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I. Panin

A SHORT EXACT SEQUENCE

Abstract. Let R be a semi-local integral Dedekind domain and K be its fraction field. Let $\mu : \mathbf{G} \to \mathbf{T}$ be an R-group schemes morphism between reductive R-group schemes, which is smooth as a scheme morphism. Suppose that T is an R-torus. Then the map $\mathbf{T}(R)/\mu(\mathbf{G}(R)) \to \mathbf{T}(K)/\mu(\mathbf{G}(K))$ is injective and certain purity theorem is true. These and other results are derived from an extended form of Grothendieck–Serre conjecture proven in the present paper for rings R as above.

§1. Main results

Let R be a commutative unital ring. Recall that an R-group scheme $\mathbf{G}$ is called reductive, (respectively, semi-simple or simple), if it is affine and smooth as an R-scheme and if, moreover, for each algebraically closed field Ω and for each ring homomorphism $R \to \Omega$ the scalar extension $\mathbf{G}_\Omega$ is a connected reductive (respectively, semi-simple or simple) algebraic group over Ω. The class of reductive group schemes contains the class of semi-simple group schemes which in turn contains the class of simple group schemes. This notion of a reductive R-group scheme coincides with [4, Exp. XIX, Definition 2.7]. This notion of a simple R-group scheme coincides with the notion of a simple semi-simple R-group scheme from Demazure and Grothendieck [4, Exp. XIX, Definition 2.7 and Exp. XXIV, 5.3]. Here is our first main result

Theorem 1.1. *Let R be a semi-local integral Dedekind domain. Let K be the fraction field of R. Let $\mu : \mathbf{G} \to \mathbf{T}$ be an R-group scheme morphism between reductive R-group schemes, which is smooth as a scheme morphism. Suppose T is an R-torus. Then the map* $\mathbf{T}(R)/\mu(\mathbf{G}(R)) \to$

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$\mathbf{T}(K)/\mu(\mathbf{G}(K))$ *is injective and the sequence*

$$\{1\} \to \mathbf{T}(R)/\mu(\mathbf{G}(R)) \to \mathbf{T}(K)/\mu(\mathbf{G}(K)) \xrightarrow{\sum r_{\mathfrak{p}}} \bigoplus_{\mathfrak{p}} \mathbf{T}(K)/[\mathbf{T}(R_{\mathfrak{p}}) \cdot \mu(\mathbf{G}(K))] \to \{1\} \tag{1}$$

is exact, where $\mathfrak{p}$ *runs over all non-zero prime ideals of* R *and* $r_{\mathfrak{p}}$ *is the natural map (the projection to the factor group).*

Let us comment on the first assertion of the theorem. Let $\mathbf{H}$ be the kernel of μ. It turns out that $\mathbf{H}$ is a quasi-reductive R-group scheme (see Definition 1.3). There is a sequence of group sheaves $1 \to \mathbf{H} \to \mathbf{G} \to \mathbf{T} \to 1$, which is exact in the étale topology on $SpecR$. Theorem 1.4 yields now the injectivity of the map $\mathbf{T}(R)/\mu(\mathbf{G}(R)) \to \mathbf{T}(K)/\mu(\mathbf{G}(K))$.

Theorem 1.2. *Let* R *be a semi-local integral Dedekind domain. Let* K *be the fraction field of* R*. Let* $\mathbf{G}_1$ *and* $\mathbf{G}_2$ *be two semi-simple* R*-group schemes. Suppose the generic fibres* $\mathbf{G}_{1,K}$ *and* $\mathbf{G}_{2,K}$ *are isomorphic as algebraic* K*-groups. Then the* R*-group schemes* $\mathbf{G}_1$ *and* $\mathbf{G}_2$ *are isomorphic.*

This theorem can not be directly derived from [12] and [13]. Indeed, only geometrically connected group schemes are regarded there. However, to prove Theorem 1.2 we need to work with the automorphism group scheme of a semi-simple R-group scheme. The latter group scheme is not geometrically connected in general.

We state right below a theorem, which asserts that an extended version of Grothendieck–Serre conjecture holds for rings R as above. This latter theorem is proved in this paper. Theorem 1.2 and the first assertion of Theorem 1 are derived from it. To state the mentioned theorem it is convenient to give the following.

Definition 1.3 (quasi-reductive)**.** *Assume that* S *is a Noetherian commutative ring. An* S*-group scheme* $\mathbf{H}$ *is called* quasi-reductive *if there is a finite étale* S*-group scheme* $\mathbf{C}$ *and a smooth* S*-group scheme morphism* $\lambda\colon \mathbf{H} \to \mathbf{C}$ *such that its kernel is a reductive* S*-group scheme and* λ *is surjecive locally in the étale topology on* S*.*

Clearly, reductive S-group schemes are quasi-reductive. Quasi-reductive S-group schemes are *affine and smooth* as S-schemes. There are two types of quasi-reductive S-group schemes, which we are focusing on in the present paper. The first one is the automorphism group scheme of a semi-simple S-group scheme. The second one is obtained as follows: take a reductive

S-group scheme $\mathbf{G}$, an S-torus $\mathbf{T}$ and a smooth S-group morphism $\mu : \mathbf{G} \to \mathbf{T}$. Then one can check that the kernel $\mathbf{H}$ of μ is quasi-reductive. It is an extension of a finite étale S-group scheme $\mathbf{C}$ of multiplicative type via a reductive S-group scheme $\mathbf{G}_0$.

Assume that U is a regular scheme, $\mathbf{H}$ is a quasi-reductive U-group scheme. Recall that a U-scheme $\mathcal{H}$ with an action of $\mathbf{H}$ is called *a principal* $\mathbf{H}$*-bundle over* U, if $\mathcal{H}$ is faithfully flat and quasi-compact over U and the action is simple transitive, that is, the natural morphism $\mathbf{H} \times_U \mathcal{H} \to \mathcal{H} \times_U \mathcal{H}$ is an isomorphism, see [9, Section 6]. Since $\mathbf{H}$ is S-smooth, such a bundle is trivial locally in étale topology but in general not in Zariski topology. Grothendieck and Serre conjectured that for a reductive U-group scheme $\mathbf{H}$ a principal $\mathbf{H}$-bundle $\mathcal{H}$ over U is trivial locally in Zariski topology, if it is trivial generically. A *survey paper* on the topic is [15].

The conjecture is true, if $\Gamma(U, \mathcal{O}_U)$ contains a field (see [7] and [18]). It is proved in [12] that the conjecture is true in general for discrete valuation rings. This result is extended in [19] to the case of semi-local Dedekind integral domains assuming that $\mathbf{G}$ is simple simply connected and isotropic in a certain precise sense. In [13] results of [12] and [19] are extended further. It is proved there that the conjecture is true in general for the case of semi-local Dedekind integral domains. The following result is a further extension of the main theorem of [13].

Theorem 1.4. *Let R be a semi-local integral Dedekind domain. Let K be the fraction field of R. Let $\mathbf{H}$ be a quasi-reductive group scheme over R. Then the map*

$$H^1_{\acute{e}t}(R, \mathbf{H}) \to H^1_{\acute{e}t}(K, \mathbf{H}),$$

induced by the inclusion of R into K, has a trivial kernel. In other words, under the above assumptions on R and $\mathbf{G}$, each principal $\mathbf{H}$-bundle over R having a K-rational point is trivial.

Corollary 1.5. *Under the hypothesis of Theorem 1.4, the map*

$$H^1_{\acute{e}t}(R, \mathbf{H}) \to H^1_{\acute{e}t}(K, \mathbf{H}),$$

induced by the inclusion of R into K, is injective. Equivalently, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are two principal $\mathbf{H}$-bundles isomorphic over $\mathrm{Spec}K$*, then they are isomorphic.*

Proof. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two principal $\mathbf{H}$-bundles isomorphic over $\mathrm{Spec}\, K$. Let $\mathrm{Iso}(\mathcal{H}_1, \mathcal{H}_2)$ be the scheme of isomorphisms of principal $\mathbf{H}$-bundles. This scheme is a principal $\mathrm{Aut}\mathcal{H}_1$-bundle. By Theorem 1.4 it is trivial, and we see that $\mathcal{H}_1 \cong \mathcal{H}_2$. □

Theorems 1.4 and 1.2 are proved in Section 2. Theorem 1.1 is proved in Section 4.

§2. Proof of Theorems 1.4 and 1.2

We begin with the following general

Lemma 2.1. *Let X be a semi-local irreducible Dedekind scheme. Let $\pi : X' \to X$ be a finite étale morphism. Let $\eta \in X$ be the generic point of X. Then sections of π over X are in the bijection with sections of π over η.*

Proof. Clearly, $\mathbf{P}^n(X) = \mathbf{P}^n(\eta)$. Since π is finite it is projective. Hence $X'(X) = X'(\eta)$. □

Corollary 2.2. *Let X, $\eta \in X$ be as in the previous lemma and $\mathbf{E}$ be a finite étale group X-scheme. Then the η-points of $\mathbf{E}$ coincides with the X-points of $\mathbf{E}$.*

Corollary 2.3. *Under the hypothesis of Corollary 2.2 the kernel of the pointed set map $H^1_{\acute{e}t}(X, \mathbf{E}) \to H^1_{\acute{e}t}(\eta, \mathbf{E})$ is trivial.*

Proof. Let $\mathcal{E}$ be a principal $\mathbf{E}$-bundle over X. The standard descent arguments shows that the X-scheme $\mathcal{E}$ is finite and étale. Thus, $\mathcal{E}(X) = \mathcal{E}(\eta)$. This proves the corollary. □

Proof of Theorem 1.4. Since $\mathbf{H}$ is quasi-reductive R-group scheme, there is a finite étale R-group scheme $\mathbf{C}$ and a smooth R-group scheme morphism $\lambda : \mathbf{H} \to \mathbf{C}$ such that its kernel $\mathbf{G}$ is a reductive R-group scheme and λ is surjecive locally in the étale topology on S. The sequence of the étale sheaves $1 \to \mathbf{G} \to \mathbf{H} \to \mathbf{C} \to 1$ is exact. Thus, it induces a commutative diagram of pointed set maps with exact rows

$$\begin{array}{ccccccc}
\mathbf{C}(R) & \xrightarrow{\partial} & H^1_{\text{ét}}(R, \mathbf{G}) & \longrightarrow & H^1_{\text{ét}}(R, \mathbf{H}) & \longrightarrow & H^1_{\text{ét}}(R, \mathbf{C}) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} \\
\mathbf{C}(K) & \xrightarrow{\partial} & H^1_{\text{ét}}(K, \mathbf{G}) & \longrightarrow & H^1_{\text{ét}}(K, \mathbf{H}) & \longrightarrow & H^1_{\text{ét}}(K, \mathbf{C})
\end{array}$$

The map α is bijective by Corollary 2.2, the map δ has the trivial kernel by Corollary 2.3, the map β is injective by 1.5. Now a simple diagram chase shows that $ker(\gamma) = *$. This proves the theorem. □

Remark 2.4. The statement of [1, Lemma 3.7] and its proof are non-acurate both. The authors are forthed to assume the injectivity of the map $H^1_{\text{ét}}(R, \mathbf{G}^0) \to H^1_{\text{ét}}(K, \mathbf{G}^0_R)$.

Proof of Theorem 1.2. The R-group scheme $\underline{\text{Aut}} := \underline{\text{Aut}}_{R-gr-sch}(\mathbf{G}_1)$ is quasi-reductive by [5]. The R-scheme $\underline{\text{Iso}} := \underline{\text{Iso}}_{R-gr-sch}(\mathbf{G_1}, \mathbf{G_2})$ is a principal $\underline{\text{Aut}}$-bundle. An isomorphism $\varphi : \mathbf{G}_{1,K} \to \mathbf{G}_{2,K}$ of algebraic K-groups gives a section of $\underline{\text{Iso}}$ over K. So, $\underline{\text{Iso}}_K$ is a trivial principal $\underline{\text{Aut}}_K$-bundle. Hence $\underline{\text{Iso}}$ is a trivial principal $\underline{\text{Aut}}$-bundle by Theorem 1.4. Thus, it has a section over R. So, there is an R-group scheme isomorphism $\mathbf{G}_1 \cong \mathbf{G}_2$. □

§3. One lemma

Lemma 3.1. *Let X be a regular irreducible affine scheme. Let $\mathbf{G}$ be a reductive X-group scheme and $\mathbf{T}$ be an X-torus. Let $\mu : \mathbf{G} \to \mathbf{T}$ be an X-group schemes morphism, which is smooth as a scheme morphism. Then the kernel of μ is a quasi-reductive X-group scheme.*

Proof. Consider the coradical $Corad(\mathbf{G})$ of $\mathbf{G}$ together with the canonical X-group morphism $\alpha : \mathbf{G} \to Corad(\mathbf{G})$. By the universal property of the X-group morphism α there is a unique X-group morphism $\bar{\mu} : Corad(\mathbf{G}) \to T$ such that $\mu = \bar{\mu} \circ \alpha$. Since μ is surjective locally for the étale topology, hence so is $\bar{\mu}$. Let $\ker(\bar{\mu})$ be the kernel of $\bar{\mu}$ and let $\mathbf{H} := \alpha^{-1}(\ker(\bar{\mu}))$ be the scheme theoretic pre-image of $\ker(\bar{\mu})$. Clearly, $\mathbf{H}$ is a closed X-subgroup scheme of $\mathbf{G}$, which is the kernel of μ. We must check that $\mathbf{H}$ is a quasi-reductive.

The X-group scheme $\ker(\bar{\mu})$ is of multiplicative type. Hence there is a finite X-group scheme $\mathbf{M}$ of multiplicative type and a faithfully flat X-group scheme morphism $can : \ker(\bar{\mu}) \to \mathbf{M}$, which has the following property: for any finite X-group scheme $\mathbf{M}'$ of multiplicative type and an X-group morphism $\varphi : \ker(\bar{\mu}) \to \mathbf{M}'$ there is a unique X-group morphism $\psi : \mathbf{M} \to \mathbf{M}'$ with $\psi \circ can = \varphi$. It is known that the kernel of can is an X-torus. Call it $\mathbf{T}^0$. Since μ is smooth, hence so is $\bar{\mu}$. Thus, the X-group scheme $\ker(\bar{\mu})$ is an X-smooth scheme. This yields that M is étale over X.

Let $\beta = \alpha|_{\mathbf{H}} : \mathbf{H} \to \ker(\bar{\mu})$ and let $\mathbf{G}^0 := \beta^{-1}(\mathbf{T}^0)$ be the scheme theoretic pre-image of $\mathbf{T}^0$. Clearly, $\mathbf{G}^0$ is a closed X-subgroup scheme of $\mathbf{H}$, which is the kernel of the morphism $can \circ \beta : \mathbf{H} \to \mathbf{M}$. Let $\gamma = \beta|_{\mathbf{G}^0} : \mathbf{G}^0 \to \mathbf{T}^0$.

The X-group scheme $\mathbf{M}$ is finite and étale. The morphism *can* is smooth. The morphism β is smooth as a base change of the smooth morphism α. Thus, $\lambda := can \circ \beta$ is smooth. It is also surjective locally in the étale topology on X, because *can* and β have this property. By the construction $\mathbf{G}^0 = \ker(\lambda)$. So, to prove that $\mathbf{H}$ is quasi-reductive it remains to check the reductivity of $\mathbf{G}^0$.

The X-group scheme $\mathbf{G}^0$ is affine as a closed X-subgroup scheme of the reductive X-group scheme $\mathbf{G}$. Prove now that $\mathbf{G}^0$ is smooth over X. Indeed, the morphism γ is smooth as a base change of the smooth morphism α. The X-scheme $\mathbf{T}^0$ is smooth, since it is an X-torus. Thus, the X-scheme $\mathbf{G}^0$ is smooth.

Write X as $SpecS$ for a regular integral domain S. It remains to verify that for each algebraically closed field Ω and for each ring homomorphism $S \to \Omega$ the scalar extension $\mathbf{G}^0_\Omega$ is a connected reductive algebraic group over Ω. Firstly, recall that $\ker(\alpha)$ is a semi-simple S-group scheme. It is the S-group scheme $\mathbf{G}^{ss}$ under the notation of [5]. Clearly, $\ker(\gamma) = \ker(\alpha)$. Thus, $\ker(\gamma) = \mathbf{G}^{ss}$ is a semi-simple S-group scheme. Since the morphism γ is smooth for each algebraically closed field Ω and for each ring homomorphism $S \to \Omega$ we have an exact sequence of smooth algebraic groups over Ω

$$1 \to \mathbf{G}^{ss}_\Omega \to \mathbf{G}^0_\Omega \to \mathbf{T}^0_\Omega \to 1.$$

The groups $\mathbf{T}^0_\Omega$, $\mathbf{G}^{ss}_\Omega$ are connected. Hence the group $\mathbf{G}^0_\Omega$ is connected too. We know already that it is affine.

Finally, check that its unipotent radical $\mathbf{U}$ of $\mathbf{G}^0_\Omega$ is trivial. Since there is no non-trivial Ω-group morphisms $\mathbf{U} \to \mathbf{T}^0_\Omega$, we conclude that $\mathbf{U} \subset \mathbf{G}^{ss}_\Omega$. Since $\mathbf{G}^{ss}_\Omega$ is semi-simple one has $\mathbf{U} = \{1\}$. This completes the proof of the reductivity of the R-group scheme $\mathbf{G}^0$. Thus, the R-group scheme $\mathbf{H}$ is quasi-reductive. This proves the lemma. □

§4. Proof of Theorem 1.1

Proof of the first assertion of Theorem 1.1. Let $\mathbf{H}$ be the kernel of μ. Since μ is smooth, the group scheme sequence

$$1 \to \mathbf{H} \to \mathbf{G} \to \mathbf{T} \to 1$$

gives rise to an short exact sequence of group sheaves in the étale topology. In turn that sequence of sheaves induces a long exact sequence of pointed sets. So, the boundary map $\partial : \mathbf{T}(R) \to \mathrm{H}^1_{\text{ét}}(R, \mathbf{H})$ fits in a commutative diagram

$$\begin{array}{ccc} \mathbf{T}(R)/\mu(\mathbf{G}(R)) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(R,\mathbf{H}) \\ \downarrow & & \downarrow \\ \mathbf{T}(K)/\mu(\mathbf{G}(K)) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(K,\mathbf{H}). \end{array}$$

Clearly, the horizontal arrows have trivial kernels. The right vertical arrow has trivial kernel by Lemma 3.1 and Theorem 1.4. Thus the left vertical arrow has trivial kernel too. Since it is a group homomorphism, it is injective. □

To prove that the sequence (1) is exact in its middle term we need some preparations. Firstly, consider a covariant functor $\mathcal{F}$ on the category of commutative R-algebras, which takes an R-algebra S to $\mathcal{F}(S) := \mathbf{T}(S)/\mu(\mathbf{G}(S))$. There is the following result. Its proof repeats literally the proof of [14, Lemma 4.0.9].

Lemma 4.1. *Under the notation and the hypothesis of Theorem* 1.1 *put* $\mathbf{H} = \ker(\mu)$. *Then the boundary map* $\partial : \mathbf{T}(K)/\mu(\mathbf{G}(K)) \to H^1_{ét}(K, \mathbf{H}_K)$ *is injective.*

Consider the following group and the following pointed set

$$\mathcal{F}_{nr,R}(K) = \bigcap_{\mathfrak{p}} Im[\mathcal{F}(R_{\mathfrak{p}}) \to \mathcal{F}(K)],$$
$$\mathrm{H}^1_{\text{ét}}(K,\mathbf{H}_K)_{nr,R} = \bigcap_{\mathfrak{p}} Im[\mathrm{H}^1_{\text{ét}}(R_{\mathfrak{p}},\mathbf{H}) \to \mathrm{H}^1_{\text{ét}}(K,\mathbf{H}_K)],$$

where $\mathfrak{p}$ runs over all non-zero prime ideals of R. Clearly, one has the following inclusion $\partial_K(\mathcal{F}_{nr,R}(K)) \subseteq \mathrm{H}^1_{\text{ét}}(K,\mathbf{H}_K)_{nr,R}$. Consider now a commutative diagram of the form

$$\begin{array}{ccccccc} & & \tilde{a} & \longmapsto & \tilde{\xi} & \longmapsto & \tilde{\zeta} \\ 1 & \longrightarrow & \mathcal{F}(R) & \xrightarrow{\partial} & \mathrm{H}^1_{\text{ét}}(R,\mathbf{H}) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(R,\mathbf{G}) \\ & & \downarrow{\scriptstyle\epsilon} & & \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\eta} \\ 1 & \longrightarrow & \mathcal{F}(K) & \xrightarrow{\partial_K} & \mathrm{H}^1_{\text{ét}}(K,\mathbf{H}_K) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(K,\mathbf{G}_K) \\ & & a & \longmapsto & \xi & \longmapsto & * \end{array}$$

in which all the maps are canonical, the horizontal lines are exact sequences of pointed sets. The map η has a trivial kernel by the main result of [13], since $\mathbf{G}$ is reductive. The map ∂_K is injective by Lemma 4.1. Using Zariski patching on $Spec(R)$ of principal bundles, we conclude that the image of ρ coincides with $\mathrm{H}^1_{\text{ét}}(K, \mathbf{H}_K)_{nr,R}$.

Proof of the exactness of the sequence (1) in its middle term. We must prove the following equality: $\mathrm{Im}[\mathcal{F}(R) \to \mathcal{F}(K)] = \mathcal{F}_{nr,R}(K)$. Obviously, $\mathrm{Im}[\mathcal{F}(R) \to \mathcal{F}(K)] \subseteq \mathcal{F}_{nr,R}(K)$. It remains to check the opposite inclusion. Take an element $a \in \mathcal{F}_{nr,R}(K)$ and set $\xi = \partial_K(a)$. As mentioned above ξ is in $\mathrm{H}^1_{\text{ét}}(K, \mathbf{H}_K)_{nr,R}$. We already know that ξ can be lifted to an element $\tilde{\xi}$ in $\mathrm{H}^1_{\text{ét}}(R, \mathbf{H})$. Let $\tilde{\zeta}$ be the image of $\tilde{\xi}$ in $\mathrm{H}^1_{\text{ét}}(R, \mathbf{G})$. Note that $\eta(\tilde{\zeta}) = *$. Since the kernel of η is trivial we see that $\tilde{\zeta} = *$. Hence there is an element $\tilde{a}$ in $\mathcal{F}(R)$ such that $\partial(\tilde{a}) = \tilde{\xi}$. The injectivity of ∂_K yields an equality $\epsilon(\tilde{a}) = a$. The exactness of the sequence (1) in its middle term is proved. □

In the rest of the proof we *establish the surjectivity of the map* $\sum r_{\mathfrak{p}}$. Clearly, it is sufficient to prove the surjectivity of the map

$$\mathbf{T}(K) \xrightarrow{\sum r'_{\mathfrak{p}}} \bigoplus_{\mathfrak{p}} \mathbf{T}(K)/\mathbf{T}(R_{\mathfrak{p}}),$$

where $\mathfrak{p}$ runs over all non-zero prime ideal of R and $r'_{\mathfrak{p}}$ is the factorisation map. The rest of the proof will be given in scheme theoretic notation. Namely, put $X = SpecR$, $\mathcal{O} = \Gamma(X, \mathcal{O}_X)$. Thus, $\mathcal{O} = R$. For each closed point x in X write $\mathcal{O}_x$ for $\mathcal{O}_{X,x}$ (the local ring of the point x on the scheme X).

Consider a finite étale Galois morphism $\pi : \tilde{X} \to X$ such that the torus $\mathbf{T}$ splits over $\tilde{X}$ and $\tilde{X}$ is irreducible. Put $\tilde{\mathcal{O}} = \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}})$ and let $\tilde{K}$ be the fraction field of the ring $\tilde{\mathcal{O}}$. For each closed point $x \in X$ consider a ring $\tilde{\mathcal{O}}_x$, which is the semi-local ring $\mathcal{O}_{\tilde{X},\tilde{x}}$ of the finite closed set $\tilde{x} = \pi^{-1}(x)$ in $\tilde{X}$. Let $Gal := Aut(\tilde{X}/X)$ be the Galois group of $\tilde{X}/X$.

Since the torus $\mathbf{T}$ splits over $\tilde{X}$ we have a short exact sequence of *Gal*-modules

$$\{1\} \to \mathbf{T}(\tilde{\mathcal{O}}) \to \mathbf{T}(\tilde{K}) \to \oplus_x \mathbf{T}(\tilde{K})/\mathbf{T}(\tilde{\mathcal{O}}_x) \to \{1\},$$

where x runs over the set of all closed points of the scheme X. This short exact sequence of Gal-modules gives rise to a long exact sequence of Gal-cohomology groups of the form

$$\{1\} \to \mathbf{T}(\mathcal{O}) \xrightarrow{in} \mathbf{T}(K) \to \oplus_x[\mathbf{T}(\tilde{K})/\mathbf{T}(\tilde{\mathcal{O}}_x)]^{Gal}$$
$$\to H^1(Gal, \mathbf{T}(\tilde{\mathcal{O}})) \xrightarrow{H^1(in)} H^1(Gal, \mathbf{T}(\tilde{K})).$$

We claim that the map $H^1(in)$ is a monomorphism. Indeed, the group $H^1(Gal, \mathbf{T}(\tilde{\mathcal{O}}))$ is a subgroup of the group $H^1_{et}(X, \mathbf{T})$ and the group $H^1(Gal, \mathbf{T}(\tilde{K}))$ is a subgroup of the group $H^1_{et}(SpecK, \mathbf{T}_K)$. By Theorem 1.4 the group map $H^1_{et}(X, \mathbf{T}) \to H^1_{et}(SpecK, \mathbf{T}_K)$ is injective. Thus, $H^1(in)$ is injective also. So, we have a short exact sequence of the form $\{1\} \to \mathbf{T}(\mathcal{O}) \xrightarrow{in} \mathbf{T}(K) \to \oplus_x[\mathbf{T}(\tilde{K})/\mathbf{T}(\tilde{\mathcal{O}}_x)]^{Gal} \to \{1\}$.

There is also the complex $\{1\} \to \mathbf{T}(\mathcal{O}) \xrightarrow{in} \mathbf{T}(K) \to \oplus_x \mathbf{T}(K)/\mathbf{T}(\mathcal{O}_x)$. Set $\alpha = id_{\mathbf{T}(\mathcal{O})}$, $\beta = id_{\mathbf{T}(K)}$ and let $\gamma = \oplus_x \gamma_x$, where $\gamma_x : \mathbf{T}(K)/\mathbf{T}(\mathcal{O}_x) \to [\mathbf{T}(\tilde{K})/\mathbf{T}(\tilde{\mathcal{O}}_x)]^{Gal}$ is induced by the inclusion $K \subset \tilde{K}$. The maps α, β and γ form a morphism between this complex and the above short exact sequence. We claim that this morphism is an isomorphism. This claim completes the proof of the theorem.

To prove this claim it is sufficient to prove that γ is an isomophism. Since the map $\mathbf{T}(K) \to \oplus_x[\mathbf{T}(\tilde{K})/\mathbf{T}(\tilde{\mathcal{O}}_x)]^{Gal}$ is an epimorphism, hence so is the map γ. It remains to prove that γ is a monomorphism. To do this it is sufficient to check that for any closed point $x \in X$ the map $\mathbf{T}(K)/\mathbf{T}(\mathcal{O}_x) \to \mathbf{T}(\tilde{K})/\mathbf{T}(\tilde{\mathcal{O}}_x)$ is a monomorphism. We will write ϵ_x for the latter map. We prove below that $ker(\epsilon_x)$ is a torsion group and the group $\mathbf{T}(K)/\mathbf{T}(\mathcal{O}_x)$ has no torsion. These two claims show that the map ϵ_x is injective indeed.

To prove that $ker(\epsilon_x)$ is a torsion group recall that there are norm maps $N_{\tilde{\mathcal{O}}_x/\mathcal{O}_x} : \mathbf{T}(\tilde{\mathcal{O}}_x) \to \mathbf{T}(\mathcal{O}_x)$ and $N_{\tilde{K}/K} : \mathbf{T}(\tilde{K}) \to \mathbf{T}(K)$ (see [14, Section 2]). Those maps induce a homomorphism

$$N_x : \mathbf{T}(\tilde{K})/\mathbf{T}(\tilde{\mathcal{O}}_x) \to \mathbf{T}(K)/\mathbf{T}(\mathcal{O}_x)$$

such that $N_x \circ \epsilon_x =$ the multiplication by d, where d is the degree of $\tilde{K}$ over K. Thus, $ker(\epsilon_x)$ is killed by the integer d.

Show now that the group $\mathbf{T}(K)/\mathbf{T}(\mathcal{O}_x)$ has no torsion. Take an element $a_K \in \mathbf{T}(K)$ and suppose that its class in $\mathbf{T}(K)/\mathbf{T}(\mathcal{O}_x)$ is a torsion element. Let $\tilde{a}_K$ be the image of a_K in $\mathbf{T}(\tilde{K})$. Since $\mathbf{T}$ splits over $\tilde{K}$ we see that $\mathbf{T}(\tilde{K})/\mathbf{T}(\tilde{\mathcal{O}}_x)$ is torsion free. Thus, the class of $\tilde{a}_K$ in $\mathbf{T}(\tilde{K})/\mathbf{T}(\tilde{\mathcal{O}}_x)$ vanishes. So, there is a unique element $\tilde{a}$ in $\mathbf{T}(\tilde{\mathcal{O}}_x)$ whose image in $\mathbf{T}(\tilde{K})$ is $\tilde{a}_K$. Moreover, $\tilde{a}$ is a Gal-invariant element in $\mathbf{T}(\tilde{\mathcal{O}}_x)$, because $\tilde{a}_K$ comes from $\mathbf{T}(K)$. Since $\mathbf{T}(\tilde{\mathcal{O}}_x)^{Gal} = \mathbf{T}(\mathcal{O}_x)$, there is a unique element $a \in \mathbf{T}(\mathcal{O}_x)$ whose image in $\mathbf{T}(\tilde{\mathcal{O}}_x)$ is $\tilde{a}$. Clearly, the image of a into $\mathbf{T}(K)$ is the element a_K. Thus, the class of a_K in $\mathbf{T}(K)/\mathbf{T}(\mathcal{O}_x)$ vanishes. So, the group $\mathbf{T}(K)/\mathbf{T}(\mathcal{O}_x)$ is torsion free.

The injectivity of ϵ_x is proved. This completes the proof of Theorem 1.1.

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St.Petersburg Department
of Steklov Institute of Mathematics
St.Petersburg, Russia
E-mail: panin@pdmi.ras.ru

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