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NOTES ON A GROTHENDIECK–SERRE CONJECTURE IN MIXED CHARACTERISTIC CASE

Abstract. Let R be a discrete valuation ring with an infinite residue field, X be a smooth projective curve over R. Let $\mathbf{G}$ be a simple simply-connected group scheme over R and E be a principal $\mathbf{G}$-bundle over X. We prove that E is trivial locally for the Zariski topology on X providing it is trivial over the generic point of X. The main aim of the present paper is to develop a method rather than to get a very strong concrete result.

In honour of 80-th birthday
of academician V.P. Platonov

§1. Main results

Let S be a commutative unital ring. Recall that an S-group scheme $\mathbf{G}$ is called *reductive*, if it is affine and smooth as an S-scheme and if, moreover, for each algebraically closed field Ω and for each ring homomorphism $S \to \Omega$ the scalar extension $\mathbf{G}_\Omega$ is a connected reductive algebraic group over Ω. This definition of a reductive R-group scheme coincides with [2, Exp. XIX, Definition 2.7].

Assume that U is a regular scheme, $\mathbf{G}$ is a reductive U-group scheme. Recall that a U-scheme $\mathcal{G}$ with an action of $\mathbf{G}$ is called *a principal* $\mathbf{G}$*-bundle over* U, if $\mathcal{G}$ is faithfully flat and quasi-compact over U and the action is simple transitive, that is, the natural morphism $\mathbf{G} \times_U \mathcal{G} \to \mathcal{G} \times_U \mathcal{G}$ is an isomorphism, see [11, Section 6]. It is well known that such a bundle is trivial locally in étale topology but in general not in Zariski topology. Grothendieck and Serre conjectured that $\mathcal{G}$ is trivial locally in Zariski topology, if it is trivial generically.

More precisely, a well-known conjecture due to J.-P. Serre and A. Grothendieck (see [24, Remarque, p. 31], [8, Remarque 3, p. 26–27], and [14,

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Remarque 1.11.a]) asserts that given a regular local ring S and its field of fractions L and given a reductive group scheme $\mathbf{G}$ over S, the map

$$H^1_{\text{ét}}(S, \mathbf{G}) \to H^1_{\text{ét}}(L, \mathbf{G}),$$

induced by the inclusion of S into L, has a trivial kernel. The conjecture is true if S contains a field (see [5] and [22]). A *survey article* [19] on the topic is published in proceedings of ICM-2018.

The following theorem is the main result of the present paper.

Theorem 1.1. *Let R be a discrete valuation ring with an infinite residue field, $\bar{X}$ be a smooth projective irreducible curve over R. Let $\mathbf{G}$ be a simple simply-connected group scheme over R and E be a principal $\mathbf{G}$-bundle over $\bar{X}$, which is trivial over the generic point of $\bar{X}$. Then E is trivial locally for the Zariski topology on $\bar{X}$.*

Theorem 1.2 (Geometric)**.** *Let R be a discrete valuation ring with an infinite residue field, $\bar{X}$ be a smooth projective irreducible curve over R. Let $Z \subseteq \bar{X}$ be a closed subset of pure codimension one, which is finite over R. Then there is an affine open X^0 in X containing Z, a monic polynomial* $\mathrm{h} \in R[t]$, *an étale morphism $\tau : X^0 \to \mathbb{A}^1_R$, a function* $\mathrm{f} \in R[X^0]$ *such that* $\mathrm{f} = \tau^*(\mathrm{h})$ *and the following square*

$$\begin{array}{ccc} X^0_{\mathrm{f}} & \xrightarrow{inc} & X^0 \\ \downarrow{\scriptstyle \tau|_{X^0_{\mathrm{f}}}} & & \downarrow{\scriptstyle \tau} \\ (\mathbb{A}^1_R)_h & \xrightarrow{inc} & \mathbb{A}^1_R \end{array} \qquad (1)$$

is an elementary distinguished square in the category of affine R-smooth schemes in the sense of [15, Defn.3.1.3]. *Particularly, Z is the vanishing locus of the function* f *in X^0.*

Proposition 1.3. *Let R, $\bar{X}$, $\mathbf{G}$ and E be as in Theorem* 1.1. *Let $\bar{X}_v$ be the closed fibre of $\bar{X}$. Let $\eta_1, \dots, \eta_r$ be generic points of all irreducible components of $\bar{X}_v$. Let $\mathcal{O}$ be the semi-local ring of points $\eta_1, \dots, \eta_r$ on $\bar{X}$. Then $E|_{Spec\ \mathcal{O}}$ is trivial.*

This is a particular case of [18, Theorem 1.2], since $\mathcal{O}$ is a semi-local Dedekind integral domain. The Proposition has the following obvious consequence.

Corollary 1.4. *Under the notation and the hypothesis of Theorem* 1.1 *there exists a closed subset Z of pure codimention one in $\bar{X}$, which is finite over R and such that the* $\mathbf{G}$*-bundle $E|_{\bar{X}-Z}$ is trivial.*

Theorem 1.5. *Let R and* $\mathbf{G}$ *be as in Theorem* 1.2. *Let* $\mathrm{h} \in R[t]$ *be a monic polynomial and $Z(\mathrm{h})$ be the closed subset $\{\mathrm{h} = 0\}$ in $\mathbb{A}^1_R$. Let $\mathcal{E}$ be a principal* $\mathbf{G}$*-bundle over $\mathbb{P}^1_R$ such that its restriction to $\mathbb{P}^1_R - Z(\mathrm{h})$ is trivial. Then there exists a closed subscheme Y in $\mathbb{A}^1_R$ finite and étale over* Spec R *such that*

(i) *the restriction of $\mathcal{E}$ to $\mathbb{P}^1_R - Y$ is trivial,*

(ii) $Y \cap Z(\mathrm{h}) = \varnothing$.

In particular, the principal $\mathbf{G}$*-bundle $\mathcal{E}$ is trivial locally for the Zarisky topology.*

Derivation of Theorem 1.1 from Theorems 1.2, 1.5 **and Corollary** 1.4**.** The $\mathbf{G}$-bundle E is trivial over the generic point of $\bar{X}$. By Corollary 1.4 there exists a closed subset Z of pure codimention one in $\bar{X}$, which is finite over R and such that the $\mathbf{G}$-bundle $E|_{\bar{X}-Z}$ is trivial. To prove the theorem it is sufficient to check that E is trivial over a Zariski neighborhood of Z.

Take an affine open X^0 in $\bar{X}$, an étale morphism $\tau : X^0 \to \mathbb{A}^1_R$, a monic polynomial $\mathrm{h} \in R[t]$ and f satisfying the conclusion of Theorem 1.2. Since Z is the vanishing locus of f and $E|_{X^0-Z}$ is trivial, and the square (1) is an elementary distinguished square, there exists a principal $\mathbf{G}$-bundle E_t over $\mathbb{A}^1_R$ such that $\tau^*(E_t) = E|_{X^0}$ and $E_t|_{(\mathbb{A}^1_R)_h}$ is trivial.

Let $Z(\mathrm{h}) \subseteq \mathbb{A}^1_R$ be the vanishing locus of h. Clearly, there is be a principal $\mathbf{G}$-bundle $\mathcal{E}$ over $\mathbb{P}^1_R$ such that its restriction to $\mathbb{P}^1_R - Z(\mathrm{h})$ is trivial and its restriction to $\mathbb{A}^1_R$ is E_t. By Theorem 1.5 there exists a closed subscheme Y in $\mathbb{A}^1_R$ finite and étale over Spec$R$ such that

(i) the restriction of $\mathcal{E}$ to $\mathbb{P}^1_R - Y$ is trivial,

(ii) $Y \cap Z(\mathrm{h}) = \varnothing$.

Hence the restriction of $E_t = \mathcal{E}|_{\mathbb{A}^1_R}$ to $\mathbb{A}^1_R - Y$ is trivial and $Z(\mathrm{h})$ is in $\mathbb{A}^1_R - Y$. Let $X^{00} = \tau^{-1}(\mathbb{A}^1_R - Y)$. Since $\tau^*(E_t) = E|_{X^0}$ we conclude that $E|_{X^{00}}$ is trivial. Moreover, Z is in X^{00}. This completes the proof of the theorem. □

§2. Proof of Theorem 1.2

Lemma 2.1. *Let R, $\bar{X}$ be as in Theorem* 1.1. *Let Z be a closed subset of pure codimention one in $\bar{X}$, which is finite over R. Then there exists* *a closed subscheme X_∞ in $\bar{X}$, which is finite, étale over R and such that $X_\infty \cap Z = \varnothing$. Additionally, one can achieve the following property of X_∞: its intersection with each irreducible component of $\bar{X}_v$ is non-empty.*

Proof. Let $V = \mathrm{Spec} R$, $v \in V$ be its closed point and $k(v)$ be the residue field of R. So, as a scheme v is $\mathrm{Spec}\, k(v)$. If T is an R-scheme we write T_v for the k-scheme $T \times_V v$ and call it the closed fibre of T. Particularly, X_v is the closed fibre of X.

Take an embedding of X into a projective space

$$\mathbb{P}^N_V = \mathrm{Proj}(R[x_0, \ldots, x_N]).$$

Since $k(v)$ is infinite, by a variant of Bertini's theorem (see [25, Exp. XI, Theorem 2.1]), there is a homogeneous quadratic polynomial

$$H \in k(v)[x_0, \ldots, x_N]$$

such that the subscheme T_v of $\mathbb{P}^N_v$ given by the equation $H = 0$ intersects the closed fiber X_v of X transversally. Moreover, we may assume that $Z_v \cap T_v = \varnothing$, because Z_v is a finite set. Take a lift of the polynomial H to a quadratic polynomial $\tilde{H} \in R[x_0, \ldots, x_N]$. Let $T \subseteq \mathbb{P}^N_V$ be the scheme given by $\tilde{H} = 0$. Then $X_\infty := T \cap X$ is the required subscheme. Indeed, we only need to check that X_∞ is étale over V. However, the closed fiber of X_∞ is étale by construction. Hence, it is enough to check that X_∞ is flat over V. The flatness follows immediately from [16, Theorem 23.1]. Since $Z_v \cap T_v = \varnothing$, we see that $Z \cap X_\infty = \varnothing$. Since $T_v \subset \mathbb{P}^N_v$ is a hypersurface, it intersects each irreducible component of X_v in a non-empty subset. Hence X_∞ intersects each irreducible component of X_v in a non-empty subset. □

Let R, $\bar{X}$ and Z be as in Lemma 2.1. Let X_∞ be from the conclusion of the Lemma. Set $X = \bar{X} - X_\infty$. The nearest aim is to check that X and its closed fibre X_v are affine R-schemes. Let $\mathcal{L} = \mathcal{O}_{\bar{X}}(X_\infty)$. Then there is an integer $n >> 0$ and a section $s \in \Gamma(\bar{X}, \mathcal{L}^{\otimes n})$ such that s has no zeros on X_∞ and s does not vanish identically on any of irreducible component of X_v. Let $s_0 \in \Gamma(\bar{X}, \mathcal{L})$ be the canonical section: its vanishing locus is the Cartier divisor X_∞. Then the sections $s_0^{\otimes n}$ and s have no common zeros. Thus, they define a regular morphism of V-schemes

$$\bar{\Pi} := [s_0^{\otimes n} : s] : \bar{X} \to \mathbb{P}^1_V.$$

It is easy to check that this morphism is finite surjective. Clearly, $\bar{\Pi}^{-1}(\infty \times V) = X_\infty$ (set theoretically). Thus, $X = \bar{\Pi}^{-1}(\mathbb{A}^1_V)$. Consider a morphism

$\Pi := \bar{\Pi}|_X : X \to \mathbb{A}^1_V$. It is finite surjective, since Π is finite surjective. Hence X is an affine R-scheme. We write $R[X]$ for $\Gamma(X, \mathcal{O}_X)$. By the same reason X_v is an affine $k(v)$-scheme. We write $k(v)[X_v]$ for $\Gamma(X_v, \mathcal{O}_{X_v})$.

The following result is a simplified version of [20, Theorem 3.8]. Its proof is very closed to the proof of [20, Theorem 3.8]. However we decided to give its proof in details for convenience of the reader.

Proposition 2.2. *Let R, $\bar{X}$ and Z be as in Lemma* 2.1. *Let X_∞ be as in the conclusion of that Lemma. Then there is a finite morphism $\bar{\sigma} : \bar{X} \to \mathbb{P}^1_V$ of V-schemes such that*

(i) $X_\infty = \bar{\sigma}^{-1}(\infty \times V)$ (*an equality of closed subsets*);
(ii) $\bar{\sigma}|_Z : Z \to \mathbb{P}^1_V$ *is a closed embedding*;
(iii) $\bar{\sigma}$ *is étale along Z.*

Proof of Proposition 2.2. Step (i). For the closed point $v \in V$ and any point $z \in Z_v$ there is a closed embedding $z^{(2)} \hookrightarrow \mathbb{A}^1_v$, where $z^{(2)} := Spec\ k(v)[X_v]/\mathfrak{m}_z^2$ for the maximal ideal $\mathfrak{m}_z \subset k(v)[X_v]$ of the point z regarded as a point of X_v. This holds, since the $k(v)$-scheme X_v is equi-dimensional of dimension one, affine and $k(v)$-smooth.

Step (ii). There is a closed embedding $i_v : \coprod_{z \in Z_v} z^{(2)} \hookrightarrow \mathbb{A}^1_v$ of the $k(v)$-schemes. To see this apply Step (i) and use that the field $k(v)$ is infinite.

Step (iii). Introduce some notation. For these consider the closed subscheme X_∞ as a Cartier divisor on $\bar{X}$. Let $\mathcal{L} := \mathcal{O}_{\bar{X}}(X_\infty)$ be the corresponding invertible sheaf and let $s_0 \in \Gamma(\bar{X}, \mathcal{L})$ be its section vanishing exactly on X_∞. One has a Cartesian square of V-schemes

$$\begin{array}{ccc} X_{\infty,v} & \xrightarrow{j_\infty} & X_\infty \\ {\scriptstyle in_v}\downarrow & & \downarrow{\scriptstyle in} \\ \bar{X}_v & \xrightarrow{j} & \bar{X}, \end{array} \qquad (2)$$

which shows that the closed embedding $in_v : X_{\infty,v} \hookrightarrow \bar{X}_v$ is a Cartier divisor on $\bar{X}_v$. Set $\mathcal{L}_{\mathbf{u}} = j^*(\mathcal{L})$ and $s_{0,v} = s_0|_{\bar{X}_v} \in \Gamma(\bar{X}_v, \mathcal{L}_v)$.

Step (iv). Since Z and X_∞ are finite over V, there is a closed reduced subscheme $W \subset \bar{X}_v$ of dimension zero such that $W \cap X_\infty = \varnothing = W \cap Z$ and W has exactly one point on each irreducible component of the closed fibre $\bar{X}_v$. Let $s_\infty : \mathcal{O}_{X_\infty} \to \mathcal{L}|_{X_\infty}$ be a trivialization of $\mathcal{L}|_{X_\infty}$. Let t be the coordinate function on $\mathbb{A}^1_v$ and i_v be the closed embedding from the

Step (ii). Let $J \subset \mathcal{O}_{\bar{X}}$ be a sheaf of $\mathcal{O}_{\bar{X}}$ ideals such that

$$\mathcal{O}_{\bar{X}}/J = \mathcal{O}_{X_\infty} \times \mathcal{O}_W \times \prod_{z \in Z_v} k(v)[X_v]/\mathfrak{m}_z^2.$$

Claim. There exists an integer $n \geqslant 0$ and a section $s_1 \in H^0(\bar{X}, \mathcal{L}^{\otimes n})$ such that

$$s_1 \mathrm{mod}\ J \otimes \mathcal{L}^{\otimes n} = (s_\infty^n, 0, i_v^*(t) \cdot s_0^n) \in \Gamma(\bar{X}, \mathcal{L}^{\otimes n} \otimes (\mathcal{O}_{\bar{X}}/J)).$$

Prove the Claim. For the coherent sheaf J on $\bar{X}$ there is an integer $n(J) \geqslant 0$ such that for any integer $n \geqslant n(J)$ one has $H^1(\bar{X}, J \otimes \mathcal{L}^{\otimes n}) = 0$. Thus the map $H^0(\bar{X}, \mathcal{L}^{\otimes n}) = H^0(\bar{X}, \mathcal{L}^{\otimes n} \otimes \mathcal{O}_{\bar{X}}) \to H^0(\bar{X}, \mathcal{L}^{\otimes n} \otimes (\mathcal{O}_{\bar{X}/J}))$ is surjective. This proves the Claim.

Step (v). Set $s_{1,v} = s_1|_{\bar{X}_v}$. Then $s_{1,v} \in \Gamma(\bar{X}_v, \mathcal{L}_v^{\otimes n})$ has no zeros on $X_{\infty,v}$ and hence the morphism

$$\bar{\sigma}_v := [s_{0,v}^n : s_{1,v}] : \bar{X}_v \to \mathbb{P}_v^1$$

is well-defined. We claim that it is finite surjective. To see this take an irreducible component Y of $\bar{X}_v$ and recall that $\varnothing \neq Y \cap X_\infty = Y \cap X_{\infty,v}$. The morphism $[s_{0,v}^n : s_{1,v}]$ takes $Y \cap X_{\infty,v}$ to the point $[0:1] \in \mathbb{P}_v^1$ and takes points of $W \cap Y$ to the point $[1:0] \in \mathbb{P}_v^1$. Thus $[s_{0,v}^n : s_{1,v}]$ *is finite and surjective.* Conclude now the step (v) with an observation:
$(*)$ there is an equality $\bar{\sigma}_v|_{\coprod_{z \in Z_v} z^{(2)}} = i_v : \coprod_{z \in Z_v} z^{(2)} \hookrightarrow \mathbb{A}_v^1 \subset \mathbb{P}_v^1$, where i_v is from the step (ii); in particular, $\bar{\sigma}_v$ is étale at every point $z \in Z_v$.

Step (vi). Let $s_1 \in H^0(\bar{X}, \mathcal{L}^{\otimes n})$ be as in the Claim. Then the morphism

$$\bar{\sigma} := [s_0^n : s_1] : \bar{X} \to \mathbb{P}_V^1$$

is finite surjective. As we already know the morphism $[s_{0,v}^n : s_{1,v}] : \bar{X}_v \to \mathbb{P}_v^1$ is finite. Thus the morphism $[s_0^n : s_1]$ is quasi-finite over a neighborhood of $\mathbb{P}_v^1$. Since V is local, hence any Zariski neighborhood of $\mathbb{P}_v^1$ coincides with $\mathbb{P}_V^1$. Since the morphism $[s_0^n : s_1]$ is projective, hence it is finite and surjective.

By the step (vi) *the resulting morphism* $\bar{\sigma} : \bar{X} \to \mathbb{P}_V^1$ is a V-scheme morphism, finite and surjective. The property $(*)$ from step (v) shows that the morphism

$$\bar{\sigma}_v\big|_{\coprod_{z \in Z_v} z^{(2)}} : \coprod_{z \in Z_v} z^{(2)} \hookrightarrow \mathbb{A}_v^1 \subset \mathbb{P}_v^1$$

is a closed embedding and $\bar{\sigma}_v : X_v \to \mathbb{A}^1_v$ is étale at every point $z \in Z_v$. It is checked above that $\bar{\sigma}_v$ is finite and surjective.

We are ready now to complete the proof of the Proposition. Obviously, $\bar{\sigma}^{-1}(\infty \times V) = X_\infty$ (set theoretically). This proves the assertion (i) of the proposition and shows an equality $X = \bar{\sigma}^{-1}(\mathbb{A}^1_V)$.

Prove now the assertion (iii) of the proposition. To do this consider a morphism $\sigma := \bar{\sigma}|_X : X \to \mathbb{A}^1_V$. It is finite surjective, since $\bar{\sigma}$ is finite surjective. Since the schemes X and $\mathbb{A}^1_V$ are regular irreducible and the morphism σ is finite surjective, the morphism σ is flat by a theorem of Grothendieck [4, Theorem 18.17]. So, to check that σ is étale at a closed point $z \in Z$ it suffices to check that the morphism $\sigma_v : X_v \to \mathbf{A}^1_v$ is étale at the point z. The latter does hold by the step (v). Whence σ is étale at all the closed points of Z. The scheme Z is semi-local. Thus, σ is étale in a neighborhood of Z. This proves the assertion (iii) of the proposition.

Prove finally the assertion (ii) of the proposition. By the step (v) for each point $z \in Z_v$ the $k(v)$-algebra homomorphism $\sigma^*_v : k(v)[t] \to k(v)[X_v]$ is étale at the maximal ideal $\mathfrak{m}_z$ of the $k(v)$-algebra $k(v)[X_v]$ and the composite map $k(v)[t] \xrightarrow{\sigma^*_v} k(v)[X_v] \to k(v)[X_v]/\mathfrak{m}_z$ is an epimorphism. Thus, for any integer $r > 0$ the $k(v)$-algebra homomorphism $k(v)[t] \to k(v)[X_v]/\mathfrak{m}^r_z$ is an epimorphism. The local ring $\mathcal{O}_{Z_v,z}$ of the scheme Z_v at the point z is of the form $k(v)[X_v]/\mathfrak{m}^s_z$ for an integer s. Thus, the $k(v)$-algebra homomorphism $k(v)[t] \xrightarrow{\sigma^*_v} k(v)[X_v] \to \mathcal{O}_{Z_v,z}$ is surjective. By the step (v) the morphism $\sigma_v|_{\coprod_{z \in Z'_v} z^{(2)}}$ is the closed embedding i_v from the step (ii). This yields the surjectivity of the $k(v)$-algebra homomorphism

$$k(v)[t] \to \prod_{z/v} \mathcal{O}_{Z_v,z} = \Gamma(Z_v, \mathcal{O}_{Z_v}). \tag{3}$$

The R-module $\Gamma(Z, \mathcal{O}_Z)$ is finite. Thus, the $k(v)$-module $\Gamma(Z_v, \mathcal{O}_{Z_v})$ is finite. Now the surjectivity of the $k(v)$-algebra homomorphism (3) and the Nakayama lemma show that the R-algebra homomorphism $R[t] \to \Gamma(Z, \mathcal{O}_Z)$ is surjective. Thus, $\sigma|_Z : Z \to \mathbb{A}^1_V$ is a closed embedding. The proposition is proved. □

Proof of Theorem 1.2. Let X_∞ be the closed subscheme in $\bar{X}$ as in the conclusion of Lemma 2.1. Set $X = \bar{X} - X_\infty$ and note that $X = \bar{\sigma}^{-1}(\mathbb{A}^1_V)$. Set $\sigma = \bar{\sigma}|_X : X \to \mathbb{A}^1_V$. Then σ is a finite surjective V-scheme morphism, because $\bar{\sigma}$ is finite and surjective. The following hold by Proposition 2.2

(a) $\sigma|_Z : Z \to \mathbb{A}^1_R$ is a closed embedding;
(b) σ is étale along Z.
Properties (a) and (b) yield a scheme theoretic equality

$$\sigma^{-1}(\sigma(Z)) = Z \sqcup Z'.$$

Thus, there is an affine open subset $X^0 \subseteq X$ containing Z such that
(i′) $\tau := \sigma|_{X^0} : X^0 \to \mathbb{A}^1_R$ is étale;
(ii′) Z is in X^0 and $\tau|_Z : Z \to \mathbb{A}^1_V$ is a closed embedding;
(iii′) $\tau^{-1}(\tau(Z)) = Z$;
By items (ii') and (iii') there is a monic polynomial $\mathrm{h} \in R[t]$ such that $\tau(Z)$ is a closed subscheme in $\mathbb{A}^1_V$ defined by the principal ideal (h). Put $\mathrm{f} = \tau^*(\mathrm{h}) \in R[X^0]$. Then the closed subscheme Z in X^0 is defined by the principal ideal (f). We constructed the affine open X^0 in X containing Z, the monic polynomial $\mathrm{h} \in R[t]$, the étale morphism $\tau : X^0 \to \mathbb{A}^1_V$, the function $\mathrm{f} \in R[X^0]$ such that the square (1) is an elementary distinguished square in the category of affine R-smooth schemes in the sense of [15, Defn.3.1.3]. □

§3. Proof of Theorem 1.5

Proof of Theorem 1.5. Let $V = \operatorname{Spec} R$, $v \in V$ be its closed point and $k(v)$ be the residue field of R. So, as a scheme v is $\operatorname{Spec} k(v)$. If T is an R-scheme we write T_v for the k-scheme $T \times_V v$ and call it the closed fibre of T.

Particularly, one has the simple algebraic $k(v)$-group $\mathbf{G}_v = \mathbf{G} \times_V v$ and the principal $\mathbf{G}_v$-bundle $\mathcal{E}_v$, where $\mathbf{G}$ and $\mathcal{E}$ are as in Theorem 1.5. The restriction of $\mathcal{E}_v$ to $\mathbb{P}^1_v - Z(\mathrm{h})_v$ is trivial. We begin with the following

Claim 1. If the $k(v)$-simple group $\mathbf{G}_v$ is anisotropic, then the principal $\mathbf{G}$-bundle $\mathcal{E}$ is trivial itself.

To prove this Claim recall that $\mathcal{E}_v$ is trivial over $\mathbb{P}^1_v$ by [6, Corollary 3.10(a)]. Since the restriction of $\mathcal{E}$ to $\mathbb{P}^1_V - Z(\mathrm{h})$ is trivial, the $\mathbf{G}$-bundle $\mathcal{E}$ is trivial itself by [26, Теорема 1]. This proves the Claim 1.

Claim 2. If the $k(v)$-simple group $\mathbf{G}_v$ is isotropic, then there exists a closed subscheme Y in $\mathbb{A}^1_V$ finite and étale over V such that
(i) the Y-group scheme $\mathbf{G}_Y := \mathbf{G} \times_V Y$ contains a parabolic Y-subgroup scheme;
(ii) the closed fibre Y_v of Y contains a $k(v)$-rational point;
(iii) $Y \cap Z(\mathrm{h}) = \varnothing$.
Repeat literally the proof of [5, Proposition 4.1] in order to find a closed

subscheme Y in $\mathbb{A}^1_V$ finite and étale over V satisfying the conditions (i) and (ii). To achieve the third condition just apply an appropriate affine shift. These prove the Claim 2.

Claim 3. Suppose the $k(v)$-simple group $\mathbf{G}_v$ is isotropic. Then for each closed subscheme Y in $\mathbb{A}^1_V$ finite and étale over V subjecting to conditions (i) to (iii) the restriction of $\mathcal{E}$ to $\mathbb{P}^1_V - Y$ is trivial.

To prove this Claim recall the following result which is not stated in [5] as a separate

Lemma 3.1. *Suppose the $k(v)$-simple group $\mathbf{G}_v$ is isotropic and $Y \subseteq \mathbb{A}^1_V$ is a closed subscheme finite and étale over V subjecting to conditions (i) to (iii). Then $E_v|_{\mathbb{P}^1_v - Y_v}$ is trivial.*

Proof of Lemma 3.1. The $\mathbf{G}_u$-bundle $\mathcal{E}_u$ is trivial over $\mathbb{P}^1_v - Z(h)_v$. Thus, by [6, Corollary 3.10(a)] it is trivial locally for Zariski topology on $\mathbb{P}^1_v$. Using again [6, Corollary 3.10(a)] and the equality $Pic(\mathbb{P}^1_v - Y_v) = 0$, we see that $\mathcal{E}_v$ is trivial over $\mathbb{P}^1_v - Y_v$. □

With Lemma 3.1 in hand the proof of the Claim 3 is literally the same as those of [5, Theorem 3]. Very briefly, we modify the $\mathbf{G}$-bundle $\mathcal{E}$ along the closed subscheme Y to get a new $\mathbf{G}$-bundle $\mathcal{E}^{mod}$ over $\mathbb{P}^1_V$. The latter $\mathbf{G}$-bundle has two properties:
(i) its restriction to $\mathbb{P}^1_V - Y$ coincides with the restriction of $\mathcal{E}$ to $\mathbb{P}^1_V - Y$;
(ii) its restriction to $\mathbb{P}^1_v$ is a trivial $\mathbf{G}_v$-bundle.
By [26, Theorem 1] and the property (ii) there is a principal $\mathbf{G}$-bundle $\mathcal{E}_0$ over V such that $\mathcal{E}^{mod}$ and $pr^*(\mathcal{E}_0)$ are isomorphic as the principal $\mathbf{G}$-bundle over $\mathbb{P}^1_V$. Here $pr : \mathbb{P}^1_V \to V$ is the projection. The restriction of $\mathcal{E}^{mod}$ and $\mathcal{E}$ to $\infty \times V$ *are isomorphic* as the principal $\mathbf{G}$-bundles due to the property (i). The restriction of $\mathcal{E}$ to $\infty \times V$ is trivial, because $\infty \times V \subset \mathbb{P}^1_V - Z(h)$ and the restriction of $\mathcal{E}$ to $\mathbb{P}^1_V - Z(h)$ is trivial. Thus the restriction of $\mathcal{E}^{mod}$ to $\infty \times V$ is trivial. Hence so is the restriction of $pr^*(\mathcal{E}_0)$ to $\infty \times V$. Thus, $\mathcal{E}_0$ is trivial over V. This yields the triviality of $pr^*(\mathcal{E}_0)$ and $\mathcal{E}^{mod}$ over $\mathbb{P}^1_V$. By the property (i) the restriction of $\mathcal{E}$ to $\mathbb{P}^1_V - Y$ is a trivial $\mathbf{G}$-bundle. These prove the Claim 3.

Claims 1, 2 and 3 complete the proof of Theorem 1.5. □

Remark 3.2. Note for the reader that the desired $\mathbf{G}$-bundle $\mathcal{E}^{mod}$ is the $\mathbf{G}$-bundle of the form $\mathrm{Gl}(\mathcal{E}', \varphi \circ \alpha)$ as in [5, Section 5.8, Claim], where $\mathcal{E}' = \mathcal{E}|_{\mathbb{P}^1 - Y}$. Point out that Lemma 3.1 is used to get an analog of [5, Lemma 5.21] in our setting.

Theorem [26, Theorem 1] replaces in our setting a reference to [23, Proposition 9.6] in the proof of [5, Proposition 5.1]. We are not able refer to [23, Proposition 9.6], since we work in the mixed characteristic case.

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