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A REMARK ON NILPOTENT LIE ALGEBRAS THAT DO NO ADMIT GRADINGS

Abstract. We explain why nilpotent Lie algebras usually are characteristically nilpotent, i.e., do not admit $\mathbb{Z}$-gradings.

Below we consider only Lie algebras over the complex field $\mathbb{C}$. We denote by $\operatorname{ad} x$ the adjoint operator in a Lie algebra, $\operatorname{ad} x(y) = [x, y]$.

By $\mathbb{C}^\times$ we denote the multiplicative group of $\mathbb{C}$.

Recall that a nilpotent Lie algebra $\mathfrak{g}$ is *characteristically nilpotent* if it satisfies the following equivalent conditions:

• the algebra of derivations of $\mathfrak{g}$ is nilpotent;

• $\mathfrak{g}$ does not admit a nontrivial $\mathbb{Z}$-grading;

• there are no nontrivial holomorphic actions of $\mathbb{C}^\times$ on $\mathfrak{g}$ by automorphisms;

• an infinite reductive algebraic group cannot act nontrivially by automorphisms of $\mathfrak{g}$.

• the group $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ cannot act on $\mathfrak{g}$ nontrivially by automorphisms.

All nilpotent Lie algebras of dimension $\leqslant 6$ admit $\mathbb{Z}$-gradings. Also, nilpotent Lie algebras of large dimensions that arise in mathematical literature are usually graded. However, in 1957 Dixmier and Lister [11] found an 8-dimensional example of a characteristically nilpotent algebra. After this, many works were published on constructions of characteristically nilpotent algebras, some references are [1, 16, 19, 23, 24].

1. Several remarks on nilpotent Lie algebras. In textbooks, nilpotent Lie algebras seem to be a simple topic. On the other hand, there are attractive general theorems about nilpotent Lie algebras/groups, such as Malcev's rigidity theorem for cocompact lattices [28] and Kirillov's theorem [20] on the correspondence between unitary representations and coadjoint orbits (for extensions of both theorems, see the book [10] by Corwin and Greenleaf; see also [17, 34]). However, nilpotent Lie algebras are a difficult and extremely viscous topic.

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A classification of such algebras is trivial up to dimension 5. The list in dimension 6 was obtained (with gaps) in Umlauf's thesis [40][1] in 1891. Morozov obtained a final classification in 1958; there are 20 indecomposable algebras and 10 algebras decomposable into directs sums of algebras of lower dimensions. Individual elements of Morozov's list are pleasant objects, but the whole list is not transparent.

In 1966, Safiullina, in her Ph.D. thesis[2] [36], obtained a classification of 7-dimensional algebras; the work was published only in a not easily available edition [37]. Her list contains gaps (which are unavoidable in such works); in the subsequent 30 years, several authors suggested their versions of the classification (see a complete collection of references in [15]). In 1993, Seeley [38] published a list which was considered to be correct; however, it also contained minor gaps. A new version, which is apparently final up to now, was obtained in Gong's Ph.D. thesis [15] in 1998. The list contains 6 one-parameter families, 119 indecomposable algebras, and 32 decomposable algebras. There remains a minor chance to cover the whole set by several transparent families with observable degenerations; in any case, the existing classification is hard to understand.

In 1961, Gerstenhaber [13] defined *varieties of structure constants*. Consider an n-dimensional Lie algebra $\mathfrak{g}$ with a fixed basis e_j. The Lie bracket in $\mathfrak{g}$ has the form $[e_k, e_l] = \sum_j c^j_{kl} e_j$, where the structure constants c^j_{kl} satisfy conditions of two types, $c^j_{kl} = -c^j_{lk}$ (anticommutativity) and

$$\sum_\alpha \bigl(c^\alpha_{ij} c^\beta_{\alpha k} + c^\alpha_{jk} c^\beta_{\alpha i} + c^\alpha_{ki} c^\beta_{\alpha j}\bigr) = 0$$

(the Jacobi identity). Thus, we get an algebraic variety Lie_n in the space with coordinates c^j_{kl}. Isomorphism classes of Lie algebras correspond to orbits of the general linear group $\mathrm{GL}(n, \mathbb{C})$ on Lie_n. This variety is reducible, i.e., consists of several components[3]. The first works on this topic were published by Vergne [41, 42]. A classification of components is known

[1]The author has not seen this thesis, the list was reproduced in [31].

[2]The author has not seen this thesis.

[3]A semisimple Lie algebra cannot be deformed, hence a $\mathrm{GL}(n)$-orbit of a semisimple algebra (its dimension is $n^2 - n$) is open dense in a certain component. On the other hand, for a semisimple Lie algebra the spectrum of an operator $\mathrm{ad}(x)$ is symmetric with respect to 0, and this property is preserved under degenerations. For solvable algebras, the symmetry of the spectrum of $\mathrm{ad}\, x$ usually does not hold; therefore, in general, solvable Lie algebras are not degenerations of semisimple algebras, hence they are contained in other components.

up to dimension 7, see [8,21,32]. The proof in [32] is based on simple arguments and is short. However, the components of Lie_7 consisting of solvable algebras are enumerated by nilpotent radicals, in particular, Morozov's list embeds into the list of components of Lie_7 (clearly, this phenomenon is not optimistic[4] from the point of view of extending the classification to higher dimensions).

In 1966, Vergne [41,42] initiated an investigation of components of the varieties $\mathrm{Nil}_n \subset \mathrm{Lie}_n$ of structure constants of nilpotent Lie algebras. For $n \leqslant 6$, these varieties are irreducible. For instance, for $n = 6$ all nilpotent Lie algebras are degenerations of the algebra with basis x_j, where $j = 1$, 2, 3, 4, 5, $\boxed{\cancel{6}}$, 7, and the relations

$$[x_k, x_l] = x_{k+l} \qquad \text{for } k < l.$$

According to Goze and Ancochea Bermudez [18], the space Nil_7 consists of two 40-dimensional components. Dense sets in these components form the following two families of algebras[5]: $\mathfrak{p}_\lambda$, with the relations

$$[x_1, x_i] = x_{i+1},\ 2 \leqslant i \leqslant 5, \qquad [x_1, x_6] = x_7, \qquad [x_2, x_3] = x_5, \tag{1}$$

$$[x_2, x_4] = x_6, \qquad [x_2, x_5] = \lambda x_7, \qquad [x_3, x_4] = (1-\lambda)x_7; \tag{2}$$

and $\mathfrak{q}_\lambda$, with the relations

$$[x_1, x_i] = x_{i+1},\ i = 2, 3, \boxed{\cancel{4}}, 5, \qquad [x_2, x_3] = x_5, \qquad [x_2, x_4] = x_6, \tag{3}$$

$$[x_1, x_4] = x_7, \tag{4}$$

$$[x_2, x_5] = \lambda x_7, \tag{5}$$

$$[x_1, x_6] = x_7, \qquad [x_3, x_4] = x_7, \tag{6}$$

$$[x_2, x_6] = x_7, \qquad [x_3, x_5] = -x_7. \tag{7}$$

The algebras $\mathfrak{p}_\lambda$ are graded, $\deg x_j = j$; the algebras $\mathfrak{q}_\lambda$ are characteristically nilpotent.

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[4]By [32], the maximum dimension of components of Lie_7 is $\frac{2}{27}n^3 + O(n^{8/3})$, the minimum dimension is $\frac{3}{4}n^2 + O(n)$. The growth of the number of components is at least exponential, see [9]. As far as I know, there are no reasonable upper estimates for this number, Bézout's theorem gives the trivial estimate $\leqslant 2^{n^4}$.

[5]We use the realizations due to Seeley [38]; in the numeration of Gong [15], they are algebras (123457I) and (12457N).

The classification of the components of Nil_8 remains unknown; certainly, this problem is solvable using a computer. However, we do not have effective tools for separating components in larger dimensions.[6]

It seems that we know almost nothing about generic nilpotent Lie algebras of large dimensions (say, $\geqslant 12$) except few general theorems.

There is another observation showing the degree of our incomprehension. By the Ado theorem, any n-dimensional Lie algebra $\mathfrak{g}$ can be realized as an algebra of matrices of a certain size $N = N(\mathfrak{g})$. The known proofs (for a simple proof, see [33]) give terrible upper estimates on $N(\mathfrak{g})$, which seem to be superexponential in $\dim\mathfrak{g}$; Burde showed that the estimates are exponential, see [4]. Problems with upper bounds are caused by nilpotent Lie algebras (passing to non-nilpotent Lie algebras causes a minor increase in the dimension[7]). However, it is difficult even to construct examples with $N - n \geqslant 2$, see [5, 6, 30]. The presence of a grading simplifies the construction of faithful representations[8]; therefore, it is reasonable to search examples with large $N(\mathfrak{g})$ among the characteristically nilpotent Lie algebras.

2. Two lemmas. Let $\mathfrak{g}$ be a nilpotent Lie algebra. Denote by

$$\mathfrak{g}^{[1]} = \mathfrak{g}, \quad \mathfrak{g}^{[2]} = [\mathfrak{g}, \mathfrak{g}^{[1]}], \quad \mathfrak{g}^{[3]} = [\mathfrak{g}, \mathfrak{g}^{[2]}], \dots$$

[6]Shafarevich formulated the following question. Consider the variety Com_n of structure constants of commutative n-dimensional algebras. It has a distinguished component $\mathrm{Com}_n^\bullet$, where the algebras isomorphic to $\mathbb{C} \oplus \dots \oplus \mathbb{C}$ form an open dense set. It is clear that there are many components of larger dimensions (for details on varieties of structure constants of commutative and associative algebras, see [26, 27]; see also [39]). However, is it possible to give an explicit example of an algebra that is not contained in $\mathrm{Com}_n^\bullet$ (without such words as "consider N algebraically independent numbers")? Arguments based on semi-continuity properties of algebras do not work here. For nilpotent Lie algebras, the problem has a similar character.

[7]Apparently, it reduces the dimension.

[8]Let $\mathfrak{g}$ be $\mathbb{Z}$-graded, $\mathfrak{g} = \oplus\mathfrak{g}_j$. Then we add a generator h to $\mathfrak{g}$ and consider a new Lie algebra $\widehat{\mathfrak{g}}$, with the relations $[h, x] = jx$ for all $x \in \mathfrak{g}_j$. The kernel of the adjoint representation of $\widehat{\mathfrak{g}}$ is contained in the subalgebra $\mathfrak{g}_0$. If $\mathfrak{g}_0 = 0$, then we get a faithful representation of $\widehat{\mathfrak{g}}$ (and hence of $\mathfrak{g}$).
If $\mathfrak{g} = \oplus_{j\geqslant 0}\mathfrak{g}_j$, then $\mathfrak{g}_+ := \oplus_{j>0}\mathfrak{g}_j$ is an ideal and $\mathfrak{g}_0 = \mathfrak{g}/\mathfrak{g}_+$. So, we can consider the direct sum of the adjoint representation of $\widehat{\mathfrak{g}}$ and a faithful representation of $\mathfrak{g}_0$. Thus, the problem is reduced to an algebra of smaller dimension.
However, if for some m both subspaces $\mathfrak{g}_m$ and $\mathfrak{g}_{-m}$ are nonzero, then such a reduction does not work.

its *lower central series*. Denote by $\mathfrak{z}(\mathfrak{g})$ the center of $\mathfrak{g}$. Let $V \subset \mathfrak{g}$ be an arbitrary subspace complementary to $\mathfrak{g}^{[2]}$. Then V generates $\mathfrak{g}$.

Consider a nilpotent Lie algebra $\mathfrak{g}$. Let $\mathrm{Aut}(\mathfrak{g})$ be its group of automorphisms,[9] and let $\mathfrak{D}(\mathfrak{g})$ be the Lie algebra of all derivations of $\mathfrak{g}$; recall that $\mathfrak{D}(\mathfrak{g})$ is the Lie algebra of $\mathrm{Aut}(\mathfrak{g})$. The group $\mathrm{Aut}(\mathfrak{g})$ is an algebraic group, therefore (see, e.g., [43, Chap. 6, Subsec. 1]), $\mathrm{Aut}(\mathfrak{g})$ is a semidirect product of a reductive subgroup $\mathcal{L}(\mathfrak{g})$ and a (connected) normal unipotent subgroup $\mathcal{N}(\mathfrak{g})$:

$$\mathrm{Aut}(\mathfrak{g}) = \mathcal{L}(\mathfrak{g}) \ltimes \mathcal{N}(\mathfrak{g}).$$

Similarly, $\mathfrak{D}(\mathfrak{g})$ is an algebraic Lie algebra, and it is a semidirect product of a reductive subalgebra $\mathfrak{L}(\mathfrak{g})$ and the nilpotent radical $\mathfrak{N}(\mathfrak{g})$:

$$\mathfrak{D}(\mathfrak{g}) = \mathfrak{L}(\mathfrak{g}) \ltimes \mathfrak{N}(\mathfrak{g}).$$

Recall that $\mathfrak{L}(\mathfrak{g})$ is the product of a semisimple algebra and an Abelian algebra, the Abelian Lie algebra acting in $\mathfrak{g}$ by diagonalizable operators[10]. The representation of $\mathcal{L}(\mathfrak{g})$ in $\mathfrak{g}$ is semisimple, i.e., it is a direct sum of irreducible representations.

The group $\mathcal{L}(\mathfrak{g})$ regarded as an abstract group is canonically defined as the quotient $\mathrm{Aut}(\mathfrak{g})/\mathcal{N}(\mathfrak{g})$. This group regarded as a subgroup $\mathcal{L}(\mathfrak{g}) \subset \mathrm{Aut}(\mathfrak{g})$ is defined up to conjugations by elements of $\mathcal{N}(\mathfrak{g})$. Moreover, for any reductive subgroup $H \subset \mathrm{Aut}(\mathfrak{g})$ there is an element $q \in \mathcal{N}(\mathfrak{g})$ such that $qHq^{-1} \subset \mathcal{L}(\mathfrak{g})$.

Lemma 1. *Let $\mathfrak{g}$ be a nilpotent Lie algebra, and let $I \subset [\mathfrak{g}, \mathfrak{g}]$ be a characteristic ideal*[11] *of $\mathfrak{g}$. Then there is a canonical (up to conjugation) embedding from $\mathcal{L}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})/\mathcal{N}(\mathfrak{g})$ to $\mathcal{L}(\mathfrak{g}/I) = \mathrm{Aut}(\mathfrak{g}/I)/\mathcal{N}(\mathfrak{g}/I)$.*

Proof. The group $\mathrm{Aut}(\mathfrak{g})$ leaves I invariant and, therefore, acts on $\mathfrak{g}/I$; hence we have a homomorphism $\mathrm{Aut}(\mathfrak{g}) \to \mathrm{Aut}(\mathfrak{g}/I)$. We must show that the induced homomorphism $\iota : \mathcal{L}(\mathfrak{g}) \to \mathcal{L}(\mathfrak{g}/I)$ is a monomorphism. Fix an embedding $\mathcal{L}(\mathfrak{g}) \to \mathrm{Aut}(\mathfrak{g})$. Since the action of $\mathcal{L}(\mathfrak{g})$ is semisimple, we can choose an $\mathcal{L}(\mathfrak{g})$-invariant complement W to I in $\mathfrak{g}$ and an $\mathcal{L}(\mathfrak{g})$-invariant complement V to $W \cap [\mathfrak{g}, \mathfrak{g}]$ in W. Assume that for some $q \in \mathcal{L}(\mathfrak{g})$, its

[9] By [3], any algebraic group can be realized as the group of automorphisms of a nilpotent Lie algebra.

[10] We emphasize that for algebraic groups and algebraic Lie algebras, these statements are stronger than the Levi–Malcev theorem for general Lie algebras.

[11] An ideal is said to be *characteristic* if it is invariant with respect to all automorphisms of $\mathfrak{g}$. A description of all characteristic ideals can be a nontrivial problem. In any case, the terms of the lower and upper central series are characteristic.

image $\iota(q)$ is trivial. Then q acts trivially on W and, therefore, on V. But V generates $\mathfrak{g}$. Hence q is trivial. $\square$

A holomorphic action ρ of $\mathbb{C}^\times$ on $\mathfrak{g}$ by automorphisms determines a $\mathbb{Z}$-grading of $\mathfrak{g}$. Namely, we define a homogeneous subspace $\mathfrak{g}_n \subset \mathfrak{g}$ as the eigenspace such that

$$\rho(z)v = z^n v;$$

then $[\mathfrak{g}_n, \mathfrak{g}_m] \subset \mathfrak{g}_{n+m}$. Conversely, a $\mathbb{Z}$-grading determines an action of $\mathbb{C}^\times$.

In a similar way, an action of a finite cyclic group $\mathbb{Z}_n$ on $\mathfrak{g}$ determines a $\mathbb{Z}_n$-grading of $\mathfrak{g}_n$.

Lemma 1 means that for a sequence of characteristic ideals

$$I_1 \supset I_2 \supset I_3 \supset \dots,$$

we have the decreasing chain

$$\mathcal{L}(\mathfrak{g}/I_1) \supset \mathcal{L}(\mathfrak{g}/I_2) \supset \mathcal{L}(\mathfrak{g}/I_3) \supset \dots \supset \mathcal{L}(\mathfrak{g}).$$

The next lemma gives a simple condition for this chain to be strictly decreasing.

Lemma 2. *Let $\mathfrak{g}$ be a nilpotent Lie algebra. Assume that*

- $\mathfrak{z}(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$;
- *the map $\iota : \mathcal{L}(\mathfrak{g}) \to \mathcal{L}(\mathfrak{g}/\mathfrak{z}(\mathfrak{g}))$ is an isomorphism.*

Let I be a subspace in $\mathfrak{z}$ such that the center of $\mathfrak{g}/I$ coincides with $\mathfrak{z}(\mathfrak{g})/I$. Let $\sigma \in \mathrm{Aut}(\mathfrak{g})$ be a semisimple element and σ° be its image in $\mathrm{Aut}\big(\mathfrak{g}/\mathfrak{z}(\mathfrak{g})\big)$. Then σ° can be extended to an element of $\mathrm{Aut}(\mathfrak{g}/I)$ if and only if I is invariant with respect to σ.

Proof. Choose a σ-invariant complement W to $\mathfrak{z}$ in $\mathfrak{g}$. Let τ be an automorphism of $\mathfrak{g}/I$ extending σ°; we may assume that

$$\tau(g) = \sigma(g)\Big|_W.$$

Take a complement S to I in $\mathfrak{z}(\mathfrak{g})$. Then we have a natural bijection between $W \oplus S$ and $\mathfrak{g}/I$; the algebra $\mathfrak{g}$ is the direct sum of subspaces $W \oplus S \oplus I$. We extend τ to a bijective linear map $\widetilde{\tau} : \mathfrak{g} \to \mathfrak{g}$ that coincides with τ on $W \oplus S$ and sends I to itself. This map satisfies the condition

$$[\widetilde{\tau}(x), \widetilde{\tau}(y)] - \widetilde{\tau}([x, y]) \in I. \tag{8}$$

For $w \in W$, we have

$$\widetilde{\tau}(w) = \sigma(w).$$

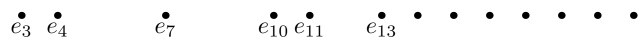

Figure 1. The basis elements of the subalgebra of the algebra of formal vector fields generated by e_3 and e_4.

Therefore, for w_1, $w_2 \in W$,

$$[\widetilde{\tau}(w_1), \widetilde{\tau}(w_2)] = [\sigma(w_1), \sigma(w_2)].$$

We have $[W, W] = [\mathfrak{g}, \mathfrak{g}]$. Keeping in mind (8), we see that for all $q \in [\mathfrak{g}, \mathfrak{g}]$,

$$\widetilde{\tau}(q) - \sigma(q) \in I.$$

On the other hand, $\widetilde{\tau}(q) = \sigma(q)$ on W and $\mathfrak{g} = W + [W, W]$. Therefore, $\widetilde{\tau}(q) - \sigma(q) \in I$ for all $q \in \mathfrak{g}$. However, $\widetilde{\tau}(q)$ sends I to itself; therefore, this holds for $\sigma(q)$. □

In the remaining part of the note, we explain the process of gradings dying as an algebra grows. In Sec. 3, we consider two examples that do not require calculations. In Sec. 4, we discuss the two families of 7-dimensional Lie algebras mentioned above.

3. Two examples. Consider a Lie algebra $\mathfrak{g}$ with basis x_α. Consider the linear subspace spanned by elements x_{i_1}, x_{i_2}, ... If this subspace is an ideal, then we denote it by $\mathfrak{J}\{x_{i_1}, x_{i_2}, \dots\}$.

Example 1. Truncated Lie algebras of formal vector fields. Consider the Lie algebra with basis e_k and the commutation relations

$$[e_k, e_l] = (l - k)e_{l+k}. \tag{9}$$

For k ranging over $\mathbb{Z}$, these relations define the Lie algebra of vector fields on the circle. For k ranging over $\mathbb{Z}_+$, this algebra can be regarded as the algebra $\mathfrak{V}$ of formal vector fields on the line[12], $e_k := x^{k+1}\frac{d}{dx}$. We take the finite-dimensional quotient $\mathfrak{V}(N)$ of $\mathfrak{V}$ by the ideal $\mathfrak{J}\{e_{N+1}, e_{N+2}, \dots\}$; in this way we get the algebra with basis e_k where $k = 0, 1, \dots, N$ and the same relations (9).

[12]Subalgebras of $\mathfrak{L}$ and their quotients were the topic of numerous works, see, e.g., [12, 14, 22, 29].

Choose $\alpha \geqslant 2$. We take the subalgebra $\mathfrak{g}$ of $\mathfrak{V}(N)$ generated by e_α, $e_{\alpha+1}$. In particular, it contains all e_k with $k > \alpha(\alpha+1)$. Note that

$$(\operatorname{ad} e_{\alpha+1})^{[N/(\alpha+1)]} = 0,$$

where $[\dots]$ denotes the integer part. If N is sufficiently large, then

$$(\operatorname{ad} e_\alpha)^{[N/(\alpha+1)]} e_{\alpha+1} \neq 0.$$

This implies that the ideal $\mathcal{J}\{e_{\alpha+1}, \dots, e_N\}$ is characteristic. Therefore, all ideals $\mathcal{J}\{e_\beta, \dots, e_N\}$ are characteristic[13]. Hence any automorphism of $\mathfrak{g}$ is a triangular matrix in the basis e_j. This easily implies that the grading $\deg e_j = j$ of $\mathfrak{g}$ is unique (up to a unipotent change of coordinates).

The center $\mathfrak{z} = \mathfrak{z}(\mathfrak{g})$ is $\mathcal{J}\{e_{N-\alpha+1}, \dots, e_N\}$. Consider a line $I = \mathbb{C} \cdot u$ in $\mathfrak{z}$ in general position. More precisely, we need the following condition:

$$u = \sum_{j=N-\alpha+1}^{N} t_j e_j \quad \text{where at least two } t_j \text{ are nonzero.}$$

The center of $\mathfrak{g}/I$ is generated by $\mathfrak{z}/I$ and the elements

$$y = \sum_{j=N-2\alpha+1}^{N-\alpha} s_j e_j \quad \text{such that } [e_\alpha, y] \in I,\ [e_{\alpha+1}, y] \in I.$$

The first condition gives $s_i = \sigma t_{i+\alpha}/(i-\alpha)$, where σ is a common factor. Therefore,

$$0 = [x_{\alpha+1}, y] = \sigma \sum_{j=N-\alpha+2}^{N} t_{j-1} \cdot \frac{j-2\alpha}{j-2\alpha+1} e_j.$$

At least one of the coefficients in the sum is not zero, hence $\sigma = 0$.

By Lemma 2, the quotient $\mathfrak{g}/I$ is characteristically nilpotent.

Next, we can take for $\mathfrak{g}$ the algebra with the basis $e_\alpha, \dots, e_N$ and the same relations (9). We can repeat the same considerations; to this end, we must take N such that all integer parts $[N/\alpha]$, $[N/(\alpha+1)]$, …, $[N/2\alpha]$ are pairwise different. Under this condition, all ideals $\mathcal{J}\{e_\beta, \dots, e_N\}$ are

[13]For an element e_γ, we choose a Lie monomial $S_\gamma(x_1, \dots, x_p; y_1, \dots, y_q)$ linear with respect to each argument such that $S_\gamma(e_\alpha, \dots, e_\alpha; e_{\alpha+1}, \dots, e_{\alpha+1}) = e_\gamma$. Consider the ideal I_{S_γ} spanned by all elements $S_\gamma(e_{\mu_1}, \dots e_{\mu_p}; e_{\nu_1}, \dots e_{\nu_q})$ such that μ_i are arbitrary and $\nu_j \geqslant \alpha+1$. By construction, I_{S_γ} is characteristic. Therefore, $\sum_{\gamma \geqslant \beta} I_{S_\gamma}$ is characteristic.

characteristic. Note that for any k, for sufficiently large N, all algebras $\mathfrak{g}^{[2]}, \ldots, \mathfrak{g}^{[k]}$ are characteristically nilpotent (cf. [23]).

Example 2. Modified relatively free nilpotent algebras. Consider the free Lie algebra $\mathfrak{f} = \mathfrak{f}[\varkappa]$ with generators $x_1, \ldots, x_\varkappa$. It is a graded Lie algebra,

$$\mathfrak{f}[\varkappa] = \bigoplus_{j \geqslant 0} \mathfrak{f}[\varkappa]_j, \tag{10}$$

where the summand $\mathfrak{f}[\varkappa]_j$ is spanned by all commutators of length j. Below, $\varkappa$ is fixed and we exclude it from the notation.

Each subspace $\mathfrak{f}_j$ is the direct sum of weight subspaces $\mathfrak{f}_{m_1,\ldots,m_\varkappa}$ consisting of the Lie polynomials of degree[14] m_i with respect to x_i. By the classical Witt formula (see, e.g., [25]),

$$\dim \mathfrak{f}_{m_1,\ldots,m_\varkappa} = \sum_{d \mid m_1,\ldots,m_\varkappa} \mu(d) \frac{(j/d)!}{(m_1/d)!\ldots m_\varkappa/d)!},$$

where $\mu(d)$ is the Möbius function.

Consider the quotient $\mathfrak{f}^N := \mathfrak{f}/\mathfrak{f}^{[N+1]}$. In other words, we assume that all commutators of length $\geqslant N+1$ are zero; equivalently, in (10) we leave the terms with $0 < j \leqslant N$. The center $\mathfrak{z}$ of our algebra is the last summand $(\mathfrak{f}^N)_N$.

The group $\mathrm{Aut}(\mathfrak{f}^N)$ is generated by the group $\mathrm{GL}(\varkappa) \simeq \mathfrak{L}(\mathfrak{f}^N)$ acting on the generators by linear transformations

$$x_m \mapsto \sum a_{ml} x_l$$

and the unipotent subgroup $\mathcal{N}(\mathfrak{f}^N)$ consisting of transformations of the form

$$x_i \mapsto x_i + \psi_i(x)$$

where $\psi_i(x)$ are linear combinations of Lie monomials of degree $\geqslant 2$. The Witt formula gives us the character of the representation of $\mathrm{GL}(\varkappa)$ in the homogeneous subspaces $(\mathfrak{f}^N)_j$.

Now let N be as large as we need. We intend to break the grading in the relatively free algebra in two steps.

[14]Thus, we have a $\mathbb{Z}^\varkappa$-grading on $\mathfrak{f}$ by the number m_j of entries of each x_j in a monomial.

1°. Consider a line $I = \mathbb{C} \cdot u \subset \mathfrak{f}^N$ in general position. We wish to apply Lemma 2. First, suppose that the center of $\mathfrak{f}^N/I$ is larger than $\mathfrak{z}/I$. Then there is an element y of degree $N-1$ such that $[y, x_j] \in I$. However, the elements $[y, x_j]$ are linearly independent[15].

According to Andreev and Popov [2], irreducible faithful representations of a given semisimple group on a space V for which the stabilizers of lines in general position are nontrivial have bounded dimensions. We apply this statement to the action of the group $\mathrm{SL}(\varkappa) \subset \mathrm{GL}(\varkappa)$ in $(\mathfrak{f}^N)_N$. In fact, in $(\mathfrak{f}^N)_N$ we have a faithful action of the quotient $\mathrm{SL}(\varkappa)/\mathbb{Z}_{\gcd(N,\varkappa)}$, where $\mathbb{Z}_{\gcd(N,\varkappa)}$ is a subgroup in the center $\mathbb{Z}_\varkappa$ of $\mathrm{SL}(\varkappa)$. Second, the Witt formula shows that for large N the representation of $\mathrm{SL}(\varkappa)$ in $\mathfrak{z}$ has large weights and, therefore, contains subrepresentations of large dimensions. Thus, we observe that $\mathrm{SL}(\varkappa) \cap \mathfrak{L}(\mathfrak{f}/I)$ is exhausted by the center of $\mathrm{SL}(\varkappa)$.

However, passing to the quotient $\mathfrak{f}^N/I$ does not break the standard $\mathbb{Z}$-grading; therefore, $\mathcal{L}(\mathfrak{f}^N/I)$ is the subgroup of scalar matrices in $\mathrm{GL}(\varkappa)$, i.e., $\mathbb{C}^\times$.

2°. Next, consider the larger relatively free algebra $\mathfrak{f}^{N+1}$. Let $u \in (\mathfrak{f}^{N+1})_N$ be the same as above. Denote by J the subspace in $(\mathfrak{f}^{N+1})_{N+1}$ spanned by $[x_1, u]$, …, $[x_\varkappa, u]$. Consider the Lie algebra

$$\mathfrak{g} := \mathfrak{f}^{N+1}/J.$$

Since the ideal J is homogeneous, the quotient is $\mathbb{Z}$-graded. The center is

$$\mathfrak{z}(\mathfrak{g}) = \mathbb{C}u \oplus (\mathfrak{f}^{N+1})_{N+1}/J.$$

The quotient $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is the Lie algebra $\mathfrak{f}^N/I$ discussed a few lines above; therefore, $\mathcal{L}(\mathfrak{g}) = \mathbb{C}^\times$.

Next, consider an element

$$w = \alpha u + \xi \in (\mathfrak{f}^{N+1})_{N+1}/J \text{ where } \alpha \neq 0 \text{ and } \xi \in (\mathfrak{f}^{N+1})_{N+1}/J \text{ is nonzero.}$$

Under this condition, the stabilizer of the line $\mathbb{C}w$ in the group $\mathbb{C}^\times$ is trivial.

It remains to check that the center of $\mathfrak{f}^{N+1}/(J + \mathbb{C} \cdot w)$ coincides with

$$\bigl(\mathbb{C}u \oplus (\mathfrak{f}^{N+1})_{N+1}/J\bigr)/\mathbb{C}w.$$

[15]Suppose that $[y, \sum a_j x_j] = 0$. Send $\sum a_j x_j$ to x_1 by an element $\tau \in \mathrm{GL}(\varkappa)$. Then $[\tau(y), x_1] = 0$. Recall that the universal enveloping algebra of $\mathfrak{f}[\varkappa]$ is the free associative algebra with $\varkappa$ generators. The equation $\tau(y)x_1 = x_1\tau(y)$ implies $\tau(y) = s \cdot x_1^{N-1}$ where $s \in \mathbb{C}$. Since $\tau(y)$ is a Lie polynomial, we have $s = 0$.

Assume that this is not the case and there exists an element h such that $[x_j, h] \in J + \mathbb{C}w$. We represent h in the form $h = h_{N-1} + h_N + h_{N+1}$ where $h_j \in (\mathfrak{f}^N)_j$. Passing to the quotients with respect to $(\mathfrak{f}^N)_N$, we get $[x_j, h_{N-1}] \in \mathbb{C}u$. This implies that $h_{N-1} = 0$. Also, we get $[x_j, h_N] \in J$. Therefore, $h_N \in \mathbb{C}u$ and $h = \beta u + h_N$ is contained in the center of $\mathfrak{f}^{N_1}/J$.

Therefore, the group $\mathcal{L}(\mathfrak{f}^{N+1}/J + \mathbb{C}w)$ is trivial, and the algebra $\mathfrak{g}/K$ does not admit a grading.

4. Two examples: 7-dimensional algebras.

Preliminaries. Central extensions and cohomology (see, e.g., [12]). Consider a Lie algebra $\mathfrak{g}$. Recall that an m-dimensional central extension of $\mathfrak{g}$ is a linear space $\mathfrak{g} \oplus \mathbb{C}^m$ with commutator $[\cdot,\cdot]^\sim$ defined by a formula of the form

$$\bigl[x \oplus t, y \oplus s\bigr]^\sim = [x, y] \oplus c(x, y)$$

where $c : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}^m$ is a skew-symmetric bilinear map. This formula determines a structure of a Lie algebra on $\mathfrak{g} \oplus \mathbb{C}^m$ if and only if the map c satisfies the following identity:

$$c\bigl([x, y], z\bigr) + c\bigl([y, z], x\bigr) + c\bigl([z, x], y\bigr) = 0. \tag{11}$$

Such a Lie algebra $\widetilde{\mathfrak{g}}$ is called an m-dimensional *central extension* of $\mathfrak{g}$.

Let $\gamma : \mathfrak{g} \to \mathbb{C}^m$ be a linear operator. A transformation of the form

$$c(x, y) \mapsto c(x, y) + \gamma([x, y])$$

does not change the resulting Lie algebra, this operation corresponds to the change of coordinates $x \oplus s \mapsto x \oplus (s + \gamma(x))$ in $\widetilde{\mathfrak{g}}$.

Clearly, the description of central extensions of $\mathfrak{g}$ reduces to one-dimensional extensions. The group of cochains $C^2(\mathfrak{g}, \mathbb{C})$ consists of the skew-symmetric maps $\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$. The group $Z^2(\mathfrak{g}, \mathbb{C})$ consisting of the cochains satisfying (11) is called the *group of cocycles*; the group $B^2(\mathfrak{g}, \mathbb{C})$ of the maps $c(x, y) = \gamma([x, y])$ where γ is a linear functional is called the *group of coboundaries*. The one-dimensional central extensions of $\mathfrak{g}$ are enumerated by the elements of the *cohomology group*

$$H^2(\mathfrak{g}, \mathbb{C}) := Z^2(\mathfrak{g}, \mathbb{C})/B^2(\mathfrak{g}, \mathbb{C}).$$

The Lie algebras $\mathfrak{q}_\lambda$. Consider the family of Lie algebras $\mathfrak{q} := \mathfrak{q}_\lambda$ defined by (3)–(7). Denote by I_k the ideal in $\mathfrak{q}$ spanned by $\{x_k, x_{k+1}, \dots, x_7\}$. The ideals I_3, I_4, I_6, I_7 are contained in the lower central series and are

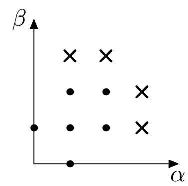

Figure 2. The algebra $\mathfrak{q}_\lambda/I_7$. The basis elements $e_{\alpha,\beta}$ (boldface points) and the nonzero homology groups $H^2_{p,q}(\mathfrak{q}_\lambda/I_7,\mathbb{C})$ (crosses $\times$).

characteristic ideals. We have

$$\mathfrak{L}(\mathfrak{g}/I_3)\simeq\mathfrak{gl}(2),\ \mathfrak{L}(\mathfrak{g}/I_4)\simeq\mathfrak{gl}(2),\ \mathfrak{L}(\mathfrak{g}/I_6)\simeq\mathfrak{gl}(2),$$
$$\mathfrak{L}(\mathfrak{g}/I_7)\simeq\mathbb{C}\oplus\mathbb{C},\ \mathfrak{L}(\mathfrak{q})=0.$$

Let us examine the last step. The Lie algebra $\mathfrak{q}/I_7$ has the commutation relations

$$[x_1,x_i]=x_{i+1},\ i=2,3,5,\qquad [x_2,x_3]=x_5,\qquad [x_2,x_4]=x_6.$$

It is $\mathbb{Z}\oplus\mathbb{Z}$-graded, the grading being determined by the following renumeration of the basis:

$$x_1=e_{1,0},\quad x_2=e_{0,1},\quad x_3=e_{1,1},\quad x_4=e_{2,1},\quad x_5=e_{1,2},\quad x_6=e_{2,2}.$$

Since the Lie algebra $\mathfrak{q}/I_7$ is $\mathbb{Z}\oplus\mathbb{Z}$-graded, the cohomology group $H^2(\mathfrak{q}/I_7,\mathbb{C})$ is also graded:

$$H^2(\mathfrak{q}/I_7,\mathbb{C})=\oplus_{(p,q)\in\mathbb{Z}\oplus\mathbb{Z}}H^2_{p,q}(\mathfrak{q}/I_7,\mathbb{C});$$

the group $H^2_{p,q}(\mathfrak{q}/I_7,\mathbb{C})$ is generated by the cocycles $c_{p,q}(\cdot,\cdot)$ such that

$$\alpha+\alpha'\neq p \text{ or } \beta+\beta'\neq q\qquad\Rightarrow\qquad c_{p,q}(e_{\alpha,\beta},e_{\alpha',\beta'})=0.$$

The number of possible nonzero cocycles $c_{p,q}(e_{\alpha,\beta},e_{\alpha',\beta'})$ is small (zero, one, or two), and we easily see that the groups $H^2_{1,3}$, $H^2_{3,1}$, $H^2_{2,3}$, $H^2_{3,2}$ are one-dimensional and the remaining groups are trivial. So, let us add to the algebra $\mathfrak{q}/I_7$ central basis elements $\xi_{1,3}$, $\xi_{3,1}$, $\xi_{2,3}$, and $\xi_{3,2}$ satisfying the following relations:

$$\begin{aligned} &[x_1, x_4] = \xi_{3,1}, \\ &[x_2, x_5] = \xi_{1,3}, \\ &[x_1, x_6] = \xi_{3,2}, \qquad [x_3, x_4] = \xi_{3,2}, \\ &[x_2, x_6] = \xi_{2,3}, \qquad [x_3, x_5] = -\xi_{2,3}. \end{aligned}$$

We get a four-dimensional central extension $\widetilde{\mathfrak{q}}$ of $\mathfrak{q}/I_7$. The Lie algebra $\mathfrak{q}$ is the quotient of this central extension by a three-dimensional subspace in general position in the center. In fact, we set

$$\xi_{1,3} = \lambda^{-1}\xi_{3,1} = \xi_{2,3} = \xi_{3,2}.$$

Example 2 (the algebras $\mathfrak{p}_\lambda$)**.** Consider the algebras $\mathfrak{p} := \mathfrak{p}_\lambda$ defined by (1)–(2). Let $I_k \subset \mathfrak{g}$ be the ideal spanned by $\{x_k, \dots, x_7\}$. The upper central series[16] of $\mathfrak{p}$ is

$$\mathfrak{p} \supset I_2 \supset I_3 \supset I_4 \supset I_5 \supset I_6 \supset I_7.$$

Therefore, all these ideals are characteristic. We have

$$\mathfrak{L}(\mathfrak{p}/I_2) = \mathfrak{L}(\mathfrak{p}/I_3) = \mathfrak{gl}(2), \quad \mathfrak{L}(\mathfrak{p}/I_4) = \mathbb{C} \oplus \mathbb{C},$$
$$\mathfrak{L}(\mathfrak{p}/I_5) = \mathfrak{L}(\mathfrak{p}/I_6) = \mathfrak{L}(\mathfrak{p}/I_7) = \mathfrak{L}(\mathfrak{p}) = \mathbb{C}.$$

Consider the group $H^2(\mathfrak{p}, \mathbb{C})$. Since $\mathfrak{p}$ is $\mathbb{Z}$-graded, the group H^2 is also $\mathbb{Z}$-graded,

$$H_2(\mathfrak{p}, \mathbb{C}) = \oplus_m H^2_m(\mathfrak{p}, \mathbb{C}).$$

Define cochains v_{kl}, where $1 \leqslant k < l \leqslant 7$, by the formula

$$v_{kl}(x_\alpha, x_\beta) = \begin{cases} 1 & \text{if } \alpha = k,\ \beta = l; \\ -1 & \text{if } \alpha = l,\ \beta = k; \\ 0 & \text{otherwise.} \end{cases}$$

They form a basis in $C^2(\mathfrak{p}, \mathbb{C})$; the elements of the basis in the cochains of degree m correspond to the pairs $k < l$ such that $k + l = m$. The equations defining the group $Z^2_m(\mathfrak{p}, \mathbb{C})$ have the form

$$c([x_k, x_l], x_j) + c([x_l, x_j], x_k) + c([x_j, x_k], x_l) = 0 \tag{12}$$

[16] Let $\mathfrak{g}$ be a nilpotent Lie algebra. Consider the quotients $\mathfrak{g}_{[2]} := \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$, $\mathfrak{g}_{[3]} := \mathfrak{g}_{[2]}/\mathfrak{z}(\mathfrak{g}_{[2]})$, etc. We get a decreasing sequence of quotients of $\mathfrak{g}$ and an increasing sequence of ideals. These ideals form the so-called *upper central series*.

where

$$k < l < j, \quad k + l + j = m$$

(conditions (12) can be dependent). The groups $B^2_p(\mathfrak{p}, \mathbb{C})$ are nonzero only if $p = 3, \ldots, 7$; in these cases, they are one-dimensional and are generated by the cochains $\sigma_m([u, v])$ defined from the expansion

$$[u, v] = \sum_{j=3}^{7} \sigma_j([u, v]) x_j.$$

After these remarks, a description of $H^2_m(\mathfrak{p}, \mathbb{C})$ for $m \leqslant 8$ is clear (larger m require explicitly writing the equations). For $m = 5, 7, 8$, the group $H^2_m(\mathfrak{p}, \mathbb{C})$ is one-dimensional. Clearly, a generic one-dimensional central extension of $\mathfrak{p}$ is characteristically nilpotent.

5. Some conjectures. Finally, we formulate several conjectures.

1. For sufficiently large n there are components of the variety Lie_n consisting only of nilpotent algebras. Note that a generic algebra $\mathfrak{g}$ of such a component must satisfy the following necessary (but not sufficient) condition: each ideal $I \subset \mathfrak{g}$ of codimension 1 must be characteristically nilpotent[17].

2. Let Ω be a component of the variety Nil_n. Denote by $\xi(\Omega)$ the number $\dim \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ for a generic algebra of Ω. Fix k. Then for sufficiently large n, the condition $\xi(\Omega) \leqslant k$ implies that generic elements of Ω are characteristically nilpotent.

3 (a stronger version). For sufficiently large n, the characteristically nilpotent Lie algebras form a Zariski-dense set in each component of Nil_n.

4. For a nilpotent Lie algebra $\mathfrak{g}$, denote by $N(\mathfrak{g})$ the minimum dimension of a faithful representation of $\mathfrak{g}$. Let

$$A(n) = \max_{\mathfrak{g} \colon \dim \mathfrak{g} = n} N(\mathfrak{g}).$$

Then the growth of $A(n)$ is exponential.

[17] Indeed, let d be a derivation of I. Let us add an element e to I and consider the new algebra $\mathfrak{n}_{I,d} \supset I$ with the commutation relations $[e, x] = d(x)$ for any $x \in I$. In this way we get a family $\mathfrak{n}_{I,d}$ in Lie_n enumerated by the differentiations d; the algebra $\mathfrak{g}$ is a point of this family.

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