Записки научных
семинаров ПОМИ
Том 484, 2019 г.

R. Lubkov, A. Stepanov

SUBGROUPS OF CHEVALLEY GROUPS OVER RINGS

Abstract. In the present paper, we study the subgroup lattice of a Chevalley group $\mathrm{G}(\Phi, R)$ over a commutative ring R, containing the subgroup $D(R)$, where D is a subfunctor of $\mathrm{G}(\Phi, \_)$. Assuming that over any field F the normalizer of the group $D(F)$ is "closed to be maximal", we formulate some technical conditions, which imply that the lattice is standard. We also study the conditions concerning the normalizer of $D(R)$ in the case, where $D(R)$ is the elementary subgroup of another Chevalley group $\mathrm{G}(\Psi, R)$ embedded into $\mathrm{G}(\Phi, R)$

Introduction

Let $G = \mathrm{G}(\Phi, \_)$ denote a Chevalley–Demazure group scheme with a reduced irreducible root system $\Phi \neq A_1$ and let $E = \mathrm{E}(\Phi, \_)$ be its elementary subgroup functor. All algebraic groups are considered as affine group schemes over a commutative ring K with a unit, which in applications can be equal to $\mathbb{Z}$ or its localization. We always assume that $E(R)$ is perfect for all K-algebras R, which amounts to say that $\Phi \neq A_1$ and either $\Phi \neq C_2$ or K has no epimorphism onto the field of 2 elements. Let D be a subfunctor (not necessarily a subscheme) of G. For an arbitrary K-algebra R with a unit let $\mathcal{L} = L\big(D(R), G(R)\big)$ be the lattice of subgroups of $G(R)$ containing $D(R)$. In the present paper, we study the lattice $\mathcal{L}$ under certain conditions on D and G, formulated in Section 2. The lattice $\mathcal{L}$ is called standard if it splits into a disjoint union of sublattices $L(E_i, N_i)$ (so-called sandwiches), where N_i is the normalizer of E_i and i ranges over some index set. The statement claiming that $\mathcal{L}$ is standard is called *sandwich classification theorem.*

Usually, the sandwich classification theorem is a broad generalization of an assertion about maximality of certain subgroups of simple linear groups. Over fields this topic was studied by M. Aschbacher, Li Shangzhi, M. W. Liebeck, J. Saxl, G. M. Seitz, and many others. Here we sketch

Key words and phrases: Chevalley group, subgroup lattice, generic element, universal localization, normalizer, transporter.

This work is supported by the Russian Science Foundation, grant №17-11-01261.

main results on sandwich classification theorem in Chevalley groups over rings. A detailed survey can be found in [27].

Subsystem subgroups. One of the first results of this kind due to Z. I. Borewich and N. A. Vavilov [4] was the description of subgroups of $\mathrm{GL}_n(R)$, containing the group of elementary block-diagonal matrices. More generally, let Δ be a (possibly reducible) root subsystem of Φ and let D be the elementary subgroup generated by all root subgroups X_α, $\alpha \in \Delta$. If Δ is large enough and has sufficiently big rank, then we can hope to prove the sandwich classification theorem. This is done for subsystems subgroups of classical groups in [4, 7, 17]. Here the sandwiches are parametrized by systems of ideals.

Subring subgroups. The first result here was obtained by N. S. Romanovski who considered the lattice $L\big(\mathrm{SL}_n(\mathbb{Z}), \mathrm{SL}_n(\mathbb{Q})\big)$. In general necessary and sufficient conditions for the lattice $L\big(E(\Phi, R), G(\Phi, A)\big)$ to be standard, where R is a subring of A, were found by Ya. N. Nuznin, A. V. Yakushevich, and the second author in [14–16, 19, 20]. In this case the sandwiches are indexed by intermediate subrings or form subrings (admissible pairs).

Chevalley groups. Here D is the elementary subgroup of another Chevalley group embedded to $G(\Phi, \_)$. The results on classical groups inside GL_n in the natural representations were obtained by You Hong [28, 29] and in a series of papers by N. A. Vavilov with V. A. Petrov [24–26]. The sandwich classification theorem in the case $D = E(F_4, \_)$ inside $G(E_6, \_)$ in the 27-dimensional representation was proved by A. Yu. Luzgarev [11]. The sandwiches correspond to ideals of the ground ring in these settings.

Tensor products and exterior powers. The group $E_m(R) \otimes E_n(R)$ maps into $\mathrm{GL}_{mn}(R)$. Similarly, considering the action of $\mathrm{GL}(V)$ on the exterior power $\Lambda^n(V)$ we obtain a map $E_m(R) \to \mathrm{GL}_N(R)$, where $N = \binom{m}{n}$. Denote by $D(R)$ the image of one of these maps. Preliminary results on these problems are obtained in the papers by A. S. Ananievskii, N. A. Vavilov, S. S. Sinchuk [2,3] and I. Nekrasov with the first author [9,10]. Strictly speaking, this item is a particular case of the previous one. For instance, if n divides m, then the action of SL_m on the exterior power is the embedding of SL_m / μ_n, which also is a Chevalley group of type A_{m-1}.

The present paper is to prepare a general framework for proving the sandwich classification theorem, basically for the last two cases, where sandwiches are parametrized by one ideal of the ground ring. Reading the paper the reader can have in mind the example $G = \mathrm{GL}_n$ and D being the elementary subgroup of the split orthogonal or symplectic group.

For an ideal $\mathfrak{a}$ of a ring R denote by $E(R,\mathfrak{a})$ the relative elementary subgroup and by $N(R,\mathfrak{a})$ the normalizer of $D(R)E(R,\mathfrak{a})$ in $G(R)$. Put $N(R) = N(R, 0)$. For an affine scheme X over a ring K the affine algebra of X is denoted by $K[X]$. For all cases, where we plan to apply our technique, all conditions are trivial or well-known except the following.

- For any field F the normalizer $N(F)$ is "closed to be maximal" in $G(F)$.
- The transporter of $D(R)$ to $N(R)$ equals $N(R)$ for any K-algebra R.
- A subgroup, containing the generic element of G and $D(K[G])$, contains a nontrivial elementary root unipotent element.

Under these conditions we prove the lattice $\mathcal{L}$ to be standard in the following sense.

Definition 1. *The lattice $\mathcal{L} = L\big(D(R), G(R)\big)$ is called standard if for any subgroup $\Gamma \in \mathcal{L}$ there exist a unique ideal $\mathfrak{a}$ of R such that*

$$D(R)E(R,\mathfrak{a}) \leqslant \Gamma \leqslant N(R,\mathfrak{a}).$$

This result is the main theorem of the present paper. It is stated and proved in Section 2. Following Bak the sublattices $L(D(R)E(R,\mathfrak{a}), N(R,\mathfrak{a}))$ are called *sandwiches.* With this terminology the lattice $\mathcal{L}$ is standard whenever it splits into a disjoint union of sandwiches.

Besides, we study the structure of the normalizer $N(R)$ in the following settings. Let H be another Chevalley group with a reduced irreducible root system Ψ and let D be its elementary subgroup functor. Let $H \to G$ be a closed immersion. We shall identify H and D with their images in G. We prove the condition on the normalizer given above (Condition 5 of Section 2) provided that R contains an infinite field. The latter condition does not seem to be essential but we cannot avoid it now. The proof is based on the fact that H is the Zariski closure of D in G and $H(R)/D(R)$ is solvable for a finite dimensional ring R. For the proof we use the classification of automorphisms of a Chevalley group. This result over rings was proved by E. Abe [1] with some gaps. In a series of papers E. Bunina obtained a correct proof and removed unnecessary conditions on invertibility of 2, although the structure constants still must be invertible for the classification. Use of the classification forces us to put extra conditions on invertibility of structure constants. Since we really use only classification of automorphisms of a *group scheme*, we believe that these extra conditions can be skipped.

The rest of the text is organized as follows. In the next section we set notation that is used throughout the paper. In Section 2 we state and prove the main theorem. The normalizer of the group $D(R)$ is studied in Sections 3–5. Namely, in Section 3 we give a sufficient condition for coincidence of scheme-theoretic and group-theoretic transporters, in Section 4 we study conditions for equality of the normalizers of $D(R)$ and $H(R)$ and the transporter $\mathrm{Tran}_{G(R)}\big(D(R), H(R)\big)$. Coincidence of the last three groups with the transporter $\mathrm{Tran}_{G(R)}\big(D(R), N(R)\big)$ is proved in Section 5.

§1. Notation

The notation is standard in Chevalley groups theory over rings, see, e. g., [18]. Let G be an arbitrary group. By a commutator of two elements we always understand *the left-normed* commutator $[x, y] = xyx^{-1}y^{-1}$, where $x, y \in G$. Multiple commutators are also left-normed; in particular, $[x, y, z] = [[x, y], z]$. By $y^x = x^{-1}yx$ we denote *the right conjugate* to y by x.

For a subset $X \subseteq G$, we denote by $\langle X\rangle$ the subgroup generated by X. The notation $H \leqslant G$ means H to be a subgroup in G. For $H \leqslant G$, we denote by $\langle X\rangle^H$ the smallest subgroup in G containing X and normalized by H. For two groups $F, H \leqslant G$, we denote by $[F, H]$ their mutual commutator: $[F, H] = \langle [f, g] \mid f \in F,\ h \in H\rangle$. The center of an abstract group G is denoted by $\mathrm{Cent}(G)$. The normalizer of a subgroup H in G is denoted by $N_G(H)$.

All rings and algebras are assumed to be commutative and to contain a unit. All homomorphisms preserve the unit elements. For an ideal $\mathfrak{a}$ of a ring R denote by $\rho_{\mathfrak{a}}$ the canonical homomorphism $R \to R/\mathfrak{a}$. This homomorphism is called the reduction homomorphism modulo $\mathfrak{a}$. If s is an element of a ring R, then $R_s = \langle s\rangle^{-1}R$ denotes the principal localization, i.e., the localization of R in the multiplicative subset, generated by s. The localization homomorphism $R \to R_s$ is denoted by λ_s.

Throughout the article the expression "group scheme" means "flat affine group scheme of finite type". Let G be an affine group scheme over K. Denote by $A = K[G]$ the affine algebra of G. By definition of an affine scheme, an element $h \in G(R)$ can be identified with a ring homomorphism $h : A \to R$. We always do this identification. Denote by $g \in G(A)$ the generic element of G, i.e., the identity map $\mathrm{id}_A : A \to A$. An element

$h \in G(R)$ induces a group homomorphism $G(h) : G(A) \to G(R)$ by the rule $G(h)(a) = h \circ a$. Thus, the image of g under $G(h)$ equals h. In the sequel, for a ring homomorphism $\varphi : R \to R'$ we denote by the same symbol φ the induced homomorphism $G(\varphi) : G(R) \to G(R')$. This cannot lead to a confusion as we always can distinguish between two different meanings of φ by the type of its argument. With this convention we have $h(g) = h \circ \mathrm{id}_A = h$. Let $\mathfrak{a}$ be an ideal of a ring R. As usual, $G(R, \mathfrak{a})$ denotes the principal congruence subgroup of $G(R)$ of level $\mathfrak{a}$, i.e., the kernel of the reduction homomorphism $\rho_{\mathfrak{a}} : G(R) \to G(R/\mathfrak{a})$.

In this article we always assume that G is a Chevalley–Demazure group scheme over a ring K with a reduced irreducible root system $\Phi \neq A_1$, and that either $\Phi \neq C_2$ or K has no epimorphisms onto the field of 2 elements. Denote by $E(\mathfrak{a})$ the subgroup of $G(R)$ generated by elementary root unipotents $x_\alpha(r)$, $\alpha \in \Phi$, $r \in \mathfrak{a}$. Then $E(R)$ is the [absolute] elementary subgroup of $G(R)$ and $E(R, \mathfrak{a}) = E(\mathfrak{a})^{E(R)}$ denotes the relative elementary subgroup.

Throughout the article we keep the notation of the affine algebra $A = K[G]$ and of the generic element $g \in G(A)$.

§2. Main theorem

In this section we state and prove the main theorem of the article under the following conditions.

List of conditions. Let R be a ring.

1. The functor D preserves surjective maps.
2. For a K-algebra R and an ideal $\mathfrak{a}$ of R, we have
$$[D(R), D(R)E(R, \mathfrak{a})] = D(R)E(R, \mathfrak{a})$$
and $D(R) \leqslant E(R)$.
3. For any $r \in R$ and $\alpha \in \Phi$ such that $x_\alpha(r) \notin D(R)$, we have
$$\langle D(R), x_\alpha(r)\rangle = D(R)E(R, rR).$$
4. The map $R \mapsto N(R)$ defines a closed subscheme in G.
5. If $D(R)^h \leqslant N(R)$, then $h \in N(R)$.
6. For any field F the subgroup $D(F)$ is an “almost maximal” subgroup in $G(F)$, i.e., if a subgroup contains $D(F)$, then it either is contained in $N(F)$ or contains $E(F)$.

7. The subgroup $\langle D(A), g\rangle \leqslant G(A)$ contains an elementary root unipotent $x_\alpha(\xi) \notin N(A)$. Moreover, for any field F there exists an element $y \in E(F)$ such that $x_\alpha\big(y(\xi)\big) \notin N(F)$.
8. If $h \in G(R, \operatorname{Rad} R) \smallsetminus N(R)$, then $\langle D(R), h\rangle$ contains a nontrivial root unipotent.

In real examples Condition 1 is trivial. For Noetherian rings of finite Bass–Serre dimension (Krull dimension works equally well here) the inclusion $D(R) \leqslant E(R)$ in Condition 2 follows from the fact that $D(R)$ is perfect. Indeed, Theorem 1 of [8] (see also [21, Corollary 12.8]) implies that $E(R)$ is the largest perfect subgroup of $G(R)$. But for infinite dimensional rings one needs some extra conditions on $D(R)$ to deduce the inclusion into $E(R)$. On the other hand, in real examples the inclusion can be easily verified.

It is not difficult to show that requirement $y \in E(F)$ can be replaced by $y \in G(F)$ in Condition 7. But to verify this condition one usually takes y to be a root unipotent element or the like, so that the inclusion $y \in E(F)$ holds automatically.

Conditions 4 and 5 for an important special case are proved in Sections 3–5.

The following statement computes the normalizer $N(R, \mathfrak{a})$ in terms of $N(R/\mathfrak{a})$. It will be used in the proof of the main theorem. The idea of the proof is borrowed from the proof of [13, Theorem 3]. In the sequel, we use the following commutator formulas. A direct computation shows that $[x, yz] = [x, y] \cdot [x, z]^{y^{-1}}$ for all elements x, y, z of an abstract group. Therefore, for subgroups X, Y, Z such that YZ is a subgroup, we have

$$[X, YZ] \leqslant [X, Y] \cdot [X, Z]^{Y}. \tag{1}$$

The second formula is the standard commutator formula obtained by G. Taddei [22] and L. Vaserstein [23]. For an ideal $\mathfrak{a}$ of a ring R, we have

$$[E(R), G(R, \mathfrak{a})] = [E(R, \mathfrak{a}), G(R)] = E(R, \mathfrak{a}). \tag{2}$$

Lemma 2.1. *Let $\mathfrak{a}$ be an ideal of a ring R. Under Condition 1, $N(R, \mathfrak{a})$ is the full preimage of $N(R/\mathfrak{a})$ under the reduction homomorphism $\rho_{\mathfrak{a}}$.*

Proof. Since the natural map $D(R) \to D(R/\mathfrak{a})$ is surjective, the group $\rho_{\mathfrak{a}}\big(N(R, \mathfrak{a})\big)$ normalizes $D(R/\mathfrak{a})$.

Conversely, let $h \in G(R)$ be such that $\bar{h} = \rho_{\mathfrak{a}}(h) \in N(R/\mathfrak{a})$. Then

$$\rho_{\mathfrak{a}}\big(D(R)^h\big) \leqslant D(R/\mathfrak{a})^{\bar{h}} \leqslant D(R/\mathfrak{a}).$$

Using the surjectivity of the map $D(R) \to D(R/\mathfrak{a})$, we obtain

$$D(R)^h \leqslant D(R)G(R,\mathfrak{a}). \tag{3}$$

By formulas (1) and (2),

$$\begin{aligned}[D(R), D(R)^h] \leqslant [D(R), D(R)G(R,\mathfrak{a})] \leqslant D(R)[D(R), G(R,\mathfrak{a})]^{D(R)} \\ = D(R)[D(R), G(R,\mathfrak{a})] \leqslant D(R)E(R,\mathfrak{a}).\end{aligned}$$

Take the mutual commutator subgroups of the both sides of inclusion (3) with $D(R)^h$. By Condition 2, we get

$$\begin{aligned}D(R)^h = [D(R)^h, D(R)^h] &\leqslant [D(R)^h, D(R)G(R,\mathfrak{a})] \\ &\leqslant [D(R)^h, D(R)] \cdot [D(R)^h, G(R,\mathfrak{a})]^{D(R)^h} \\ &\leqslant D(R)E(R,\mathfrak{a})[E(R), G(R,\mathfrak{a})] = D(R)E(R,\mathfrak{a})\end{aligned}$$

(we used also the standard commutator formula and normality of $E(R)$ in $G(R)$). Since $E(R,\mathfrak{a})$ is normal in $G(R)$ this inclusion implies that h normalizes the group $D(R)E(R,\mathfrak{a})$. □

The following lemma shows that the conditions above imply that the set $N(R,\mathfrak{a}) \smallsetminus D(R)E(R,\mathfrak{a})$ contains no elementary root unipotents and establish the uniqueness property in Definition 1.

Lemma 2.2. *Let $\mathfrak{a} \neq \mathfrak{b}$ be ideals of a K-algebra R. Conditions 2 and 3 imply that the set $N(R,\mathfrak{a}) \smallsetminus D(R)E(R,\mathfrak{a})$ contains no elementary root unipotents and that the sandwiches*

$$L\big(D(R)E(R,\mathfrak{a}), N(R,\mathfrak{a})\big) \quad \text{and} \quad L\big(D(R)E(R,\mathfrak{b}), N(R,\mathfrak{b})\big)$$

have empty intersection.

Proof. Let $\Gamma \in L\big(D(R)E(R,\mathfrak{a}), N(R,\mathfrak{a})\big)$. By Condition 2,

$$[D(R), \Gamma] \geqslant [D(R), D(R)E(R,\mathfrak{a})] = D(R)E(R,\mathfrak{a}).$$

On the other hand,

$$[D(R), \Gamma] \leqslant [D(R)E(R,\mathfrak{a}), N(R,\mathfrak{a})] \leqslant D(R)E(R,\mathfrak{a}).$$

Thus, the ideal $\mathfrak{a}$ is defined uniquely by the subgroup Γ from the sandwich.

If $x_\alpha(r) \in N(R,\mathfrak{a}) \smallsetminus D(R)E(R,\mathfrak{a})$, then by Condition 7, $E(R, rR) \leqslant N(R,\mathfrak{a})$. It follows that $D(R)E(R,\mathfrak{a}) \leqslant D(R)E(R,\mathfrak{a}+rR) \leqslant N(R,\mathfrak{a})$, which contradicts the first paragraph of the proof. □

Theorem 1. *Suppose conditions* 1–8 *are satisfied; then for any K-algebra R the lattice $\mathcal{L} = L\big(D(R), G(R)\big)$ is standard.*

Proof. Let Γ be a subgroup of $G(R)$ containing $D(R)$. Put

$$\mathfrak{a} := \{s \in R \mid \text{ there exists } \alpha \in \Phi : \ x_\alpha(s) \in \Gamma \smallsetminus D(R)\}.$$

Using Condition 3, we have $D(R)E(R, \mathfrak{a}) \leqslant \Gamma$. Let $\overline{R} = R/\mathfrak{a}$ and $\overline{\Gamma} = \rho_{\mathfrak{a}}(\Gamma)$. Suppose $x_\alpha(\bar{r}) \in \overline{\Gamma} \smallsetminus D(\overline{R})$ for some $\alpha \in \Phi$ and $\bar{r} \in \overline{R}$. Using Condition 1, we get $D(\overline{R}) \leqslant \overline{\Gamma}$ and by Condition 3, $D(\overline{R})E(\overline{R}, \bar{r}\overline{R}) \leqslant \overline{\Gamma}$. Let $r \in R$ be a preimage of the element $\bar{r}$ under $\rho_{\mathfrak{a}}$. Then

$$D(R)E(R, rR) \leqslant \Gamma G(R, \mathfrak{a}).$$

Taking the commutator subgroup of $D(R)$ with the both sides of the latter inclusion, we obtain

$$[D(R), D(R)E(R, rR)] \leqslant [D(R), \Gamma G(R, \mathfrak{a})]. \tag{4}$$

By Condition 2, the left-hand side equals $D(R)E(R, rR)$. On the other hand, by formula (1) the right-hand side of inclusion (4) is contained in $[D(R), \Gamma] \cdot [D(R), G(R, \mathfrak{a})]^\Gamma$. Since $D(R) \leqslant E(R)$, by the standard commutator formula this group is contained in $\Gamma \cdot E(R, \mathfrak{a})^\Gamma = \Gamma$. Note that $x_\alpha(\bar{r}) \notin D(\overline{R})$ implies $x_\alpha(r) \notin D(R)$. By definition of the ideal $\mathfrak{a}$, it follows that $r \in \mathfrak{a}$. Thus, $\bar{r} = 0$ and the set $\overline{\Gamma} \smallsetminus D(\overline{R})$ does not contain nontrivial root unipotents.

Now, let $\xi \in A$ be an element from Condition 7. Put $x = x_\alpha(\xi) \in G(A)$. For a K-algebra B let

$$S(B) = \{b \in G(B) \mid b(x) \in N(B)\}.$$

By Condition 4, N is a closed subscheme of G defined by some ideal $\mathfrak{q}$ of A. As usual, we denote $N = V(\mathfrak{q})$. Then, it is easy to see that $S = V\big(x(\mathfrak{q})A\big)$ is a closed subscheme of G. For an element $h \in \overline{\Gamma}$ the root unipotent element $x_\alpha\big(h(\xi)\big) = h\big(x\big)$ belongs to the subgroup generated by h and $D(\overline{R})$. By the previous paragraph of the proof it must lie in $N(\overline{R})$, hence $h \in S(\overline{R})$. Since h is an arbitrary element of $\overline{\Gamma}$, we conclude that $\overline{\Gamma} \in S(\overline{R})$.

Let $\mathfrak{m}$ be a maximal ideal of the ring $\overline{R}$. Denote by F the residue field $\overline{R}/\mathfrak{m}$. The subgroup $\widetilde{\Gamma} = \rho_{\mathfrak{m}}(\overline{\Gamma})$ is contained in $S(F)$, hence, by Condition 7, $\widetilde{\Gamma}$ does not contain $E(F)$. On the other hand, $\widetilde{\Gamma} \geqslant D(F)$. Consequently, using Condition 6, we get $\widetilde{\Gamma} \leqslant N(F)$.

Let $h \in \overline{\Gamma}$ and let $\tilde{h} = \rho_{\mathfrak{m}}(h) = \rho_{\mathfrak{m}} \circ h$. Since $\tilde{h} \in N(F)$, we see that $\tilde{h}(\mathfrak{q}) = 0$. Hence, $h(\mathfrak{q}) \subseteq \mathfrak{m}$. Since $\mathfrak{m}$ is an arbitrary maximal ideal, we obtain $h(\mathfrak{q}) \in \operatorname{Rad} \overline{R}$. Consequently, $\rho_{\operatorname{Rad} \overline{R}} \circ h(\mathfrak{q}) = 0$ and $\rho_{\operatorname{Rad} \overline{R}}(h) \in N(\overline{R}/\operatorname{Rad} \overline{R})$.

Since h is an arbitrary element of $\overline{\Gamma}$, we get $\rho_{\operatorname{Rad} \overline{R}}(\overline{\Gamma}) \in N(\overline{R}/\operatorname{Rad} \overline{R})$. This implies that

$$\rho_{\operatorname{Rad} \overline{R}}\big(D(\overline{R})^{\overline{\Gamma}}\big) \leqslant D(\overline{R}/\operatorname{Rad} \overline{R})^{\rho_{\operatorname{Rad} \overline{R}}(\overline{\Gamma})} \leqslant D(\overline{R}/\operatorname{Rad} \overline{R}),$$

Therefore, $D(\overline{R})^{\overline{\Gamma}} \leqslant D(\overline{R})G(\overline{R}, \operatorname{Rad} \overline{R})$.

If there exists an element $ab \in D(\overline{R})^{\overline{\Gamma}} \smallsetminus N(R)$, where $a \in D(\overline{R})$ and $b \in G(\overline{R}, \operatorname{Rad} \overline{R})$, then $b \in D(\overline{R})^{\overline{\Gamma}} \smallsetminus N(R)$. Since $D(\overline{R})^{\overline{\Gamma}}$ contains the subgroup generated by $D(\overline{R})$ and b, Condition 8 implies that $D(\overline{R})^{\overline{\Gamma}}$ contains a nontrivial root unipotent. But we have already proved that $\overline{\Gamma}$ does not contain such elements. The contradiction shows $D(\overline{R})^{\overline{\Gamma}} \leqslant N(R)$. Now, Condition 5 implies that $\overline{\Gamma} \leqslant N(R)$ and by Lemma 2.1, $\Gamma \leqslant N(R, \mathfrak{a})$. □

§3. Transporters

We start with studying properties of transporters. In this section G denotes an algebraic group over a ring K and X, Y are subfunctors of G. Unfortunately, the function $R \mapsto \operatorname{Tran}_{G(R)}\big(X(R), Y(R)\big)$ in general is not a subfunctor of G. Therefore, we define a scheme-theoretic transporter $\operatorname{Tran}_G(X, Y)$ as a subfunctor of G given by the formula

$$\operatorname{Tran}_G(X, Y)(R) = \{a \in G(R) \mid z^a \in Y(\tilde{R}) \text{ for all } z \in X(\tilde{R}) \text{ and all } R\text{-algebras } \tilde{R}\}$$

(we always identify elements of $G(R)$ with their canonical images in $G(\tilde{R})$).

More generally, let w be a group word in 2 letters, i.e., an element of the 2-generated free group. For elements z, a of an abstract group Γ we write $w(z, a)$ to denote the image of w in Γ under the group homomorphism, sending the first generator of the free group to z and the second to a. For subsets Δ, Ω of Γ define

$$\operatorname{Tran}^w_\Gamma(\Delta, \Omega) = \{a \in \Gamma \mid w(z, a) \in \Omega \text{ for all } z \in \Delta\}.$$

Similarly, define the scheme-theoretic w-transporter $\operatorname{Tran}^w_G(X, Y)$ as a subfunctor of G given by the formula

$$\operatorname{Tran}^w_G(X, Y)(R) = \{a \in G(R) \mid w(z, a) \in Y(\tilde{R}) \text{ for all } z \in X(\tilde{R}) \text{ and all } R\text{-algebras } \tilde{R}\}.$$

Now we give sufficient conditions for the scheme-theoretic w-transporter to be closed and discuss the equation

$$\mathrm{Tran}^w_G(X,Y)(R) = \mathrm{Tran}^w_{G(R)}\big(X(R), Y(R)\big).$$

For usual transporters (i.e., for $w = x_2^{-1}x_1x_2$) the results can be found in [12, Theorem 6.1] or [6, I, §2, 7.7 and II, §2, 3.6]). Recall that a K-scheme X is called locally free if there exists an open affine covering X_i, where i ranges over an index set, such that the affine algebra of X_i is a free K-module for each i. In our case X is affine, therefore refining the open covering we may assume that each X_i is a principal affine open subscheme of X, i.e., $K[X_i]$ is a principal localization of $K[X]$. Throughout this section $B = K[X]$ and $x \in X(B \otimes R)$ denotes the image of the generic element of X under the natural homomorphisms $X(B) \to G(B) \to G(B \otimes R)$.

Lemma 3.1. *Let X be a representable subfunctor of G and R a K-algebra. Then*

$$\mathrm{Tran}^w_G(X,Y)(R) = \{a \in G(R) \mid w(x,a) \in Y(B \otimes R)\}.$$

Proof. Let $\widetilde{R}$ be an R-algebra, $z \in X(\widetilde{R})$, and a belongs to the right-hand side of the above formula. Applying $z \otimes \mathrm{id}_R$ to the inclusion $w(x,a) \in Y(B \otimes R)$, we get $w(z,a) \in Y(\widetilde{R} \otimes R)$. Sending this element to $Y(\widetilde{R})$ by the multiplication homomorphism $\widetilde{R} \otimes R \to \widetilde{R}$, we see that $a \in \mathrm{Tran}^w_G(X,Y)(R)$. Thus, we proved that the right-hand side is contained in the transporter. The inverse implication is trivial. □

Theorem 2. *Suppose that X is a locally free K-scheme and Y is a closed subscheme of G. Then $\mathrm{Tran}^w_G(X,Y)$ is a closed subscheme.*

Proof. Identify the generic element g of the scheme G with its canonical image in $G(B \otimes A)$. Take an open covering $\operatorname{Spec} B_{s_i}$ of X such that each B_{s_i} is a free K-module and denote by h_i the canonical image of $w(g,x)$ in $G(B_{s_i} \otimes A)$. Thus, $h_i : A \to B_{s_i} \otimes A$ regarded as a map. In other words, h_i is the composition of the map $w(g,x) : A \to B \otimes A$ with the localization homomorphism λ_{s_i}. A basis of the K-module B_{s_i} defines an isomorphism $B_{s_i} \otimes A \cong \prod_{j \in J_i} A$ of A-modules. Denote by $h_{ij} : A \to A$ the composition of h_i with the projection to the j-th component of the direct product above. As usual, $I(Y)$ denotes the ideal of A that defines the subscheme Y. Let $\mathfrak{q}$ be the ideal generated by all images of $I(Y)$ under the maps h_{ij}.

Let $a \in G(R)$. By Lemma 3.1, $a \in \mathrm{Tran}_G^w(X,Y)(R)$ whenever $w(x,a) \in Y(B \otimes R)$. Since $\operatorname{Spec} B_{s_i}$ is an open covering of X, this inclusion is equivalent to the inclusions $v_i \in Y(B_{s_i} \otimes R)$, where v_i is the canonical image of $w(x,a)$ in $G(B_{s_i} \otimes R)$. Note that considered as a map v_i is the composition

$$(\lambda_{s_i} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes a) \circ w(x,g) = (\mathrm{id} \otimes a) \circ (\lambda_{s_i} \otimes \mathrm{id}) \circ w(x,g) = (\mathrm{id} \otimes a) \circ h_i.$$

The inclusion $v_i \in Y(B_{s_i} \otimes R)$ is equivalent to

$$v_i\big(I(Y)\big) = 0 \iff \Big(\mathrm{id} \otimes a\Big)\Big(h_i\big(I(Y)\big)\Big) = 0 \iff a\Big(h_{ij}\big(I(Y)\big)\Big) = 0$$

for all $j \in J_i$. Thus, $a \in \mathrm{Tran}_G^w(X,Y)(R)$ whenever $a(\mathfrak{q}) = 0$, which means that $\mathrm{Tran}_G^w(X,Y)$ is the closed subscheme of G defined by the ideal $\mathfrak{q}$. $\square$

Next, we prove that under some natural conditions group theoretic and scheme theoretic normalizers coincide. Let Z be a function from the class of rings to the class of sets such that $Z(R)$ is a subset of $G(R)$. Since the intersection of closed subschemes of G is a closed subscheme, there exists the smallest subscheme $\overline{Z}$ of G, containing Z (i.e., $\overline{Z}(R) \supseteq Z(R)$ for all rings R). This subscheme is called the closure of Z in G. In other words, $\overline{Z}$ is the closed subscheme defined by the intersection of the kernels of z over all $z \in Z(R)$ and all K-algebras R. A function Z is called dense in G if $\overline{Z} = G$. If X is a subfunctor of G and R is a K-algebra, then by the closure of $X(R)$ in G we mean the closure of the function that equals $X(R)$ at R and the empty set elsewhere.

Proposition 3.2. *Let X be a representable subfunctor of G and let Y be a closed subscheme of G. If $X(R) = X_R(R)$ is dense in X_R, then*

$$\mathrm{Tran}_{G(R)}^w\big(X(R), Y(R)\big) = \mathrm{Tran}_G^w(X,Y)(R).$$

Proof. Let $h \in G(R)$. By Lemma 3.1, $h \in \mathrm{Tran}_G^w(X,Y)(R)$ whenever $w(x,h) \in Y(B \otimes R)$. Put $u = w(x,h) : A \to B \otimes R$. If $\mathfrak{q} = I(Y)$, then the inclusion $u \in Y(B \otimes R)$ is equivalent to $u(\mathfrak{q}) = 0$. On the other hand,

$$\begin{aligned} h \in \mathrm{Tran}_{G(R)}^w\big(X(R), Y(R)\big) &\iff w(a,h) \in Y(R) \text{ for all } a \in X(R) \\ &\iff w(a,h)(\mathfrak{q}) = 0 \text{ for all } a \in X(R) \\ &\iff (a \,\overline{\otimes}\, \mathrm{id}) \circ u(\mathfrak{q}) = 0 \text{ for all } a \in X(R), \end{aligned}$$

where $a \,\overline{\otimes}\, \mathrm{id}$ is the composition $B \otimes R \xrightarrow{a \otimes \mathrm{id}} R \otimes R \xrightarrow{\mathrm{mult}} R$. The closure of $X_R(R)$ in X_R is equal to $V(\mathfrak{a})$, where $\mathfrak{a}$ is the intersection of the kernels of the R-algebra homomorphisms $\hat{a} : B \otimes R \to R$ over all $\hat{a} \in X_R(R)$. The R-algebra homomorphisms $\hat{a} : B \otimes R \to R$ are in one-to-one correspondence with the K-algebra homomorphisms $a : B \to R$, namely, $\hat{a} = a \overline{\otimes} \mathrm{id}$. Thus, $X_R(R)$ is dense in X_R if and only if the intersection of the kernels of the homomorphisms $a \overline{\otimes} \mathrm{id}$ over all $a \in X(R)$ is trivial. It follows that $u(\mathfrak{q}) = 0$ if and only if $(a \overline{\otimes} \mathrm{id}) \circ u(\mathfrak{q}) = 0$ for all $a \in X(R)$, which completes the proof. □

§4. Computation of the normalizer

Let H be a closed K-subgroup of G. Suppose that H itself is a Chevalley–Demazure group scheme with a reduced irreducible root system Ψ and denote by D its elementary subgroup functor. Suppose also that $D(R)$ is perfect for all K-algebras R, which amounts to say that $\Psi \neq A_1$ and either $\Psi \neq C_2$ or K has no epimorphisms onto the field of 2 elements. Let R be a K-algebra. For a functor X on the category of K-algebras denote by X_R its restriction to the category of R-algebras (of course, H_R and G_R are affine group schemes over R).

Let $N(R)$ be the normalizer of $D(R)$ in $G(R)$ and let $\tilde{N}(R)$ be the normalizer of $H(R)$ in $G(R)$. Put $\mathrm{Tran}(R) = \mathrm{Tran}_{G(R)}(D(R), H(R))$. Clearly, both normalizers, $N(R)$ and $\tilde{N}(R)$, are contained in $\mathrm{Tran}(R)$.

Lemma 4.1. $N(R) = \mathrm{Tran}(R) \geqslant \tilde{N}(R)$.

Proof. Let $h \in \mathrm{Tran}(R)$ and let R' be a finitely generated $\mathbb{Z}$-subalgebra of R. Then $D(R')^h \leqslant H(R'')$ for some finitely generated $\mathbb{Z}$-subalgebra $R'' \supseteq R'$ of R. By the main theorem of [8] $H(R'')/D(R'')$ is solvable, hence $D(R'')$ is the largest perfect subgroup of $H(R'')$. Since $D(R')^h \leqslant H(R'')$ and $D(R')^h$ is perfect, it is contained in $D(R'') \leqslant D(R)$. Any ring R is a direct limit of finitely generated $\mathbb{Z}$-subalgebras, hence $D(R)^h \leqslant D(R)$, i.e., $h \in N(R)$. □

In the next corollary we consider the scheme-theoretic normalizers. For a subfunctor X of G put $N_G(X) = \mathrm{Tran}_G(X, X)$.

Corollary 4.2. *Both $N_G(D)$ and $N_G(H)$ are closed in G.*

Proof. Let X_α, $\alpha \in \Psi$ be the root subgroups of H, corresponding to a chosen split maximal torus. Note that by Lemma 4.1 $N_G(D) = \mathrm{Tran}_G(D, H)$. Then $N_G(D) = \mathrm{Tran}_G(D, H) = \cap_{\alpha \in \Psi} \mathrm{Tran}_G(X_\alpha, H)$. Since H is a closed subscheme and $K[X_\alpha] = K[t]$ is a free K-module, the assertion about $N_G(D)$ follows from Theorem 2.

The Gauss decomposition in a split reductive group states that there exists an open covering of H such that each piece is isomorphic (as a scheme) to $U \times U \times T$, where U is the unipotent radical of a Borel subgroup and T is a split torus. Therefore, H is locally free and the result again follows from Theorem 2. $\square$

Proposition 4.3. $N_G(D) = \mathrm{Tran}_G(D, H) = N_G(H)$.

Proof. The inclusions $N_G(D) = \mathrm{Tran}_G(D, H) \geqslant N_G(H)$ follows immediately from Lemma 4.1 and the definition of scheme theoretic transporters. Conversely, let R be a K-algebra and $h \in N_G(D)(R) = \mathrm{Tran}_G(D, H)(R)$. Consider the functor D_R^h on the category of R-algebras given by $D_R^h(R') = D(R')^h$ with obvious action on morphisms (we still identify h with its image in $G(R')$ under the structure homomorphism $R \to R'$). Clearly, $D_R^h \leqslant H_R$, hence the closure $\overline{D_R^h}$ of D_R^h in G_R is contained in H_R as well. The conjugation by h is a scheme automorphism of G_R, therefore $\overline{D_R^h} = (\overline{D_R})^h$. Since the closure of the elementary subgroup is the whole Chevalley group, $\overline{D_R} = H_R$. It follows that $(H_R)^h \leqslant H_R$, which means that $h \in N_G(H)(R)$. $\square$

The following statement is to verify the density condition of Proposition 3.2 and establish the equality of group theoretic and ring theoretic normalizers.

Lemma 4.4. *Suppose that R is an algebra over an infinite field K. Then $\mathbb{G}_a(R)$ is dence in $\mathbb{G}_{a,R}$.*

Proof. An R-algebra homomorphism $R[\mathbb{G}_a] = R[t] \to R$ is the evaluation homomorphism. A polynomial $p \in R[t]$ is in the intersection of the kernels of all such homomorphisms whenever $p(r) = 0$ for all $r \in R$. In particular, $p(r_i) = 0$, $i = 0, \dots, \deg p$, for $\deg P + 1$ distinct elements of K. Since the Vandermonde determinant is nonzero, all coefficients of p must be zero. Thus, the intersection of the kernels of all R-algebra homomorphisms $R[\mathbb{G}_a] \to R$ is trivial, which means that the closure of $G_a(R)$ is $G_{a,R}$. $\square$

Proposition 4.5. *If K is an infinite field, then*

$$N_G(D)(R) = \mathrm{Tran}_G(D, H)(R) = N_G(H)(R) = N(R) = \tilde{N}(R) = \mathrm{Tran}(R).$$

Proof. The group $D(R)$ is generated by the root subgroups $X_\alpha(R)$, $\alpha \in \Psi$, therefore

$$\mathrm{Tran}(R) = \bigcap_{\alpha \in \Psi} \mathrm{Tran}_{G(R)}\big(X_\alpha(R), H(R)\big)$$

and
$$\mathrm{Tran}_G(D,H)(R)=\bigcap_{\alpha\in\Psi}\mathrm{Tran}_G\big(X_\alpha,H\big)(R).$$
Since $X_\alpha\cong\mathbb{G}_a$, by Proposition 3.2 and Lemma 4.4
$$\mathrm{Tran}_{G(R)}\big(X_\alpha(R),H(R)\big)=\mathrm{Tran}_G\big(X_\alpha,H\big)(R),$$
hence
$$\mathrm{Tran}(R)=\mathrm{Tran}_G(D,H)(R).$$
Then, we obtain the following chain of inclusions.
$$\begin{aligned}\tilde{N}(R)\leqslant\mathrm{Tran}(R)&=\mathrm{Tran}_G(D,H)(R)=\mathrm{Tran}_G(H,H)(R)\\&\leqslant\mathrm{Tran}_{G(R)}\big(H(R),H(R)\big)=\tilde{N}(R).\end{aligned}$$
Thus $\tilde{N}(R)=\mathrm{Tran}(R)$ and all other equalities were already proved in Lemmas 4.1 and 4.3. □

§5. Equality of the transporters and the normalizers

This section is to prove Condition 5 in the settings of sections 3–4.

Theorem 3. *Let K be an infinite field of characteristic not 2 if $\Psi=A_2,B_l,C_l,F_4,G_2$, and not 3 if $\Phi=G_2$. Suppose further that there exists an absolutely irreducible representation of H in G, i.e., a linear representation of G such that the K-submodules $KD(K)$ and $KG(K)$ of the matrix ring $\mathrm{M}_n(K)$ are equal. Then $\mathrm{Tran}(D,N)=N$.*

Proof. The condition on existence of a representation ensures that the centralizer of D in G is equal to the center of G and to the center of D. For a K-algebra R denote by $D_{\mathrm{ad}}(R)$ the quotient group of $D(R)$ modulo its center. The center of $D(R)$ is known to coincide with the (scheme theoretic) center of H, therefore $D_{\mathrm{ad}}(R)$ is the elementary subgroup of the adjoint Chevalley group of type Ψ.

Let $a\in\mathrm{Tran}(D,N)(R)$. Put $C=R[H]$ and let $h\in H(C)\subseteq G(C)$ be the generic element of H_R. By Lemma 3.1, $h^a\in N(C)$ (as usual, we identify a with its canonical image in $G(C)$). Consider the map $\theta:N(C)\to\mathrm{Aut}(D(C))$, where $N(C)$ acts on $D(C)$ by conjugation. Since $\theta(b)$ acts identically on the center of $D(C)$, we may consider θ as an automorphism of $D_{\mathrm{ad}}(C)$.

By the classification of automorphisms of an adjoint elementary Chevalley group [5, Theorem 1] $\theta(h^a)=\theta(b)\cdot\gamma\cdot\varphi$, where $b\in H_{\mathrm{ad}}(C)$, γ is a graph automorphism, and φ is induced by a ring automorphism. Then

$\varphi = \gamma^{-1}\theta(b^{-1}h^a)$ is an automorphism of the group scheme H_C. Being induced by a ring automorphism $\psi : C \to C$, φ acts on a root subgroup X_α sending $x_\alpha(r)$ to $x_\alpha(\psi(r))$. Since φ is a scheme automorphism, ψ induces an automorphism of the scheme $\mathbb{G}_{a,C}$. But a scheme automorphism of $\mathbb{G}_{a,C}$ preserving the element 1 is identity. Therefore, ψ and φ are identities as well.

Thus, we have $\theta(b^{-1}h^a) = \gamma$. The counit map $\varepsilon : C \to R$ takes h to the identity element of $H(R)$ (this is equivalent to the definition of the counit). Both sides of equation $\theta(b^{-1}h^a) = \gamma$ are natural transformations $D_{\mathrm{ad}} \to D_{\mathrm{ad}}$, therefore the diagram

$$\begin{array}{ccc} D_{\mathrm{ad}}(C) & \xrightarrow{\ \varepsilon\ } & D_{\mathrm{ad}}(R) \\ {\scriptstyle \gamma=\theta(b^{-1}h^a)}\downarrow & & \downarrow{\scriptstyle \gamma=\theta_R(\varepsilon(b)^{-1})} \\ D_{\mathrm{ad}}(C) & \xrightarrow{\ \varepsilon\ } & D_{\mathrm{ad}}(R) \end{array}$$

commutes, where $\theta_R : N(R) \to \mathrm{Aut}\big(D_{\mathrm{ad}}(R)\big)$ denotes the conjugation by the argument of θ_R. We see that $\theta(\varepsilon(b))\gamma$ acts trivially on $D_{\mathrm{ad}}(R)$. This means that γ is induced by the inner automorphism of the root system Ψ and can be substituted by conjugation by an appropriate element from the Weyl group. Multiplying b by this element we may assume that γ is identity.

Now, $\theta(h^a) = \theta(b)$, which means that $b^{-1}h^a$ acts trivially on $D_{\mathrm{ad}}(C)$, hence also on $D(C)$. It follows that $h^a = bc$ for some element c from the center of $G(C)$. Let R' be an R-algebra. Sending h to each element from $D(R')$, we see that $D(R')^a \leqslant D(R') \cdot \mathrm{Cent}(G(R'))$. Finally,

$$\begin{aligned} D(R')^a &= [D(R')^a, D(R')^a] \\ &\leqslant [D(R') \cdot \mathrm{Cent}(G(R')), D(R') \cdot \mathrm{Cent}(G(R'))] \leqslant D(R'), \end{aligned}$$

which means that $a \in N(R)$. □

References

1. E Abe, *Automorphisms of chevalley groups over commutative rings.* — St.Petersburg Math. J. **5**, no. 2 (1994), 287–300.
2. A. S. Anan'evskii, N. A. Vavilov, S. S. Sinchuk, *Overgroups of* $\mathrm{E}(l,R) \otimes \mathrm{E}(m,R)$ *I. Levels and normalizers.* — St.Petersburg Math. J. **23**, No. 5 (2012), 819–849.
3. A. S. Ananievsky, N. A. Vavilov, S. S. Sinchuk, *Overgroups of* $\mathrm{E}(l,R) \otimes \mathrm{E}(m,R)$. — J. Math. Sci. (N. Y.) **161**, No. 4 (2009), 461–473.
4. Z. I. Borevich, N. A. Vavilov, *The distribution of subgroups in the general linear group over a commutative ring.* — Proc. Steklov. Inst. Math. **165** (1985), 27–46.

5. E. I. Bunina, *Automorphisms of Chevalley groups of different types over commutative rings.* — J. Algebra **355** (2012), 154-170.
6. M. Demazure, P. Gabriel, *Introduction to algebraic geometry and algebraic groups.* Math. Stud. **39**, North-Holland, Amsterdam, 1980.
7. P. B. Gvozdevski, *Overgroups of Levi subgroups* I. *The case of abelian unipotent radical.* — St.Petersburg Math. J. **31**, No. 6 (2020).
8. R. Hazrat, N. Vavilov, K_1 *of Chevalley groups are nilpotent.* — J. Pure and Appl. Algebra **179** (2003), 99–116.
9. R. A. Lubkov, I. I. Nekrasov, *Overgroups of exterior powers of an elementary group.* I. *Levels and normalizers.* — Linear Multilinear Algebra, to appear.
10. R. A. Lubkov, I. I. Nekrasov, *Explicit equations for exterior square of the general linear group.* — Zap. Nauchn. semin. **470** (2018), 120–137.
11. A. Yu. Luzgarev, *Overgroups of* F_4 *in* E_6 *over commutative rings.* — St. Petersburg Math. J. **20** (2009), 955–981.
12. J. S. Milne, *Basic theory of affine group schemes*, Preprint `http://www.jmilne.org/math/CourseNotes/AGS.pdf`, 2012.
13. N. H. T. Nhat, T. N. Hoi, *The normalizer of the elementary linear group of a module arising under extension of the base ring.* — J. Math. Sci. **234**, No. 2 (2018), 197–202.
14. Ya. N. Nuzhin, *Groups contained between groups of Lie type over different fields.* — Algebra Logic **22** (1983), 378–389.
15. Ya. N. Nuzhin, *Intermediate subgroups in the Chevalley groups of type* B_l*,* C_l*,* F_4*, and* G_2 *over the nonperfect fields of characteristic* 2 *and* 3. — Sib. Math. J. **54**, No. 1 (2013), 119–123.
16. Ya. N. Nuzhin, A. V. Yakushevich, *Intermediate subgroups of Chevalley groups over a field of fractions of a ring of principal ideals.* — Algebra Logic **39**, No. 3 (2000), 199–206.
17. A. V. Shchegolev, *Overgroups of elementary block-diagonal subgroups in the classical symplectic group over an arbitrary commutative ring.* — St.Petersburg Math. J. **30**, No. 6 (2019), 1007–1041.
18. A. K. Stavrova, A. V. Stepanov, *Normal structure of isotropic reductive groups over rings*, Preprint `http://arxiv.org/abs/1801.08748`, 2017.
19. A. V. Stepanov, *Free product subgroups between Chevalley groups* $\mathrm{G}(\Phi, F)$ *and* $\mathrm{G}(\Phi, F[t])$. — J. Algebra **324**, No. 7 (2010), 1549–1557.
20. A. V. Stepanov, *Subring subgroups in Chevalley groups with doubly laced root systems.* — J. Algebra **362**, 12–29 (2012).
21. A. V. Stepanov, *Structure of Chevalley groups over rings via universal localization.* — J. Algebra **450**, 522–548 (2016).
22. G. Taddei, *Normalité des groupes élémentaires dans les groupes de Chevalley sur un anneau.* — Contemp. Math. **55** (1986), 693–710.
23. L. N. Vaserstein, *On normal subgroups of Chevalley groups over commutative rings.* — Tôhoku Math. J. **38** (1986), 219–230.
24. N. Vavilov, V. Petrov, *Overgroups of* $\mathrm{EO}(2l, R)$. — J. Math. Sci. (N. Y.) **116**, No. 1 (2003), 2917–2925.
25. N. Vavilov, V. Petrov, *Overgroups of* $\mathrm{Ep}(2l, R)$. — St.Petersburg Math. J. **15**, No. 4 (2004), 515–543.

26. N. Vavilov, V. Petrov, *Overgroups of* $\mathrm{EO}(n, R)$. — St.Petersburg Math. J. **19**, No. 2 (2008), 167–195.
27. N. A. Vavilov, A. V. Stepanov, *Overgroups of semisimple groups.* — Vestn. Samar. Gos. Univ. Estestvennonauchn. Ser., No. 3 (2008), 51–95.
28. H. You, *Overgroups of symplectic group in linear group over commutative rings.* — J. Algebra **282**, No. 1 (2004), 23–32.
29. H. You, *Overgroups of classical groups over commutative ring in linear group.* — Sci. China Math. **49**, No. 5 (2006), 626–638.

St.Petersburg State University;
St.Petersburg Department
of V. A. Steklov Institute of Mathematics
of the Russian Academy of Sciences
E-mail: RomanLubkov@yandex.ru

St.Petersburg State University
E-mail: stepanov239@gmail.com

Поступило 8 ноября 2019 г.