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Abstract. Let R be any associative ring with 1, $n \geqslant 3$, and let A, B be two-sided ideals of R. In the present paper we show that the mixed commutator subgroup $[E(n,R,A), E(n,R,B)]$ is generated as a group by the elements of the two following forms: 1) $z_{ij}(ab,c)$ and $z_{ij}(ba,c)$, 2) $[t_{ij}(a), t_{ji}(b)]$, where $1 \leqslant i \neq j \leqslant n$, $a \in A$, $b \in B$, $c \in R$. Moreover, for the second type of generators, it suffices to fix one pair of indices (i,j). This result is both stronger and more general than the previous results by Roozbeh Hazrat and the authors. In particular, it implies that for all associative rings one has the equality $[E(n,R,A), E(n,R,B)] = [E(n,A), E(n,B)]$ and many further corollaries can be derived for rings subject to commutativity conditions.

**To the remarkable St Petersburg algebraist
Alexander Generalov**

§1. Introduction

In the present note we generalize and strengthen the results by Roozbeh Hazrat and the authors [13, 15, 28] on generation of mutual commutator subgroups of relative and unrelative elementary subgroups in the general linear group. Namely, we both dramatically reduce the sets of generators that occur therein and either seriously weaken, or completely remove commutativity conditions.

Let R be an associative ring with 1, and $\mathrm{GL}(n,R)$ be the general linear group of degree $n \geqslant 3$ over R. As usual, e denotes the identity matrix, whereas e_{ij} denotes a standard matrix unit. For $c \in R$ and $1 \leqslant i \neq j \leqslant n$, we denote by $t_{ij}(c) = e + ce_{ij}$ the corresponding elementary transvection.

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To an ideal $A \trianglelefteq R$, we assign the elementary subgroup

$$E(n, A) = \langle t_{ij}(a),\ a \in A,\ 1 \leqslant i \neq j \leqslant n \rangle.$$

The corresponding relative elementary subgroup $E(n, R, A)$ is defined as the normal closure of $E(n, A)$ in the absolute elementary subgroup $E(n, R)$. From the work of Michael Stein, Jacques Tits, and Leonid Vaserstein it is classically known that *as a group* $E(n, R, A)$ is generated by $z_{ij}(a, c) = t_{ji}(c)t_{ij}(a)t_{ji}(-c)$, where $1 \leqslant i \neq j \leqslant n$, $a \in A$, $c \in R$.

Further, consider the reduction homomorphism $\rho_I \colon \mathrm{GL}(n, R) \longrightarrow \mathrm{GL}(n, R/I)$ modulo I. By definition, the principal congruence subgroup $\mathrm{GL}(n, I) = \mathrm{GL}(n, R, I)$ is the kernel of ρ_I. In other words, $\mathrm{GL}(n, I)$ consists of all matrices g congruent to e modulo I.

A first version of following result was discovered (in a slightly less precise form) by Roozbeh Hazrat and the second author, see [15], Lemma 12. In exactly this form it is stated in our paper [13], Theorem 3A.

Theorem A. *Let R be a quasi-finite ring with 1, let $n \geqslant 3$, and let A, B be two-sided ideals of R. Then the mixed commutator subgroup $[E(n, R, A), E(n, R, B)]$ is generated as a group by the elements of the form*

- *$z_{ij}(ab, c)$ and $z_{ij}(ba, c)$,*
- *$[t_{ij}(a), t_{ji}(b)]$,*
- *$[t_{ij}(a), z_{ij}(b, c)]$,*

where $1 \leqslant i \neq j \leqslant n$, $a \in A$, $b \in B$, $c \in R$.

In the present paper, we prove the following result, which is both *terribly* much stronger, and much more general than Theorem A and which completely solves [13], Problem 1, for the case of GL_n.

Theorem 1. *Let R be any associative ring with 1, let $n \geqslant 3$, and let A, B be two-sided ideals of R. Then the mixed commutator subgroup $[E(n, R, A), E(n, R, B)]$ is generated as a group by the elements of the form*

- *$z_{ij}(ab, c)$ and $z_{ij}(ba, c)$,*
- *$[t_{ij}(a), t_{ji}(b)]$,*

where $1 \leqslant i \neq j \leqslant n$, $a \in A$, $b \in B$, $c \in R$. Moreover, for the second type of generators, it suffices to fix one pair of indices (i, j).

Let us briefly review the sequence of events that led us to this result. It all started a decade ago with our joint papers with Alexei Stepanov and Roozbeh Hazrat [14, 30, 31], which, in particular, gave *three* completely different proofs of the following birelative standard commutator formula, under various commutativity conditions[1]. In turn, this formula generalised a great number of preceding results due to Hyman Bass, Alec Mason and Wilson Stothers, Andrei Suslin, Leonid Vaserstein, Zenon Borewicz and the first author, and many others, see, for instance [2,3,18,19,25,27], and a complete bibliography of early papers in [6,10,32]. Compare also [7–9,13] for a detailed description of the recent work in the area.

Theorem B. *Let R be a quasi-finite ring with 1, let $n \geqslant 3$, and let A, B be two-sided ideals of R. Then the following birelative standard commutator formula holds*

$$[E(n,R,A), \mathrm{GL}(n,R,B)] = [E(n,R,A), E(n,R,B)].$$

The condition in the above theorem is very general and embraces very broad class of rings, but some commutativity condition is necessary here, since it is known that the standard commutator formula may fail for general associative rings even in the absolute case, see [5].

Last year the first author noticed that for *commutative* rings an argument from his paper with Alexei Stepanov [24] implies that the standard commutator formula holds also in the following unrelativised form, see [28], Theorem 1.

Theorem C. *Let R be a commutative ring with 1, let $n \geqslant 3$, and let A, B be two-sided ideals of R. Then the following commutator formula holds*

$$[E(n,A), \mathrm{GL}(n,R,B)] = [E(n,A), E(n,B)].$$

In particular, this immediately implies the following striking equality, see [28], Theorem 2.

Theorem D. *Let R be a commutative ring with 1, let $n \geqslant 3$, and let A, B be two-sided ideals of R. Then one has*

$$[E(n,R,A), E(n,R,B)] = [E(n,A), E(n,B)].$$

Thereupon, the second author immediately suggested that since everything occurs inside the absolute elementary group $E(n,R)$, one should be able to prove Theorem D directly, by looking at the elementary generators in Theorem A and proving that the third type of generators are

[1]The third of these proofs was essentially reduction to the absolute case via level calculations, as discovered earlier by Hong You [36], of which we were not aware at the time of writing these papers.

redundant. Over *commutative* rings this was essentially accomplished in the more general context of Chevalley groups in our joint paper [33].

Immediately thereafter we started to work on the unitary sequel [34] and discovered that most of the requisite results, apart from the unitary analogue of Theorem A, hold for *arbitrary* form rings, without any commutativity conditions. This propted us to look closer inside the proofs of [13], Theorems 3A and 3B, and the Main Lemmas of [33, 34]. We discovered that all references to Theorem B or any of its special cases can be easily replaced by elementary calculations that hold over arbitrary associative rings, and only depend on Steinberg relations (so that they can be carried out already in Steinberg groups).

Finally, attempting to strengthen the birelative standard commutator formula in the arithmetic case [29], the first author was forced to look for a stronger version of Theorem 1, with demoted set of generators. But a calculation that procures such a reduction was already contained in the papers on bounded generation, see, for instance, [4, 26]. A similar calculation is hidden also in the proof of the main theorem in the recent preprint by Andrei Lavrenov and Sergei Sinchuk [17]. Observe that, as discovered by Wilberd van der Kallen [16], and amply developed by Stepanov [21,22], one could reduce also the number of requisite $z_{ij}(a, c)$'s.

Since both types of generators in Theorem 1 already belong to $[E(n, A), E(n, B)]$, we get the following generalisation of Theorem D, for *arbitrary* associative rings.

Theorem 2. *Let R be any associative ring with 1, let $n \geqslant 3$, and let A, B be two-sided ideals of R. Then one has*

$$[E(n, R, A), E(n, R, B)] = [E(n, A), E(n, B)].$$

In turn, together with Theorem B this last result immediately implies the following very broad generalisation of Theorem C.

Theorem 3. *Let R be a quasi-finite ring with 1, let $n \geqslant 3$, and let A, B be two-sided ideals of R. Then the following commutator formula holds*

$$[E(n, A), \mathrm{GL}(n, R, B)] = [E(n, A), E(n, B)].$$

In §2 we prove Theorem 1, and thus also Theorems 2 and 3. Finally, in §3 we establish some further corollaries and variations of these results and make some further related observations.

In the present paper we describe part of the astounding recent progress in the direction of unrelativisation, whose first steps were presented in our talk "Relativisation and unrelativisation" at the Polynomial Computer Algebra 2019 (see `http://pca-pdmi.ru/2019/program`, April 19).

§2. The proof of Theorem 1

Everywhere below the commutators are left-normed so that for two elements x, y of a group G one has $[x,y] = x^y \cdot y^{-1} = xyx^{-1}y^{-1}$. In the sequel we repeatedly use standard commutator identities such as $[xy, z] = {}^x[y,z] \cdot [x,z]$ or $[x, yz] = [x,y] \cdot {}^y[x,z]$ without any explicit reference.

The following lemma is a classical result due to Stein, Tits and Vaserstein, see [27].

Lemma 1. *Let R be an associative ring with 1, $n \geqslant 3$, and let A be a two-sided ideal of R. Then as a subgroup $E(n,R,A)$ is generated by $z_{ij}(a,c)$, for all $1 \leqslant i \neq j \leqslant n$, $a \in A$, $c \in R$.*

Since $E(n,R,A)$ is normal in $E(n,R)$ by the very definition, in particular this lemma implies that every elementary conjugate of $z_{ij}(a,c)$ is again a product of generators of the same type.

The following result is [15], Lemma 12, a detailed proof is also reproduced in [13], Lemma 2A.

Lemma 2. *Let R be an associative ring with 1, let $n \geqslant 3$, and let A, B be two-sided ideals of R. Then the mixed commutator subgroup $[E(n,R,A), E(n,R,B)]$ is generated as a group by the elements of the form*

- ${}^x z_{ij}(ab,c)$ *and* ${}^x z_{ij}(ba,c)$,
- ${}^x[t_{ij}(a), t_{ji}(b)]$,
- ${}^x[t_{ij}(a), z_{ij}(b,c)]$,

where $1 \leqslant i \neq j \leqslant n$, $a \in A$, $b \in B$, $c \in R$, and $x \in E(n,R)$.

Both types of generators in the first item belong to $E(n,R,AB+BA)$ and Lemma 1 implies that for them x can be removed, without affecting the subgroup they generate. The first type of generators listed in Theorem 1 still generate the same group $E(n,R,AB+BA)$.

An obvious level calculation (see, for instance, [30], Lemma 3 or [13], Lemma 1A) shows the other two types of generators listed in Theorem A still belong to the congruence subgroup $\mathrm{GL}(n,R,AB+BA)$. Now, in the conditions of Theorem B, an elementary conjugate of these generators is again the same generator, modulo the subgroup $E(n,R,AB+BA)$.

However, it is very easy to get rid of any reference to Theorem B here. Let us start with the second type of generators $z = [t_{ij}(a), t_{ji}(b)]$.

Lemma 3. *Let R be an associative ring with 1, $n \geqslant 3$, and let A, B be two-sided ideals of R. Then for any $1 \leqslant i \neq j \leqslant n$, $a \in A$, $b \in B$, and any $x \in E(n,R)$ the conjugate ${}^x[t_{ij}(a), t_{ji}(b)]$ is congruent to $[t_{ij}(a), t_{ji}(b)]$ modulo $E(n,R,AB+BA)$.*

Proof. Clearly, $z = [t_{ij}(a), t_{ji}(b)]$ resides in the image of the fundamental embedding of $E(2,R)$ into $E(n,R)$ in the i-th and j-th rows and columns, where one has

$$z = \left[\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\right] = \begin{pmatrix} 1+ab+abab & -aba \\ bab & 1-ba \end{pmatrix}$$

and

$$z^{-1} = \left[\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}, \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1-ab & aba \\ -bab & 1+ba+baba \end{pmatrix}.$$

Consider the elementary conjugate ${}^x z$. We argue by induction on the length of $x \in E(n,R)$ in elementary generators. Let $x = y t_{kl}(c)$, where $y \in E(n,R)$ is shorter than x, whereas $1 \leqslant k \neq l \leqslant n$, $c \in R$.

• If $k, l \neq i, j$, then $t_{kl}(c)$ commutes with z and can be discarded.

• On the other hand, for any $h \neq i, j$ the above formulas for z and z^{-1} immediately imply that

$$[t_{ih}(c), z] = t_{ih}(-abc-ababc) t_{jh}(-babc), \quad [t_{jh}(c), z] = t_{ih}(abac) t_{jh}(bac),$$
$$[t_{hi}(c), z] = t_{hi}(cab) t_{hj}(-caba), \quad [t_{hj}(c), z] = t_{hi}(cbab) t_{hj}(-cba-cbaba).$$

All factors on the right hand side belong already to $E(n, AB+BA)$ This means that

$${}^x z \equiv {}^y z \pmod{E(n,R,AB+BA)}.$$

• Finally, for $(k,l) = (i,j), (j,i)$ we can take an $h \neq i, j$ and rewrite $t_{kl}(c)$ as a commutator $t_{ij}(c) = [t_{ih}(c), t_{hj}(1)]$ or $t_{ji}(c) = [t_h(c), t_{hi}(1)]$ and apply the previous item to get the same congruence modulo $E(n,R,AB+BA)$.

By induction we get that ${}^x z \equiv z \pmod{E(n,R,AB+BA)}$. □

Thus, to prove the first claim of Theorem 1 it only remains to establish the following lemma. For commutative rings it is essentially the simplest special case of the Main Lemma of [33]. However, there it is expressed in the language of root elements. Even though commutativity is not used in the proof in any material way, it is formally assumed. For completeness, we reproduce the proof of a somewhat stronger fact, in matrix notation.

Lemma 4. *Let R be an associative ring with 1, $n \geqslant 3$, and let A, B be two-sided ideals of R. Then for any $1 \leqslant i \neq j \leqslant n$, $a \in A$, $b \in B$, $c \in R$ and any $x \in E(n,R)$ the conjugate ${}^x[t_{ij}(a), z_{ij}(b,c)]$ of a generator of the third type is congruent to an elementary conjugate of some generator $[t_{kl}(a'), t_{lk}(b')]$, $1 \leqslant k \neq l \leqslant n$, $a' \in A$, $b' \in B$, of the second type, modulo $E(n,R,AB+BA)$.*

Proof. Indeed, let $z = [t_{ij}(a), z_{ij}(b,c)]$. Take any $h \neq i,j$. Then

$$z = [t_{ij}(a), z_{ij}(b,c)] = t_{ij}(a) \cdot {}^{z_{ij}(b,c)}t_{ij}(-a) = t_{ij}(a) \cdot {}^{z_{ij}(b,c)}[t_{ih}(1), t_{hj}(-a)].$$

Thus,

$$\begin{aligned} z &= t_{ij}(a) \cdot [{}^{z_{ij}(b,c)}t_{ih}(1), {}^{z_{ij}(b,c)}t_{hj}(-a)] \\ &= t_{ij}(a) \cdot [t_{ih}(1-bc)t_{jh}(-cbc), t_{hi}(-acbc)t_{hj}(-a(1-cb))] \\ &= t_{ij}(a) \cdot [t_{ih}(1)u, t_{hj}(-a)v], \end{aligned}$$

where

$$u = t_{jh}(-cbc)t_{ih}(-bc) \in E(n,B), \quad v = t_{hi}(-acbc)t_{hj}(acb) \in E(n,AB).$$

Thus,

$$z \equiv t_{ij}(a) \cdot [t_{ih}(1)u, t_{hj}(-a)] \pmod{E(n,R,AB+BA)}.$$

On the other hand,

$$t_{ij}(a) \cdot [t_{ih}(1)u, t_{hj}(-a)] = t_{ij}(a) \cdot {}^{t_{ih}(1)}[u, t_{hj}(-a)] \cdot t_{ij}(-a),$$

whereas

$$\begin{aligned} [u, t_{hj}(-a)] = {}^{t_{jh}(-cbc)}t_{ih}(bca) \cdot [t_{jh}(-cbc), t_{hj}(-a)] \equiv \\ [t_{jh}(-cbc), t_{hj}(-a)] \pmod{E(n,R,AB+BA)}. \end{aligned}$$

Summarising the above, we see that

$${}^x z \equiv {}^{x t_{ij}(a) t_{ih}(1)}[t_{jh}(-cbc), t_{hj}(-a)] \pmod{E(n,R,AB+BA)},$$

where $[t_{jh}(-cbc), t_{hj}(-a)]$ is the second type generator, as claimed. □

At this point we have already established the first claim of Theorem 1 – and thus also Theorems 2 and 3. The rest is a bonus, that we need for more sophisticated applications. The proof of the final claim of Theorem 1 is in fact a refinement of the proof of [29], Theorem 3. Again, formally commutativity was assumed there, but can be easily circumvented.

Lemma 5. *Let R be an associative ring with 1, $n \geqslant 3$, and let A, B be two-sided ideals of R. Then for any $1 \leqslant i \neq j \leqslant n$, any $1 \leqslant k \neq l \leqslant n$, and $a \in A$, $b \in B$, the elementary commutator $[t_{ij}(a), t_{ji}(b)]$ is congruent to $[t_{kl}(a), t_{lk}(b)]$ modulo $E(n, R, AB + BA)$.*

Proof. Take any $h \neq i, j$ and rewrite the elementary commutator $z = [t_{ij}(a), t_{ji}(b)]$ as

$$z = t_{ij}(a) \cdot {}^{t_{ji}(b)}t_{ij}(-a) = t_{ij}(a) \cdot {}^{t_{ji}(b)}[t_{ih}(a), t_{hj}(-1)].$$

Expanding the conjugation by $t_{ji}(b)$, we see that

$$z = t_{ij}(a) \cdot [{}^{t_{ji}(b)}t_{ih}(a), {}^{t_{ji}(b)}t_{hj}(-1)] = t_{ij}(a) \cdot [t_{jh}(ba)t_{ih}(a), t_{hj}(-1)t_{hi}(b)].$$

Now, the first factor $t_{jh}(ba)$ of the first argument in this last commutator already belongs to the group $E(n, BA)$ which is contained in $E(n, R, AB + BA)$. Thus, as above,

$$z \equiv t_{ij}(a) \cdot [t_{ih}(a), t_{hj}(-1)t_{hi}(b)] \,(\bmod\, E(n, R, AB + BA)) \,.$$

Using multiplicativity of the commutator w.r.t. the second argument, cancelling the first two factors of the resulting expression, and then applying Lemma 3 we see that

$$z \equiv {}^{t_{hj}(-1)}[t_{ih}(a), t_{hi}(b)] \equiv [t_{ih}(a), t_{hi}(b)] \,(\bmod\, E(n, R, AB + BA)) \,.$$

Similarly, rewriting the commutator z differently, as

$$z = [t_{ij}(a), t_{ji}(b)] = {}^{t_{ij}(a)}t_{ji}(b) \cdot t_{ji}(-b) = {}^{t_{ij}(a)}[t_{jh}(b), t_{hi}(1)] \cdot t_{ji}(-b),$$

we get the congruence

$$z \equiv [t_{hj}(a), t_{jh}(b)] \,(\bmod\, E(n, R, AB + BA)) \,.$$

Obviously, for $n \geqslant 3$ we can pass from any position (i, j), $i \neq j$, to any other such position (k, l), $k \neq l$, by a sequence of at most three such elementary moves. □

This finishes the proof of Theorem 1.

§3. Further variations and final remarks

The following result is a generalisation of the unrelative normality theorem by Bogdan Nica, see [20], Theorem 2, which pertained to the commutative case. It is an immediate corollary of our Theorem 3.

Theorem 4. *Let R be a quasi-finite ring with 1, let $n \geqslant 3$, and let A be a two sided ideal of R. Then $E(n, A)$ is normal in $\mathrm{GL}(n, R, A)$.*

Let us mention another amazing corollary of Theorem 1, in the style of stability results without stability conditions by Tony Bak, see [1]. With this end, observe that Lemma 3 implies that the quotient

$$[E(n,A),E(n,B)]/E(n,R,AB+BA)$$

is central in $E(n,R)/E(n,R,AB+BA)$. In other words, the following holds.

Lemma 6. *Let R be an associative ring with 1, $n \geqslant 3$, and let A, B be two-sided ideals of R. Then*

$$[[E(n,A),E(n,B)],E(n,R)] = E(n,R,AB+BA).$$

But now Theorem 1 implies surjective stability of such quotients, which is a generalisation of the first half of [13], Lemma 15, to arbitrary associative rings, without any stability conditions, or commutativity conditions. Indeed, in view of Theorem 1 and Lemma 6 as a normal subgroup of $E(n,R)$ the group $[E(n,A),E(n,B)]$ is generated by $[E(3,A),E(3,B)]$. This can be restated as follows.

Theorem 5. *Let R be any associative ring with 1, and let A and B be two sided ideals of R. Then for all $n \geqslant 3$ the stability map*

$$[E(n,A),E(n,B)]/E(n,R,AB+BA) \longrightarrow [E(n+1,A),E(n+1,B)]/E(n+1,R,AB+BA)$$

is surjective.

This quotient occurs surprisingly often in seemingly unrelated problems, and is so interesting in itself, that we are now hatching the idea to definitively comprehend its structure. Lemmas 3 and 5 assert that some elementary commutators and their conjugates are congruent modulo $E(n,R,AB+BA)$. We have several further results in the same spirit. For instance, one has

- $[t_{ij}(ac),t_{ji}(b)] \equiv [t_{ij}(a),t_{ji}(cb)] \pmod{E(n,R,AB+BA)}$,
- $[t_{ij}(a_1+a_2),t_{ji}(b)] \equiv [t_{ij}(a_1),t_{ji}(b)]\cdot[t_{ij}(a_2),t_{ji}(b)] \pmod{E(n,R,AB+BA)}$,
- $[t_{ij}(a),t_{ji}(b_1+b_2)] \equiv [t_{ij}(a),t_{ji}(b_1)]\cdot[t_{ij}(a),t_{ji}(b_2)] \pmod{E(n,R,AB+BA)}$,
- $[t_{ij}(a),t_{ji}(b)]^{-1} \equiv [t_{ij}(-a),t_{ji}(b)] \equiv [t_{ij}(a),t_{ji}(-b)] \pmod{E(n,R,AB+BA)}$,

- $[t_{ij}(a_1), t_{ji}(b)] \equiv [t_{ij}(a_2), t_{ji}(b)] \pmod{E(n, R, AB + BA)}$,
  if $a_1 \equiv a_2 \pmod{AB + BA + A^2}$
- $[t_{ij}(a), t_{ji}(b_1)] \equiv [t_{ij}(a), t_{ji}(b_1)] \pmod{E(n, R, AB + BA)}$,
  if $b_1 \equiv b_2 \pmod{AB + BA + B^2}$

etc. We have not made any attempt to systematically collect all such congruences in the present article, since they are not directly needed to prove Theorem 1. But they may turn out very useful to control the quotient $[E(n, A), E(n, B)]/E(n, R, AB + BA)$. Observe that by Lemma 6 this quotient is central in $E(n, R)/E(n, R, AB + BA)$, and thus, in particular, it is itself abelian. We intend to list all such properties in a subsequent paper, where we propose to assail the following tantalising problem.

Problem 1. *Give a presentation of*

$$[E(n, A), E(n, B)]/\operatorname{EE}(n, R, AB + BA)$$

by generators and relations.

Let us mention yet another corollary of Theorem 1. Let $U(n, R)$ and $U^-(n, R)$ be the groups of upper unitriangular and lower unitriangular matrices, respectively. These are unipotent radicals of the standard Borel subgroup, and its opposite Borel subgroup. Further, set

$$U(n, I) = U(n, R) \cap \operatorname{GL}(n, R, I), \quad U^-(n, I) = U^-(n, R) \cap \operatorname{GL}(n, R, I).$$

In [28] we considered another birelative group

$$\operatorname{EE}(n, A, B) = \langle U(n, A), U^-(n, B) \rangle,$$

and established that for *commutative* rings this group contains $[E(n, A), E(n, B)]$, see [28], Theorem 3. Since in the case $\operatorname{EE}(n, A, B)$ contains $E(n, R, AB)$ by [29], Lemma 8, this theorem immediately follows from our Theorem 1. It is natural to expect that an analogue of this result holds over arbitrary associative rings.

Problem 2. *Let R be any associative ring with 1, let $n \geqslant 3$, and let A, B be two-sided ideals of R. Prove that*

$$[E(n, A), E(n, B)] \leqslant \operatorname{EE}(n, A, B).$$

The difficulty now is exactly to prove that

$$E(n, R, AB + BA) \leqslant \operatorname{EE}(n, A, B).$$

In the non-commutative case the argument used in the proof of [29], Lemma 8, only works in the following form.

If $i, j \neq n$, there exists an $h > i, j$ so that one can express $t_{ij}(ab)$ as

$$t_{ij}(ab) = [t_{ih}(a), t_{hj}(b)] \in [U(n, A), U^-(n, B)],$$

and conclude that $z_{ij}(ab, c) \in \mathrm{EE}(n, A, B)$.

Similarly, $i, j \neq 1$, there exists an $h < i, j$ so that one can express $t_{ij}(ba)$ as

$$t_{ij}(ba) = [t_{ih}(b), t_{hj}(a)] \in [U^-(n, B), U(n, A)],$$

and conclude that $z_{ij}(ba, c) \in \mathrm{EE}(n, A, B)$.

In the case of commutative rings (or, more generally, when $AB = BA$) this implies that $E(n, R, AB + BA) \in \mathrm{EE}(n, A, B)$, see [16, 21, 22]. But in the general case this would require some additional reasoning.

Let us mention an even more challenging related question. Namely, let P be a proper standard parabolic subgroup of $\mathrm{GL}(n, R)$. We can define the corresponding subgroup of $\mathrm{EE}(n, A, B)$ as follows:

$$\mathrm{EE}_P(n, A, B) = \langle U_P(A), U_P^-(B)\rangle,$$

where $U_P(A)$ and $U_P^-(B)$ are the intersections of $U(n, A)$ and $U^-(n, B)$ with the unipotent radicals U_P and U_P^- of P and its opposite standard parabolic P^-, respectively. In the definition of $\mathrm{EE}(n, A, B)$ itself $P = B(n, R)$ is the standard Borel subgroup. However, in many cases it is technically much more expedient to work with the maximal standard parabolics instead, see, for instance, the works by Alexei Stepanov [21, 22].

Problem 3. *Let R be any associative ring with* 1*, let $n \geqslant 3$, and let A, B be two-sided ideals of R. Prove that*

$$[E(n, A), E(n, B)] \leqslant \mathrm{EE}_P(n, A, B).$$

The next problem proposes to generalise [9], Theorem 8A, and [13], Theorem 5A, from quasi-finite rings, to arbitrary associative rings. In other words, to prove that any multiple commutator of relative or unrelative elementary subgroups is equal to some double such commutator, see [9, 12, 13, 15, 23]

Here $A \circ B = AB + BA$ stands for the symmetrised product of two sided ideals A and B. In general, the symmetrised product is not associative. Thus, when writing something like $A \circ B \circ C$, we have to specify the order in which products are formed. for notation pertaining to multiple commutators.

Let G be a group and $H_1, \ldots, H_m \leqslant G$ be its subgroups. There are many ways to form a higher commutator of these groups, depending on where we put the brackets. Thus, for three subgroups $F, H, K \leqslant G$ one can form two triple commutators $[[F, H], K]$ and $[F, [H, K]]$. Usually, we write $[H_1, H_2, \dots, H_m]$ for the *left-normed* commutator, defined inductively by

$$[H_1, \dots, H_{m-1}, H_m] = [[H_1, \dots, H_{m-1}], H_m].$$

To stress that here we consider *any* commutator of these subgroups, with an arbitrary placement of brackets, we write $[\![H_1, H_2, \dots, H_m]\!]$. Thus, for instance, $[\![F, H, K]\!]$ refers to any of the two arrangements above.

Actually, a specific arrangment of brackets usually does not play major role in our results – apart from one important attribute. Namely, what will matter a lot is the position of the outermost pairs of inner brackets. Namely, every higher commutator subgroup $[\![H_1, H_2, \dots, H_m]\!]$ can be uniquely written as

$$[\![H_1, H_2, \dots, H_m]\!] = [[\![H_1, \dots, H_h]\!], [\![H_{h+1}, \dots, H_m]\!]],$$

for some $h = 1, \dots, m-1$. This h will be called the cut point of our multiple commutator.

Problem 4. *Let R be any associative ring with 1, let $n \geqslant 3$, and let $A_i \trianglelefteq R$, $i = 1, \dots, m$, be two-sided ideals of R. Consider an arbitrary arrangment of brackets $[\![\dots]\!]$ with the cut point h. Then one has*

$$[\![E(n, I_1), E(n, I_2), \dots, E(n, I_m)]\!] = [E(n, I_1 \circ \dots \circ I_h), E(n, I_{h+1} \circ \dots \circ I_m)],$$

where the bracketing of symmetrised products on the right hand side coincides with the bracketing of the commutators on the left hand side.

Observe that Theorem A and its analogues were used by Alexei Stepanov in his remarkable results on bounded width of commutators with respect to elementary generators, see [23], and our survey [8]. Now, it would be natural to refer in these results to our new reduced set of generators from Theorem 1.

Analogues of our Theorems 1 and 2 hold for Bak's unitary groups over *arbitrary* form rings. In particular, this generalises [12], Theorem 9 and [13], Theorem 3B. Also, it solves [13], Problem 1 for the unitary case. These results are now incorporated in our unitary paper [34]. A full analogue of Theorem 1 for Chevalley groups is much more difficult even in the commutative case, and will be published in [35].

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