Записки научных
семинаров ПОМИ
Том 481, 2019 г.

P. Nikitin

THE ABSOLUTE OF THE COMB GRAPH

Abstract. In the 1970s R. Stanley introduced the comb graph $\mathbb{E}$ whose vertices are indexed by the set of compositions of positive integers and branching reflects the ordering of compositions by inclusion. A. Vershik defined the absolute of a $\mathbb{Z}_+$-graded graph as the set of all ergodic probability central measures on it. We show that the absolute of $\mathbb{E}$ is naturally parametrized by the space $\Omega = \{(\alpha_1, \alpha_2, \dots) : \alpha_i \geqslant 0, \sum_i \alpha_i \leqslant 1\}$.

§1. Introduction

The Young graph is the Hasse diagram of the set of all Young diagrams (or partitions) partially ordered by inclusion (see [6] for details). If, instead of partitions, we consider the set Comp of all *compositions* of positive integers, we obtain the so-called *comb graph* $\mathbb{E}$. It was introduced by R. Stanley in [9, 10], and it corresponds to the following partial order on $\mathbb{N}^2$ (see Fig. 1):

$$(i,j) \prec (k,l) \iff \begin{cases} i = k \text{ and } j < k; \\ j = l = 1 \text{ and } i < k. \end{cases} \tag{1}$$

Both the Young graph and comb graph are $\mathbb{Z}_+$-graded graphs without multiple edges, where the grading $|\mu| = \sum_i \mu_i$ of a diagram $\mu = (\mu_1, \mu_2, \dots)$ is the number of boxes in the diagram (the *weight* of μ).

Definition 1.1. *The comb graph $\mathbb{E}$ can be constructed as follows. The vertices of the nth level $\mathbb{E}_n$ of $\mathbb{E}$ are identified with the set* Comp_n *of compositions of weight n. Denote by $\ell(\mu)$ the number of parts in $\mu \in \mathbb{E}_n$. There is an edge between vertices $\mu \in \mathbb{E}_n$ and $\boldsymbol{\mu} \in \mathbb{E}_{n+1}$ if one of the two situations occur:*

- *either there exists $j \in \{1, \dots, \ell(\mu)\}$ with* $\boldsymbol{\mu} = \big(\mu_1, \dots, \mu_j + 1, \dots, \mu_{\ell(\mu)}\big)$,
- *or* $\boldsymbol{\mu} = \big(\mu_1, \dots, \mu_{\ell(\mu)}, 1\big)$.

Key words and phrases: comb graph, compositions, Martin boundary, ergodic central measures, absolute.

The work is supported by the RSF grant 17-71-20153.

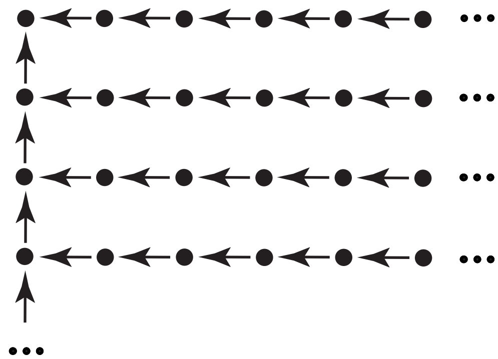

Figure 1. The “comb” partial order on $\mathbb{N}^2$.

Denote any of these two situations by $\mu \nearrow \boldsymbol{\mu}$.

The first levels of the comb graph are shown in Fig. 2.

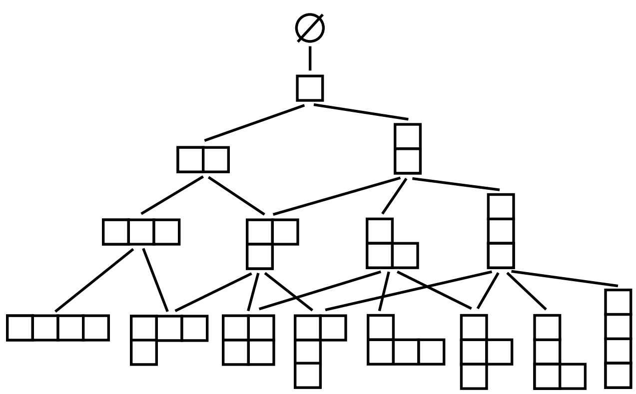

Figure 2. The comb graph.

A function $h : \bigcup\limits_{n=0}^{\infty} \mathbb{E}_n \mapsto \mathbb{R}_+$ is called a *normalized harmonic function* if $h(\varnothing) = 1$ and

$$h(\mu) = \sum_{\mu \nearrow \boldsymbol{\mu}} h(\boldsymbol{\mu}) \quad \text{for any} \quad \mu \in \bigcup_{n=0}^{\infty} \mathbb{E}_n.$$

Denote the space of such functions by $\mathcal{H}(\mathbb{E})$. The characterization of the harmonic functions for particular examples of $\mathbb{Z}_+$-graded graphs (or *Bratteli diagrams*) is one of the basic questions in the asymptotic theory of such graphs, see [5, 6, 8, 11, 12] and the references therein. Such a function is said to be *extremal* if it cannot be written as a nontrivial convex combination $ah_1(\mu) + (1-a)h_2(\mu)$, where $a \in (0,1)$, $h_1, h_2 \in \mathcal{H}(\mathbb{E})$, $h_1 \neq h_2$. The set $E_{\min}(\mathbb{E})$ of normalized nonnegative extremal harmonic functions is called the *minimal boundary* of the graph $\mathbb{E}$. In general, it is a proper subset of the *Martin boundary* $E_{\mathrm{Mart}}(\mathbb{E})$, see Definition 2.2. For the comb graph, we show that $E_{\min}(\mathbb{E}) = E_{\mathrm{Mart}}(\mathbb{E})$ and give an explicit description of the boundaries.

Consider the topological space

$$\Omega = \{(\alpha_1, \alpha_2, \dots) : \alpha_i \geqslant 0, \sum_i \alpha_i \leqslant 1\}$$

with the topology of pointwise convergence, and the set of polynomials $\{Q_\mu(\alpha_1, \alpha_2, \dots)\}$, $\mu \in \mathrm{Comp}$,

$$Q_\mu(\alpha_1, \alpha_2, \dots) = \prod_{i=1}^{\ell(\mu)} \left(1 - \sum_{k=1}^{i-1} \alpha_k\right) \alpha_i^{\mu_i - 1}. \tag{2}$$

Theorem (Main theorem)**.** (i) $E_{\mathrm{Mart}}(\mathbb{E}) = E_{\min}(\mathbb{E}) \cong \Omega$.

(ii) *The integral representation*

$$\phi(\mu) = \int_\Omega Q_\mu(\omega) dP_\phi$$

gives a one-to-one correspondence between the harmonic functions $\phi \in \mathcal{H}(\mathbb{E})$ and the probability measures P_ϕ on Ω.

Recall (see [6, 8]) that there is a natural bijection between the nonnegative harmonic functions and the central measures on a $\mathbb{Z}_+$-graded graph. Under this bijection, the boundary $E_{\min}(\mathbb{E})$ corresponds to the set of all ergodic probability central measures on the graph (the *absolute* of $\mathbb{E}$).

Corollary 1.1. *The absolute of $\mathbb{E}$ is parametrized by Ω.*

Remark 1.2. In [13], A. Vershik described the Plancherel measure on the Young graph as the unique *nondegenerate* central measure on it; this can be deduced from the fact that the Plancherel measure is the unique measure with *zero frequencies.* He inspired the present work by conjecturing that the family of nondegenerate central measures for the comb graph is richer. We see from Corollary 1.1 and Proposition 2.4 that there is only one harmonic function h_0 with zero frequencies (see Definition 2.1):

$$h_0(\lambda) = \begin{cases} 1 & \text{if } \lambda = (1^r),\ r \in \mathbb{N}; \\ 0 & \text{otherwise}; \end{cases}$$

therefore, the conjecture does not hold.

For any sequence $L = \{l_i : l_i \in \mathbb{N}\}_{i=1}^{\infty}$, consider the subset

$$\bigcup_{i=1}^{\infty} \{(i,j) : 1 \leqslant j \leqslant l_i\} \subset \mathbb{N}^2$$

and restrict the partial order (1) to it. The corresponding branching graph $\mathbb{E}^L$ is a subgraph of the comb graph.

Corollary 1.3. *For any sequence $L = \{l_i : l_i \in \mathbb{N}\}_{i=1}^{\infty}$, the absolute of the graph $\mathbb{E}^L$ consists of the unique central measure with zero frequencies.*

Remark 1.4. One can easily see from Proposition 2.4 and formulas (2) that the central measure corresponding to $(\alpha_1, \alpha_2, \dots) \in \Omega$ can be described as a random walk on $\mathbb{E}$ with transition probabilities

$$p\Big((\mu_1, \mu_2, \dots), \boldsymbol{\mu}\Big) = \begin{cases} \alpha_j & \text{if } \boldsymbol{\mu} = \big(\mu_1, \dots, \mu_j + 1, \dots, \mu_{\ell(\mu)}\big); \\ 1 - \sum_{j=1}^{\ell(\mu)} \alpha_j & \text{if } \boldsymbol{\mu} = \big(\mu_1, \dots, \mu_{\ell(\mu)}, 1\big). \end{cases}$$

Remark 1.5. A similar description for the boundaries of the *Kingman graph* $\mathbb{K}$ was given in [4, 5, 7]. In this case, the boundaries $E_{\mathrm{Mart}}(\mathbb{K}) = E_{\min}(\mathbb{K})$ are parametrized by the ordered version of Ω:

$$\Omega_{\mathbb{K}} = \{(\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant 0) : \sum_i \alpha_i \leqslant 1\}.$$

Another nonsymmetric generalization of the Kingman graph, the *refined Kingman graph*, was considered in [2, 3]. In this case, the Martin boundary and the minimal boundary of the graph also coincide, but their parametrization is more subtle.

§2. Proofs

For any two vertices $\mu = \mu(n) \in \mathbb{E}_n$, $\boldsymbol{\mu} = \mu(N) \in \mathbb{E}_N$, denote by $\dim(\mu, \boldsymbol{\mu})$ the number of paths $\mu = \mu(n) \nearrow \cdots \nearrow \mu(N) = \boldsymbol{\mu}$, $|\mu(i)| = i$, and denote by $\dim(\boldsymbol{\mu})$ the number of paths from $\varnothing \in \mathbb{E}_0$ to $\boldsymbol{\mu}$.

Lemma 2.1. *For* $\mu = (\mu_1, \mu_2, \dots, \mu_l) \in \mathbb{E}_n$, $\boldsymbol{\mu} = (\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_L) \in \mathbb{E}_N$, $n \leqslant N$, *we have*

$$\dim(\boldsymbol{\mu}) = N!\left(\prod_{i=1}^{L}(N - \sum_{k=1}^{i-1}\boldsymbol{\mu}_k)(\boldsymbol{\mu}_i - 1)!\right)^{-1}, \tag{3}$$

$$\dim(\mu, \boldsymbol{\mu}) = (N-n)!\left(\prod_{i=l+1}^{L}(N - \sum_{k=1}^{i-1}\boldsymbol{\mu}_k)\prod_{i=1}^{l}(\boldsymbol{\mu}_i - \mu_i)!\prod_{i=l+1}^{L}(\boldsymbol{\mu}_i - 1)!\right)^{-1}, \tag{4}$$

$$\frac{\dim(\mu, \boldsymbol{\mu})}{\dim \boldsymbol{\mu}} = \frac{(N-n)!}{N!}\prod_{i=1}^{l}\left(N - \sum_{k=1}^{i-1}\boldsymbol{\mu}_k\right)(\boldsymbol{\mu}_i - 1)_{\mu_i - 1}, \tag{5}$$

where $(x)_n$ *denotes the Pochhammer symbol (falling factorial):*

$$(x)_n = x(x-1)\dots(x-n+1).$$

Denote by $\dim'(\mu, \boldsymbol{\mu})$ the right-hand side of (4). The only difficulty is to prove the branching rule for $\dim'(\mu, \boldsymbol{\mu})$. For the convenience of the reader, we first show it for a representative special case.

Example 2.1. Let $\ell(\mu) = 2$, $\ell(\boldsymbol{\mu}) = 5$. We show that

$$\dim'(\mu, \boldsymbol{\mu}) = \sum_{j=1}^{5}\dim'(\mu, (\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_j - 1, \dots, \boldsymbol{\mu}_5)).$$

We multiply both sides by the common denominator to obtain

$$\begin{aligned}
&(N-n)(N - \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 - 1)(N - \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 - \boldsymbol{\mu}_3 - 1)\\
&= \Big((\boldsymbol{\mu}_1 - \mu_1) + (\boldsymbol{\mu}_2 - \mu_2)\Big)(N - \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 - 1)(N - \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 - \boldsymbol{\mu}_3 - 1)\\
&\qquad + (N - \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)(N - \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 - \boldsymbol{\mu}_3 - 1)(\boldsymbol{\mu}_3 - 1)\\
&\qquad + (N - \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)(N - \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 - \boldsymbol{\mu}_3)(\boldsymbol{\mu}_4 - 1)\\
&\qquad + (N - \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)(N - \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 - \boldsymbol{\mu}_3)(N - \sum_{i=1}^{4}\boldsymbol{\mu}_i) = 0 \Longleftrightarrow
\end{aligned}$$

$$\Longleftrightarrow (N-\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)\Big((N-\boldsymbol{\mu}_1-\boldsymbol{\mu}_2-1)(N-\boldsymbol{\mu}_1-\boldsymbol{\mu}_2-\boldsymbol{\mu}_3-1)$$
$$-(N-\boldsymbol{\mu}_1-\boldsymbol{\mu}_2-\boldsymbol{\mu}_3-1)(\boldsymbol{\mu}_3-1)-(N-\boldsymbol{\mu}_1-\boldsymbol{\mu}_2-\boldsymbol{\mu}_3)(\boldsymbol{\mu}_4-1)$$
$$(N-\boldsymbol{\mu}_1-\boldsymbol{\mu}_2-\boldsymbol{\mu}_3)(N-\sum_{i=1}^{4}\boldsymbol{\mu}_i)\Big)=0\Longleftrightarrow (N-\boldsymbol{\mu}_1-\boldsymbol{\mu}_2-\boldsymbol{\mu}_3)$$
$$\times\Big((N-\boldsymbol{\mu}_1-\boldsymbol{\mu}_2-\boldsymbol{\mu}_3-1)-(\boldsymbol{\mu}_4-1)-(N-\sum_{i=1}^{4}\boldsymbol{\mu}_i)\Big)=0.$$

Proof of lemma 2.1. Formulas (3) and (5) follow from (4). We prove (4) by induction. Obviously, $\dim'(\mu,\mu)=1$ and $\dim'(\mu,\boldsymbol{\mu})=0$ if $\mu\not\subset\boldsymbol{\mu}$. Therefore, it suffices to prove that

$$\dim'(\mu,\boldsymbol{\mu})=\sum_{j=1}^{\ell(\boldsymbol{\mu})}\dim'(\mu,(\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_{j-1},\boldsymbol{\mu}_j-1,\boldsymbol{\mu}_{j+1},\dots,\boldsymbol{\mu}_{\ell(\boldsymbol{\mu})})). \tag{6}$$

We set $l=\ell(\mu)$, $L=\ell(\boldsymbol{\mu})$ and rewrite (6) as

$$(N-n)\prod_{k=l+1}^{L-1}(N-\sum_{m=1}^{k-1}\boldsymbol{\mu}_m-1)-\Big(\sum_{i=1}^{l}(\boldsymbol{\mu}_i-\mu_i)\Big)\prod_{k=l+1}^{L-1}(N-\sum_{m=1}^{k-1}\boldsymbol{\mu}_m-1)$$
$$-\sum_{j=l+1}^{L-1}(\boldsymbol{\mu}_j-1)\prod_{k=l+1}^{j}(N-\sum_{m=1}^{k-1}\boldsymbol{\mu}_m)\prod_{k=j+1}^{L-1}(N-\sum_{m=1}^{k-1}\boldsymbol{\mu}_m-1)$$
$$-\prod_{k=l+1}^{L}(N-\sum_{m=1}^{k-1}\boldsymbol{\mu}_m)=0\Longleftrightarrow(N-\sum_{m=1}^{l}\boldsymbol{\mu}_m)\Big(\prod_{k=l+1}^{L-1}(N-\sum_{m=1}^{k-1}\boldsymbol{\mu}_m-1)$$
$$-\sum_{j=l+1}^{L-1}(\boldsymbol{\mu}_j-1)\prod_{k=l+2}^{j}(N-\sum_{m=1}^{k-1}\boldsymbol{\mu}_m)\prod_{k=j+1}^{L-1}(N-\sum_{m=1}^{k-1}\boldsymbol{\mu}_m-1)$$
$$-\prod_{k=l+2}^{L}(N-\sum_{m=1}^{k-1}\boldsymbol{\mu}_m)\Big)=0\Longleftrightarrow(N-\sum_{m=1}^{l+1}\boldsymbol{\mu}_m)\Big(\prod_{k=l+2}^{L-1}(N-\sum_{m=1}^{k-1}\boldsymbol{\mu}_m-1)$$
$$-\sum_{j=l+2}^{L-1}(\boldsymbol{\mu}_j-1)\prod_{k=l+3}^{j}(N-\sum_{m=1}^{k-1}\boldsymbol{\mu}_m)\prod_{k=j+1}^{L-1}(N-\sum_{m=1}^{k-1}\boldsymbol{\mu}_m-1)$$
$$-\prod_{k=l+3}^{L}(N-\sum_{m=1}^{k-1}\boldsymbol{\mu}_m)\Big)=0\Longleftrightarrow\cdots\Longleftrightarrow(N-\sum_{m=1}^{L-2}\boldsymbol{\mu}_m)$$

$$\times\Big((N-\sum_{m=1}^{L-2}\boldsymbol{\mu}_m-1)-(\boldsymbol{\mu}_{L-1}-1)-(N-\sum_{m=1}^{L-1}\boldsymbol{\mu}_m)\Big)=0. \qquad \square$$

Lemma 2.2. *In the notation of the previous lemma, we have*

$$\left|\frac{\dim(\mu,\boldsymbol{\mu})}{\dim\boldsymbol{\mu}}-Q_\mu\Big(\frac{\boldsymbol{\mu}_1}{N},\frac{\boldsymbol{\mu}_2}{N},\dots\Big)\right|\leqslant\frac{C(\mu)}{N},$$

where the constant $C(\mu)$ depends on μ only.

Proof. By the previous lemma, we have

$$\begin{aligned}
&\left|\frac{\dim(\mu,\boldsymbol{\mu})}{\dim\boldsymbol{\mu}}-Q_\mu\Big(\frac{\boldsymbol{\mu}_1}{N},\frac{\boldsymbol{\mu}_2}{N},\dots\Big)\right|\\
&\quad=\left|\frac{(N-n)!}{N!}\prod_{i=1}^{\ell(\mu)}\Big(N-\sum_{k=1}^{i-1}\boldsymbol{\mu}_k\Big)(\boldsymbol{\mu}_i-1)_{\mu_i-1}-Q_\mu\Big(\frac{\boldsymbol{\mu}_1}{N},\frac{\boldsymbol{\mu}_2}{N},\dots\Big)\right|\\
&\quad=\left|\frac{(N-n)!}{N!}\sum_{1^{\ell(\mu)}\subset\mu'\subsetneq\mu}c(\mu,\mu')N^{|\mu'|}Q_{\mu'}\Big(\frac{\boldsymbol{\mu}_1}{N},\frac{\boldsymbol{\mu}_2}{N},\dots\Big)\right|\leqslant\frac{C(\mu)}{N},
\end{aligned}$$

where $c(\mu,\mu')$ is a combinatorial factor depending on μ and μ' only. Here we have used the obvious estimate $|Q_{\mu'}(x_1,x_2,\dots)|\leqslant 1$ for any $\mu'\in\mathrm{Comp}$ if $\sum x_i\leqslant 1$ and $x_i\geqslant 0$. $\square$

Lemma 2.3. *The linear space spanned by the polynomials $\{Q_\mu(\omega)\}_{\mu\in\mathrm{Comp}}$ is uniformly dense in the space of continuous functions on Ω.*

Proof. The topological space Ω is compact as a closed subspace of $[0,1]^{\mathbb{N}}$. The polynomials $\{Q_\mu(\omega)\}$ are, obviously, continuous. We show that the closure of their linear span contains all the monomials $\prod_{i=1}^{\ell}\alpha_i^{\mu_i}$, $\ell\in\mathbb{N}$, $\mu_1,\mu_2,\dots\mu_{\ell-1}\in\mathbb{N}\cup\{0\}$, $\mu_\ell\in\mathbb{N}$, and then apply the Stone–Weierstrass theorem. Indeed, for $\ell=2$ we have

$$\begin{aligned}
\alpha_1^{\mu_1}\alpha_2^{\mu_2}&=(1-\alpha_1)\alpha_1^{\mu_1}\alpha_2^{\mu_2}\sum_{j=0}^{\infty}\alpha_1^j=\sum_{j=0}^{\infty}(1-\alpha_1)\alpha_1^{\mu_1+j}\alpha_2^{\mu_2}\\
&=\sum_{j=0}^{\infty}Q_{(1+\mu_1+j,1+\mu_2)}(\alpha_1,\alpha_2,\dots),
\end{aligned}$$

where the sum is uniformly convergent, because we have

$$\sum_{j=j_0}^{\infty}(1-x_1)x_1^{\mu_1+j}x_2^{\mu_2} \leqslant x_1^{\mu_1+j_0}(1-x_1) \leqslant \frac{1}{\mu_1+j_0} \tag{7}$$

if $x_1 \geqslant 0$, $x_2 \geqslant 0$, $1 \geqslant x_1+x_2$. In the general case, we write

$$\begin{aligned}\prod_{i=1}^{\ell}\alpha_i^{\mu_i} &= \prod_{n=0}^{\ell-1}\alpha_{n+1}^{\mu_{n+1}} = \prod_{n=0}^{\ell-1}\Bigl(1-\sum_{k=1}^{n}\alpha_k\Bigr)\alpha_{n+1}^{\mu_{n+1}}\sum_{j=0}^{\infty}\Bigl(\sum_{k=1}^{n}\alpha_k\Bigr)^j \\ &= \prod_{n=0}^{\ell-1}\Bigl(1-\sum_{k=1}^{n}\alpha_k\Bigr)\alpha_{n+1}^{\mu_{n+1}}\sum_{j_{n,1},j_{n,2},\dots,j_{n,n}=0}^{\infty}\binom{j_{n,1}+\dots+j_{n,n}}{j_{n,1},\dots,j_{n,n}}\prod_{m=1}^{n}\alpha_m^{j_{n,m}} \\ &= \sum\binom{j_{2,1}+j_{2,2}}{j_{2,1},j_{2,2}}\dots\binom{j_{n,1}+\dots+j_{n,n}}{j_{n,1},\dots,j_{n,n}}\dots \\ &\quad\times Q_{(1+\mu_1+j_{1,1}+j_{2,1}+\dots,1+\mu_2+j_{2,2}+j_{3,2}+\dots,\dots)}(\alpha_1,\alpha_2,\dots),\end{aligned}$$

where the last sum is over all $j_{p,q} \in \mathbb{N}\cup\{0\}$, $p,q\in\mathbb{N}$, $p \geqslant q$. This sum is uniformly convergent because of the same uniform bound (7). $\square$

For a fixed $n \in \mathbb{N}$, there is a natural mapping from Comp_n to Ω:

$$\mu = (\mu_1,\mu_2,\dots) \;\mapsto\; \omega_\mu = \Bigl(\frac{\mu_1}{n},\frac{\mu_1}{n},\dots\Bigr).$$

We identify the set Comp of all compositions with the set

$$\widetilde{\mathbb{E}} = \bigcup_{n=1}^{\infty}\bigcup_{\mu\in\mathbb{E}_n}\Bigl(\frac{1}{n},\omega_\mu\Bigr) \subset [0,1]\times\Omega,$$

and put

$$\widetilde{\Omega} = \widetilde{\mathbb{E}}\cup(\{0\}\times\Omega).$$

For a sequence of compositions $(\mu(k))$, we say that

$$\Bigl(1/|\mu(k)|,\omega_{\mu(k)}\Bigr)\to(q,\omega)\in[0,1]\times\Omega$$

as $k\to\infty$ if and only if $1/|\mu(k)|\to q$ in $[0,1]$ and $\omega_{\mu(k)}\to\omega$ in Ω. Note that the boundary of the subset $\widetilde{\mathbb{E}}$ in $[0,1]\times\Omega$ is $\{0\}\times\Omega\cong\Omega$, and, following [5], we call Ω the *geometric boundary* of the graph $\mathbb{E}$.

Definition 2.1. *If the limit* $\lim_{k\to\infty}\mu(k)_i/|\mu(k)|$ *exists, then we call it the* ith frequency *of the sequence* $(\mu(k))$.

It turns out that harmonic functions are fully characterized by their frequencies. The following argument is standard once the density in $C(\Omega)$ is proved.

Proposition 2.4. *Let* $\left\{\left(|\boldsymbol{\mu}(k)|^{-1}, \omega_{\boldsymbol{\mu}(k)}\right)\right\}_{k=1}^{\infty}$ *be a sequence of elements of* $\widetilde{\Omega}$ *with* $\lim|\boldsymbol{\mu}(k)|=\infty$. *The following two conditions are equivalent:*

(1) *There exists* $(0,\omega)\in\widetilde{\Omega}$ *such that*

$$\left(\frac{1}{|\boldsymbol{\mu}(k)|}, \omega_{\boldsymbol{\mu}(k)}\right) \xrightarrow[k\to\infty]{} (0,\omega) \text{ in } \widetilde{\Omega}. \tag{8}$$

(2) *For each* $\mu\in\mathrm{Comp}$, *the limit*

$$\lim_{k\to\infty}\frac{\dim(\mu,\boldsymbol{\mu}(k))}{\dim\boldsymbol{\mu}(k)} \tag{9}$$

exists.

The limit in (9) *equals* $Q_\mu(\omega)$.

Proof. If the limit (8) exist, then we use Lemma 2.2 to see that

$$\lim_{k\to\infty}\frac{\dim(\mu,\boldsymbol{\mu}(k))}{\dim\boldsymbol{\mu}(k)} = Q_\mu(\omega).$$

Conversely, assume that (9) holds, and suppose that there are two subsequences in (8) with different limits $(0,\omega_1)$, $(0,\omega_2)$. We construct a function $f\in C(\Omega)$ with $f(\omega_1)\neq f(\omega_2)$ and use the density of the space spanned by $\{Q_\mu(\omega)\}$ in $C(\Omega)$ to see that $f(\omega_1)=f(\omega_2)$, a contradiction. □

Definition 2.2. *Consider the image* $\widetilde{\Delta}$ *of a* $\mathbb{Z}_+$*-graded graph* Δ *under the following mapping to* $\mathbb{R}_+^{\Delta}$:

$$B\mapsto\left(\beta\mapsto\frac{\dim(\beta,B)}{\dim B}\right),$$

where the space of functions is endowed with the topology of pointwise convergence. Let $\widetilde{E}$ *be the closure of* $\widetilde{\Delta}$, *and denote by* $E_{\mathrm{Mart}}(\Delta)$ *the corresponding boundary,* $E_{\mathrm{Mart}}(\Delta)=\widetilde{E}\backslash\widetilde{\Delta}$. *It is called the* Martin boundary *of the branching graph* Δ.

Every point $\omega\in E_{\mathrm{Mart}}(\Delta)$ of the Martin boundary corresponds to a normalized nonnegative harmonic function $K(\,\cdot\,,\omega)\colon \mu\mapsto K(\mu,\omega)$. We have $E_{\min}(\Delta)\subset E_{\mathrm{Mart}}(\Delta)$, and the following integral representation holds.

Theorem 2.5 ([1])**.** *Every normalized nonnegative harmonic function $\phi \in \mathcal{H}(\Delta)$ admits a unique integral representation*

$$\phi(\mu) = \int\limits_{E_{\min}(\Delta)} K(\mu, \omega) dP_\phi,$$

where P_ϕ is a probability measure. Conversely, every probability measure P on $E_{\min}(\Delta)$ corresponds to a normalized nonnegative harmonic function.

Theorem (Main theorem)**.** (i) $E_{\text{Mart}}(\mathbb{E}) = E_{\min}(\mathbb{E}) \cong \Omega$.

(ii) *The integral representation*

$$\phi(\mu) = \int\limits_{\Omega} Q_\mu(\omega) dP_\phi$$

gives a one-to-one correspondence between the harmonic functions $\phi \in \mathcal{H}(\mathbb{E})$ and the probability measures P_ϕ on Ω.

Proof. We see from Proposition 2.4 that $E_{\text{Mart}}(\mathbb{E}) \cong \Omega$. Moreover, the functions $Q_{\cdot}(\omega) \colon \mu \mapsto Q_\mu(\omega)$ are normalized nonnegative harmonic functions on $\mathbb{E}$. Therefore, it suffices to check that all these functions are extremal.

Assume that $Q_{\cdot}(\omega_0)$ is not extremal for $\omega_0 \in \Omega$. By Theorem 2.5, there exists a probability measure dP_{ω_0} such that

$$Q_\mu(\omega_0) = \int\limits_{E_{\min}(\mathbb{E})} Q_\mu(\omega) dP_{\omega_0}$$

for any $\mu \in \text{Comp}$. By Lemma 2.3, the linear space spanned by the polynomials $\{Q_\mu(\omega)\}$ is uniformly dense in $C(\Omega)$; therefore, the equality

$$f(\omega_0) = \int\limits_{E_{\min}(\mathbb{E})} f(\omega) dP_{\omega_0}$$

holds for any $f \in C(\Omega)$. However, it is easy to construct a nonnegative function $f_0 \in C(\Omega)$ such that $f(\omega_0) = 1$ and $f(\omega) < 1$ for $\omega \neq \omega_0$. We have

$$1 = f_0(\omega_0) = \int\limits_{E_{\min}(\mathbb{E})} f_0(\omega) dP_{\omega_0} < \int\limits_{E_{\min}(\mathbb{E})} dP_{\omega_0} = 1,$$

a contradiction.

Part (2) of the theorem follows from (2) and Theorem 2.5. $\square$

The author is deeply grateful to Anatoly Vershik and Maksim Karev for useful discussions.

References

1. J. L. Doob, *Discrete potential theory and boundaries.* — J. Math. Mech. **8** (1959), 433–458.
2. A. V. Gnedin, *The representation of composition structures.* — Ann. Probab. **25**, No. 3 (1997), 1437–1450.
3. M. V. Karev, P. P. Nikitin, *The boundary of the refined Kingman graph.* — Zap. Nauchn. Semin. POMI **468** (2018), 58–74.
4. S. V. Kerov, *Combinatorial examples in the theory of AF-algebras.* — Zap. Nauchn. Semin. LOMI **172** (1989), 55–67.
5. S. Kerov, A. Okounkov, G. Olshanski, *The boundary of the Young graph with Jack edge multiplicities.* — Int. Math. Res. Not. **4** (1998), 173–199.
6. S. Kerov, A. Vershik, *The Grothendieck group of the infinite symmetric group and symmetric functions with the elements of the K_0-functor theory of AF-algebras.* In: Representation of Lie Groups and Related Topics, Adv. Stud. Contemp. Math. **7**, Gordon and Breach, 1990, pp. 36–114.
7. J. F. C. Kingman, *Random partitions in population genetics.* — Proc. Roy. Soc. London A **361** (1978), 1–20.
8. A. M. Vershik, A. V. Malyutin, *The absolute of finitely generated groups: I. Commutative (semi)groups.* — Eur. J. Math. **4**, No. 4 (2018), 1476–1490.
9. R. P. Stanley, *Ordered Structures and Partitions*, Amer. Math. Soc., 1972.
10. R. P. Stanley, *The Fibonacci lattice.* — Fib. Quart. **13** (1975), 215–232.
11. A. M. Vershik, *Equipped graded graphs, projective limits of simplices, and their boundaries.* — Zap. Nauchn. Semin. POMI **432** (2015), 83–104.
12. A. M. Vershik, *The problem of describing central measures on the path spaces of graded graphs.* — Funct. Anal. Appl. **48**, No. 4 (2014), 256–271.
13. A. M. Vershik, *Three theorems on the uniqueness of the Plancherel measure from different viewpoints.* — Trudy Mat. Inst. Steklov **305** (2019), to appear.

St. Petersburg Department of
Steklov Institute of Mathematics;
St.Petersburg State University,
St. Petersburg, Russia
E-mail: `pnikitin0103@yahoo.co.uk`

Поступило 15 сентября 2019 г.