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A COMBINATORIAL FORMULA FOR MONOMIALS IN KONTSEVICH'S ψ-CLASSES

Abstract. Diagonal complexes provide a simplicial model for the Kontsevich's tautological bundles over $\mathcal{M}_{g,n}$. Local combinatorial formula for the first Chern class yields a combinatorial formula for the ψ-classes (that is, first Chern classes of the tautological bundles). In the present paper we derive a formula for arbitrary monomials in ψ-classes.

§1. Preliminaries

Following Harer [3], in [1] we introduced and studied complexes of pairwise non-intersecting curves (called *diagonals*) on an oriented surface. They provide combinatorial simplicial models for the space $\mathcal{M}_{g,n}$ and for tautological bundles over $\mathcal{M}_{g,n}$. In particular, an application of N.Mnev and G. Sharygin's local combinatorial formula has led to and explicitly represent the Chern class of a tautological bundle as a cochain.

The *ψ-classes* $\psi_1, \ldots, \psi_n$, that is, *Chern classes* of the tautological bundles over $\mathcal{M}_{g,n}$ and their products are of a particular interest, see [4]. In the present paper we derive a combinatorial formula for arbitrary monomials in ψ-classes.

Combinatorial models. Let us briefly review the constructions of [1].

Assume that we have an oriented surface F of genus g with b labeled boundary components $B_1, \ldots, B_b$. We fix n distinct labeled points on F not lying on the boundary (*free vertices*). Besides, for each $i = 1, .., b$ we fix one labeled point on each of the boundary component B_i. We assume that F can be triangulated with vertices at the marked points. That is, we exclude all "small" cases (like sphere with two marked points).

Altogether we have $N = n + b$ marked points.

A *pure diffeomorphism* $F \to F$ is an orientation preserving diffeomorphism which maps each marked point to itself.

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A *diagonal* is a simple (that is, not self-intersecting) smooth curve d on F whose endpoints are some of the (possibly the same) vertices such that

(1) d contains no vertices (except for the endpoints).
(2) d does not intersect the boundary (except for its endpoints),
(3) d is homotopic to no edge of the boundary.
Here and in the sequel, we mean homotopy with fixed endpoints in the complement of the vertices $F \setminus \ Vert$. In other words, a homotopy never hits a vertex.
(4) d is non-contractible.

An *admissible diagonal arrangement* (or an *admissible arrangement*, for short) is a non-empty collection of diagonals $\{d_j\}$ with the properties:

(1) Each free vertex is an endpoint of at least one diagonal.
(2) No two diagonals intersect (except for their endpoints).
(3) No two diagonals are homotopic.
(4) The complement of the arrangement and the boundary components $(F \setminus \bigcup d_j) \setminus \bigcup B_i$ is a disjoint union of open disks.

Definition 1. *Two admissible arrangements A_1 and A_2 are weakly equivalent if there exists a composition of a homotopy and a pure diffeomorphism taking A_1 to A_2.*

A pair (g, b) is *stable* if no admissible arrangement has a non-trivial automorphism (that is, each pure diffeomorphism which maps an arrangement to itself, maps each germ of each of d_i to itself).

Throughout the paper we assume that (g, b) **is stable**.

Now we are ready to describe the complex $\mathcal{BD} = \mathcal{BD}_{g,b,n}$

Each element of the poset $\mathcal{BD}_{g,b,n}$ is (the strong equivalence class of) some admissible arrangement $A = \{d_1, \dots, d_m\}$ with a linearly ordered partition $A = \bigsqcup S_i$ into some non-empty sets S_i such that the first set S_1 in the partition is an admissible arrangement.

The partial order on $\mathcal{BD}$ is generated by the following rule:
$(S_1, \dots, S_p) \leqslant (S'_1, \dots, S'_{p'})$ whenever one of the two conditions holds:

(1) We have one and the same arrangement A, and $(S'_1, \dots, S'_{p'})$ is an orientation preserving refinement of $(S_1, \dots, S_p)$.

(2) $p \leqslant p'$, and for all $i = 1, 2, \dots, p$, we have $S_i = S'_i$. That is, $(S_1,\dots, S_p)$ is obtained from $(S'_1,\dots, S'_{p'})$ by removal of $S'_{p+1},\dots, S'_{p'}$.

Let us look at the partial ordering in more details. Given $(S_1, \dots, S_p)$, to list all the elements of $\widetilde{BD}$ that are smaller than $(S_1, \dots, S_p)$ one has

(1) to eliminate some (but not all!) of S_i from the end of the string, and (2) to replace some consecutive collections of sets by their unions.

Example:

$$(\{d_5, d_2\}, \{d_3\}, \{d_1, d_6\}, \{d_4\}, \{d_7\}, \{d_8\}) > (\{d_5, d_2\}, \{d_3, d_1, d_6\}, \{d_4, d_7\}).$$

Assume $b = 1$, that is, there is only one boundary component. Its contraction turns F to a closed surface.

Turning the boundary component B_1 to a new free vertex v labeled by $n+1$ yields a forgetful simplicial mapping

$$\pi : BD_{g,1,n} \to \mathcal{BD}_{g,0,n+1}$$

whose defining rule can be described in the following way. Simplex in $\mathcal{BD}_{g,b,n;1,\dots,n_b}$ corresponds to some admissible arrangement. After contraction of B_1 some of the diagonals may become contractible. Eliminate them. Some of the diagonals may become homotopy equivalent. In each homotopy equivalence class we leave exactly one that belongs to S_i with the smallest index i.

The complex $\mathcal{BD}_{g,0,n+1}$ is homotopy equivalent to the space $\mathcal{M}_{g,n+1}$, and the above map is homotopy equivalent to the Kontsevich's tautological circle bundle L_{n+1} over $\mathcal{M}_{g,n+1}$.

Combinatorial formula for the ψ-classes. The above provides a combinatorial model for the tautological S^1-bundle: we have triangulated base and triangulated total space of the bundle such that the projection is a simplicial map. Thus the local combinatorial formulae for the first Chern class and its powers [5] are applicable.

Let us start with some auxiliary constructions. An *oriented necklace* (or a *necklace*, for short) on letters $a_1, \dots, a_{k+1}$ is an orbit of a word (on the same letters) under cyclic permutations. One thinks of a necklace as of a number of beads colored by numbers $a_1, \dots, a_{k+1}$ on an oriented cyclic thread.

For a necklace ν with odd $k+1$, let $N_{\text{odd}}(\nu)$ (respectively, $N_{\text{even}}(\nu)$) be the number of ways to choose exactly $k+1$ beads of the necklace ν, one bead out of each of the colors, in such a way that the resulted permutation of the chosen beads is odd (respectively, even). The *parity of the necklace* is defined as

$$\mathfrak{p}(\nu) = N_{\text{even}}(\nu) - N_{\text{odd}}(\nu).$$

Set also $N_{a_i}(\nu)$ to be the number of beads colored by a_i, and set

$$\mathfrak{N}(\nu) = N_{a_1}(N_{a_1} + N_{a_2})(N_{a_1} + N_{a_2} + N_{a_3}) \dots (N_{a_1} + N_{a_2} + \dots + N_{a_{k+1}}).$$

In our construction, a k-dimensional simplex $\sigma = \sigma^k$ in $\mathcal{BD}_{g,0,n+1}$ is labeled by some $(S_1, \dots, S_{k+1})$. Therefore the germs of diagonals emanating from the free marked point v_i have associated numbers, and thus give a necklace $\nu(\sigma)$ on letters $1, \dots, k+1$. Although some of the colors might be missing in a particular necklace $\nu(v_i, \sigma)$, the color "1" is always present.

Proposition 1. [1] *The cochain*

$$\psi_i^h(\sigma^{2h}) = \frac{(-1)^h h! \cdot \mathfrak{p}(\nu(v_i, \sigma))}{(2h)! \cdot \mathfrak{N}(\nu))}$$

represents the h-th power of the psi-class.

§2. Monomials in psi-classes

Let us first derive a formula for a two-term monomial $\psi_1^{h_1}\psi_2^{h_2}$.

Given a simplex $\sigma = \sigma^{2h_1+2h_2} \in \mathcal{BD}$, follow the algorithm (illustrated in Figure 1):

(1) Take the necklaces $\nu(v_1, \sigma)$ and $\nu(v_2, \sigma)$ obtained from the germs of edges emanating from the free marked points v_1 and v_2.
(2) Eliminate all the entries from $\nu(v_1, \sigma)$ numbered by $2h_1+2, 2h_1+3$, etc. Denote the resulted necklace by $\overline{\nu}_1$.
(3) Relabel all the beads of $\nu(v_2, \sigma)$ that are labeled by $1, 2, \dots, 2h_1+1$. Their new label should be 1. Denote the resulted necklace by $\overline{\nu}_2$.

Proposition 2. *The cochain*

$$\psi_1^{h_1}\psi_2^{h_2}(\sigma) = \frac{(-1)^{h_1+h_2} h_1! h_2! \cdot \mathfrak{p}(\overline{\nu}_1)\mathfrak{p}(\overline{\nu}_2)}{(2h_1)!(2h_2)! \cdot \mathfrak{N}(\overline{\nu}_1)\mathfrak{N}(\overline{\nu}_2)}$$

represents the desired monomial.

Proof. We shall use the definition of cup product of cochains, see [2].

The simplex $\sigma^{2h_1+2h_2}$ is labeled by some $(S_1, \dots, S_{2h_1+2h_2+1})$. Its vertices have a natural ordering: (S_1), $(S_1 \cup S_2)$, $(S_1 \cup S_2 \cup S_3)$, etc. The "front face" of the simplex, generated by first $2h_1+1$ vertices, is labeled by

$$(S_1, \dots, S_{2h_1+1}),$$

and the back face, generated by last $2h_2+1$ vertices, is

$$(S_1 \cup \dots \cup S_{2h_1+1}, S_{2h_1+2}, \dots, S_{2h_1+2h_2+1}).$$

The front face corresponds to eliminating all the diagonals labeled by an index bigger than $2h_1+1$, whereas back face corresponds to assigning index 1 to all diagonals with indices less or equal to $2h_1+1$.

It remains to combine Proposition 2 with the cup product rules. □

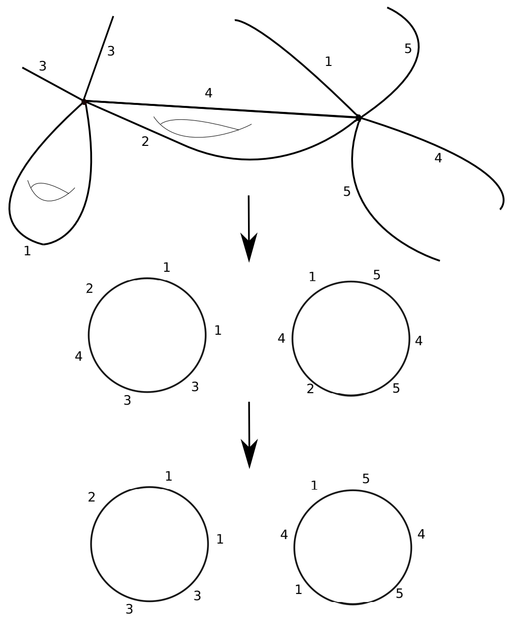

Figure 1. A part of F with two marked points, the associated necklaces $\nu(v_1,\sigma)$ and $\nu(v_2,\sigma)$, and their modifications $\overline{\nu}_1$ and $\overline{\nu}_2$. $h_1 = h_2 = 1$.

The general case is captured by the following theorem, which has an analogous proof:

Theorem 1. *For a monomial*

$$\psi_1^{h_1}\dots\psi_m^{h_m}$$

and a simplex $\sigma = \sigma^{2h_1+\dots+2h_m}$*, do the following:*

For $i = 1,\dots,m$*,*

(1) *Take the necklaces* $\nu(v_i,\sigma)$ *obtained from the germs of edges emanating from the free marked points* v_i *and* v_2*.*
(2) *Eliminate all the entries from* $\nu(v_i,\sigma)$ *numbered by numerics greater than* $2\sum_{j=1}^{i} h_j + 1$*.*

(3) *Relabel all the beads of* $\nu(v_i, \sigma)$ *with labels less or equal to* $2\sum_{j=1}^{i-1} h_j+1$. *Their new label should be* 1.

(4) *Denote the resulted necklace by* $\overline{\nu}_i$.

In these notation, the cochain

$$\psi_1^{h_1} \dots \psi_m^{h_m}(\sigma) = \frac{(-1)^{h_1+..+h_m} h_1! \dots h_m! \cdot \mathfrak{p}(\overline{\nu}_1) \dots \mathfrak{p}(\overline{\nu}_m)}{(2h_1)! \dots (2h_m)! \cdot \mathfrak{N}(\overline{\nu}_1) \dots \mathfrak{N}(\overline{\nu}_m)}$$

represents the desired monomial $\psi_1^{h_1} \dots \psi_m^{h_m}$. □

Remark. *Note that* $\overline{\nu}_i = \overline{\nu}(v_i, \sigma, A_{i-1}, A_i)$ *with* $A_i = 2\sum_{j=1}^{i} h_j + 1$, *i.e. depend on these and only these parameters.*

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