-1} \subset V_{k-1}$ is $\dim F_k - \dim F_{k-1} =:$ n_k-dimensional subspace of V_{k-1}. It is just F_k considered as the subspace of V_{k-1}.

To define coordinates on the open subsets of these manifolds let us consider the fixed from the very begining basis $\{\mathbf{e}\}$ in $V \simeq \mathbb{C}^N$ and introduce models of the quotient-spaces V_k as the coordinate subspaces $E_k := \mathscr{L}(\mathbf{e}_N, \mathbf{e}_{N-1}, \mathbf{e}_{N-2}, \dots)$ of the corresponding dimension. The coordinate subspaces that complete E_k to E_{k-1} we denote by L_k:

$$E_{k-1} = E_k \oplus L_k, \quad L_k \simeq [F_k].$$

Here $\mathbf{e}_i \in \{\mathbf{e}\}$ are vectors of the basis.

The position of flag $\{\mathscr{F}\}$ is uniquely determined by the projection of the fixed flag made from the coordinate subspaces, to $\{\mathscr{F}\}$ parallel to the spaces of the correspodning models E_k of V_k's. It gives a map $\mathcal{Q} : A \to \mathbf{Q}$, where

$$\mathbf{Q} =: \bigoplus_k \mathrm{Hom}([F_k], V_k).$$

Here the components $Q_k(A) \in \mathrm{Hom}([F_k], V_k)$ of $\mathcal{Q}$ are the projections of the points of $[F_k]$ to L_k parallel to E_k:

$$Q_k(x) + x \in L_k, \quad x \in [F_k], \quad Q_k(x) \in E_k.$$

The matrix-elements of these maps in basis $\{\mathbf{e}\}$ form the affine coordinates on the (open subset of the) flag manifold $\mathrm{Gr}^{\Phi}_{\{n_k\}}$.

Consider linear spaces

$$\mathbf{P}_k = \mathrm{Hom}(V_k, [F_k])$$

of the linear maps in the opposite direction. We combine them to one linear space $\mathbf{P}$:

$$\mathbf{P} =: \bigoplus_k \mathbf{P}_k.$$

We get components $P_k(A)$ in the following way. We contract $\mathcal{A}_{k-1}$ on V_k and set P_k as a projection of this contraction to $[F_k] := \ker(\mathcal{A}_{k-1} - \lambda_k \mathrm{I})$ parallel to E_k.

A linear space $\mathbf{P} \oplus \mathbf{Q}$ has natural symplectic structure because it is a direct sum of two mutually conjugated linear spaces. Any bases in the corresponding spaces $[F_k], E_k$ give the matrix representations and the canonical form of the symplectic form on the orbit $\omega_{\mathcal{O}} = \mathrm{tr}\, dP_k \wedge dQ_k$, see [3].

The matrix elements $(P_k)_{ij}, (Q_k)_{ji}$ are called *flag coordinates* on the orbit. To construct an atlas of these coordinate maps we consider all permutations of the basic vectors $\mathbf{e}_i$.

§4. Fibration of orbit over Grassmanian of flags

Let us fix the ordering of λ_k's once and forever and consider a structure of the fiber bundle on the orbit:

$$\mathcal{O} \to \mathrm{Gr}^{\Phi}_{\{n_k\}},$$

where $\mathrm{Gr}^{\Phi}_{\{n_k\}} \ni \{\mathscr{F}\}$ is a corresponding Grassmanian of the flags.

Our aim is the construction of the principal bundle associated with $\mathcal{O} \to \mathrm{Gr}^{\Phi}_{\{n_k\}}$. The charts of the atlas of this fiber bundle can be parameterized by the elements of the symmetric group $\Sigma_N \ni \alpha$ that permutes the basic vectors. Different permutations form different models of V_k.

Let us consider two different maps $\alpha, \beta \in \Sigma_N$. The corresponding components of the coordinates on the fiber over some $\{\mathscr{F}\}$ we denote by P_k^{α} and P_k^{β}. They differ by the directions of the projections of the correctly defined values of $(A_k - \lambda_k \mathrm{I})|_{V_k}(v)$ on the correctly defined subspace $[F]$. Let the directions of the projections be the directions of the coordinate subspaces E_k^{α} and E_k^{β}.

The differences $P_k^{\beta}(A) - P_k^{\alpha}(A)$ have the same values for all A with the same projection $A_k \in \mathrm{End}\ V_k$ due to Thales theorem. Let us denote

$$\Phi_k^{\alpha\beta}(A_k) := P_k^{\beta}(A) - P_k^{\alpha}(A) \in \mathrm{Hom}(V_k, [F_k]).$$

We treat this transformation as a transition function. It transforms the fiber that is $\mathbf{P} =: \bigoplus_k \mathrm{Hom}\ (V_k, [F_k])$ in the following way: $P_k^{\beta}(A) = P_k^{\alpha}(A) + \Phi_k^{\alpha\beta}(A_k)$, i.e. each component P_k is shifted on the vector depending on the projection of $A \in \mathrm{End}\ V$ to $\mathrm{End}\ V_k$. The image of this projection of A is denoted by $\mathcal{A}_k$.

There is a unique eigenvector of $\mathcal{A}_{k-1}$ corresponding λ_k that connects the points of the coordinate subspaces E_k^{β} and E_k^{α} over the point $\mathcal{A}_k(v) \in V_k \ni v$. The value of the shift is this eigenvector of A_{k-1}. It is the point of V_{k-1} where A_{k-1} acts.

The fiber of the principal bundle over Grassmanian $\mathrm{Gr}^{\Phi}_{\{n_k\}}$ is the affine group. The transition functions are the shifts on vectors with the components $\Phi_k^{\alpha\beta} \in \mathrm{Hom}\ (V_k, [F_k])$:

$$\{v \to f\} \xrightarrow{\Phi_k^{\alpha\beta}} \{v \to f + \Phi_k^{\alpha\beta}(A_k(v))\},$$

We need the values λ_k to restore A itself. To add these values to the model we consider the homothetic dilatation of $[F_k]$ with the coefficient of the dilatation λ_k. We associate the dilatations with the point $(id, \mathrm{I}\lambda_k) \in \mathrm{T}^*\mathrm{GL}([F_k])$ of the cotangent bundle $\mathrm{T}^*\mathrm{GL}([F_k])$.

Let us add a direct summand $\mathrm{T}^*\mathrm{GL}([F_k])$ to the fiber $\mathrm{Hom}(V/F_k, [F_k])$ of the associated bundle. Group $\mathrm{GL}([F_k])$ acts on the linear space $[F_k] = F_{k-1}/F_k$ by the changes of basis. The transition functions change basis on $[F_k]$ that implies the adjoint action on the matrices that represent the points of $\mathrm{GL}([F_k])$ in different charts α and β.

Consider the fiber

$$\mathrm{T}^*\mathrm{GL}([F_k]) \oplus \mathrm{Hom}(V/F_k, [F_k])$$

itself. Any coordinate map $\{\mathbf{e}\}_\alpha$ induces its own coordinates on $[F_k]$. The coordinates are defined by projection of the basis from the coordinate subspace L_k^α to $[F_k]$ parallel to E_k^α. Different charts induce different coordinates on $[F_k]$. The transition function $\phi_{\mathrm{GL}}^{\alpha\beta} \in \mathrm{GL}([F_k])$ is defined as a linear transformation of the basis constructed using E_k^α, L_k^α to the basis constructed by E_k^β, L_k^β on $[F_k]$. The bases are the projections of the bases from L^α and from L^β parallel to E_k^α and to E_k^β correspondingly. It is evident that such transformations define a cocycle: $\phi_{\mathrm{GL}}^{\alpha\beta}\phi_{\mathrm{GL}}^{\beta\gamma}\phi_{\mathrm{GL}}^{\gamma\alpha} = id \in \mathrm{T}^*\mathrm{GL}([F_k])$.

These mappings $\phi_{\mathrm{GL}}^{\alpha\beta}$ act on the points of the fiber $\mathrm{GL}([F_k])$ by the similarity transformations. If $\mathbf{e}^\alpha \to \mathbf{e}^\alpha g^\alpha$ and $\mathbf{e}^\beta \to \mathbf{e}^\beta g^\beta$ is the same point of $\mathrm{GL}([F_k])$ in different charts α and β: $\mathbf{e}^\beta = \mathbf{e}^\alpha \phi_{\mathrm{GL}}^{\alpha\beta}$, then

$$g^\alpha = \phi_{\mathrm{GL}}^{\alpha\beta} g^\beta \phi_{\mathrm{GL}}^{\beta\alpha}.$$

Let us combine the mappings $\phi_{\mathrm{GL}}^{\alpha\beta}$ and $\Phi^{\alpha\beta}$ to one object, namely the set of the transition functions marked by the couples $\alpha, \beta \in \Sigma_N$:

$$\begin{aligned} &\phi^{\alpha\beta} = (\phi_{\mathrm{GL}}^{\alpha\beta}, \Phi^{\alpha\beta}) \ni \otimes_{k=1}^N \mathrm{GL}([F_k]) \times \mathrm{Hom}(V/F_k, [F_k]), \\ &[F_k] := F_k/F_{k-1}. \end{aligned} \tag{1}$$

It is our goal, namely it is a set of the transition functions of the bundle over the Grassmanian $\mathrm{Gr}_{\{n_k\}}^\Phi$. A fiber of the bundle is $\otimes_{k=1}^N \mathrm{T}^*\mathrm{GL}([F_k]) \times \mathrm{Hom}(V/F_k, [F_k])$.

This manifold has evident symplectic structure. The symplectic forms living on each map coincide on the overlappings because the canonical form on the manifold $\mathrm{GL}([F_k])$ does not depend on the coordinates, particularly on the coordinates induced on $\mathrm{GL}([F_k])$ by the bases $\mathbf{e}^\alpha$ or $\mathbf{e}^\beta$.

The orbit $\mathcal{O}$ itself is embedded into this fiberbundle as the submanifold with the fixed coordinates $(id, \mathrm{I}\lambda_k) \in \mathrm{T}^*\mathrm{GL}([F_k])$. These coordinates are the same in all maps.

§5. Extension of LPKK-structure on manifold over all $\mathrm{GL}(N, \mathbb{C})$

The model of the orbit that equipped with the LPKK structure was constructed as the quotion with respect to the subgroup of the commuting with Λ matrices of the right-invariant section $(g, g^{-1}\Lambda)$. We will extend the form on the fibers and construct the form over all $\mathrm{GL}(N, \mathbb{C})$.

Let us fix Young tableaux and consider the submanifold $\{\cup\mathcal{O}\}_{\mathscr{Y}}$ in the space of matrices. Let Young tableaux is standard-marked (see [5]) in a same way.

The special version of the normal Jordan form of $A \in \{\cup\mathcal{O}\}_{\mathscr{Y}}$ corresponds to the standard marking, see [4]. Its geometrical meaning is following.

Consider a basis of the cyclic vectors of A and collect the vectors to sets. Each set consists of generalized eigenvectors of the same order corresponding to some eigenvalue. The sets are ordered in ascending fashion of the orders. The first sets correspond to the not generalized but just the eigenvectors.

Let Λ be the matrix of $\mathcal{A}$ in such basis. It means that the columns of matrix g^{-1} form corresponding Jordan basis for matrix $A = g^{-1}\Lambda g$.

Consider the flag coordinates [3, 5] corresponding this (standard) marking of the diagram. It gives the representation $A = Q\rho_p Q^{-1}$, where Q is block-uni-lower-triangular matrix. Its columns are the projectors of the sets of the basic vectors of the standard Jordan basis $\{\mathbf{e}\}$ on the subspaces of the flag $\{\mathscr{F}\}$ generated by A. The projections are parallel to the coordinate subspaces E_k.

The k's set of the columns of g^{-1} completes the subspace F_k to subspace F_{k+1}. Let us project it on the coordinate subspace V_k where $\mathcal{A}_k$ acts.

The projection of the vectors forming the $k+1$-st set of columns of g^{-1} on V_k parallel to F_k gives some basis of the eigenspace of $\mathcal{A}_k$ corresponding to λ_{k+1}. Let us denote this basis by $\mathbf{g}_k$ Another basis of the same subspace forms the $k+1$-st set of columns of Q. We denote this basis by $\mathbf{q}_k$.

We have constructed two bases of one eigenspace of $\mathcal{A}_k \in \mathrm{End}\ V_k$. This subspace can be identified with $[F_{k+1}] = F_{k+1}/F_k$, and the construction defines an element of $\mathrm{GL}([F_{k+1}])$ that transforms basis $\mathbf{q}_k$ to $\mathbf{g}_k$.

Basis $\mathbf{q}_k$ is the same for all g's that differ on the left factor commuting with Λ. So we have defined local projections u_k^α of the right-invariant section $\cup_g(g, g^{-1}\Lambda)$ to the $\otimes_k \mathrm{GL}([F_k])$.

Each point $(g, g^{-1}\Lambda)$ correctly defines some basis $\mathbf{g}_k$ on each F_{k+1}/F_k. The coordinate charts $\alpha \in \Sigma_N$ define the corresponding bases $\mathbf{q}_k^\alpha$ and transformations $u_k^\alpha \in \mathrm{GL}([F_k])$ that transform $\mathbf{q}_k^\alpha$ to $\mathbf{g}_k$.

Let us consider $\mathrm{GL}([F_k])$ as a base of the cotangent bundle $\mathrm{T}^*\mathrm{GL}([F_k])$. We constructed the following objects

- fiber-bundle $(g, g^{-1}\Lambda) \to \oplus_k [F_k]$ with transition functions $\phi_{\mathrm{GL}}^{\alpha\beta}$,
- the section of the frame bundle $g \to \{\mathbf{g}_k\}_{k=1}^M \subset \oplus_k [F_k]$,
- the local sections $g \to \{\mathbf{q}_k^\alpha\}_{k=1}^M \subset \oplus_k [F_k]$.

It gives the desired local sections u_k^α of $\otimes_k \mathrm{GL}([F_k])$ defined as transformations $\{\mathbf{q}_k^\alpha\}_{k=1}^M$ to $\{\mathbf{g}_k\}_{k=1}^M$.

The different bases $\{\mathbf{q}_k^\alpha\}_{k=1}^M$ have the same values for all points of the fiber $(g, g^{-1}\Lambda) \to \mathcal{O}$ because they are correctly defined on the orbit. On the other hand the different g induce different projections $\mathbf{g}_k$ because they are collected from the columns of matrix g^{-1}. The dimension of the fiber $(g, g^{-1}\Lambda) \to \mathcal{O}$ is the dimension of the subgroup of the matrices commuting with Λ, consequently the dimension of the fiber and the dimension of $\otimes_k \mathrm{GL}([F_k])$ coincide. We have the injective map between the manifolds of the same dimensions, consequently it is a local isomorphism. This isomorphism induce an isomorphism between the tangent spaces to the fiber $g \to g^{-1}\Lambda g$ and $\mathrm{T} \otimes_k \mathrm{GL}([F_k])$.

Consider the linear spaces $[F_k]$. They equiped with the sets of bases $\mathbf{q}_k^\alpha$ that can be considered as the representatives of the points of the quotient space F_k/F_{k-1}.

Consider the cotangent bundle of $\mathrm{GL}([F_k])$. The points of its fibers $\mathrm{gl}^*([F_k])$ can be identified with the elements of the $\mathrm{gl}([F_k])$ using the product $\mathrm{tr}\, AB$, where A and B are the matrix realizations of elements of $\mathrm{gl}([F_k])$ in some basis.

Each orbit with the ordered eigenvalues λ_k defines the set $\mathcal{A}_k$ of the transformations of $V_k, k = 1, \ldots, M$. The contraction of $\mathcal{A}_k$ on $[F_{k+1}]$ is a homothetic dilatation with the coefficient λ_{k+1}. We identify these transformations with the points of $\mathrm{gl}^*([F_k])$.

We have constructed the embedding of the manifold of all right-invariant sections $g \to (g, g^{-1}\Lambda)$ with the same Young tableaux $\mathscr{Y}(\Lambda)$ to the symplectic space $\otimes_k \mathrm{T}^*\mathrm{GL}([F_k]) \times \mathcal{O}$.

The contraction of the symplectic form $\omega := \omega_{\mathcal{O}} + \omega_{\otimes \mathrm{GL}}$ on the embedded orbit is equal to $\omega_{\mathcal{O}}$ because the p-coordinate of the summand $\omega_{\otimes \mathrm{GL}}$ is equal to constant $(\lambda_1 \mathrm{I}, \ldots, \lambda_M \mathrm{I})$. So we constructed an extension of the LPKK form on the orbit.

The form is evidently symplectic as a direct sum of the symplectic spaces. The form is an extension of the form from the quotient of $\mathrm{GL}(N)$ with respect to the commuting with Λ matrices to all $\mathrm{GL}(N)$ because we constructed the isomorphism between the fibers.

§6. Case of general position

Let us consider our construction in the case of general position that is the case of N different eigenvalues λ_k. All the spaces $[F_k]$ are one-dimensional and we need to calculate the proportionality factor between the projections of the columns of g^{-1} on the coordinate subspaces E_k and vectors-columns of the lower-triangular $\mathbf{Q}$.

It is evident that the coefficients that are the elements of one-dimensional $\mathrm{GL}([F_k])$ just the diagonal entries of the factorisation of g^{-1} on the lower-triangular and upper-uni-triangular factors. These values are equal to the ratios of the principal minors $\Delta_k(g^{-1})$ of the neighbor sizes:

$$u_1 = (g^{-1})_{11}, u_2 = \Delta_2(g^{-1})/(g^{-1})_{11}, \dots, u_N = \det g^{-1}/\Delta_{n-1}(g^{-1}).$$

These coefficients are the coordinates in the chart indexed by the numeration of the basic vectors. We consider such chart that all principal minors are not vanish.

Let us consider the cotangent spaces at the corresponding points. Let ξ_k are the coordinates on these spaces in the bases du_k.

The constructed two-form is

$$\omega := \omega_{\mathcal{O}} + \sum_{k=1}^{N} d\xi_k \wedge d\frac{\Delta_k(g^{-1})}{\Delta_{k-1}(g^{-1})}, \quad \Delta_0(g^{-1}) := 1.$$

It is the symplectic form. In this simplest case it can be verified directly.

It is smooth because such permutation of the basic vectors that the matrix of not-degenerated transformation has all not-vanishing principal minors always exists. It is closed because $\omega_{\mathcal{O}}$ is closed and either ξ_k or $\frac{\Delta_k(g^{-1})}{\Delta_{k-1}(g^{-1})}$ are correctly defined functions at the chart.

Form $\omega_{\mathcal{O}}$ does not depend on the multiplication of g on the diagonal factor, consequently we can represent $g = g_{diag}(\mathbf{u})g_{\mathcal{O}}(\mathbf{P}, \mathbf{Q})$ where diagonal g_{diag} and $g_{\mathcal{O}}$ depend on different variables and $g_{\mathcal{O}}$ has constant principal minors. If form ω is identical zero for some $\dot{\mathbf{P}}, \dot{\mathbf{Q}}, \dot{\xi}, \dot{\mathbf{u}}$ and any $\mathbf{P}', \mathbf{Q}', \xi', \mathbf{u}'$ then both summands, $\omega_{\mathcal{O}}$ and the sum over k, are equal to zero separately, because the expression must be zero either for $\mathbf{P}' = 0 = \mathbf{Q}'$ or for $\xi'_k = 0 = u'_k \forall k$. Form $\omega_{\mathcal{O}}$ is not degenerated, consequently $\dot{\mathbf{P}} = 0 = \dot{\mathbf{Q}}$.

Vanishing the sum over k implies vanishing $\dot{\xi}$ and $\dot{\mathbf{u}}$, because all ξ_k and u_k are the independent variables. We see that ω is not degenerated. It completes the verification.

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