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N. Mnëv

MINIMAL TRIANGULATIONS OF CIRCLE BUNDLES, CIRCULAR PERMUTATIONS, AND THE BINARY CHERN COCYCLE

Abstract. We investigate a PL topology question: which circle bundles can be triangulated over a given triangulation of the base? The question gets a simple answer emphasizing the role of minimal triangulations encoded by local systems of circular permutations of vertices of the base simplices. The answer is based on an experimental fact: the classical Huntington transitivity axiom for cyclic orders can be expressed as the universal binary Chern cocycle.

§1. Introduction

1.1. Let B a PL polyhedron. We have the Weyl–Kostant correspondence between its integer cohomology classes $H^2(B;\mathbb{Z})$ and the isomorphism classes of circle bundles $S^1 \to E \xrightarrow{p} B$ over B. The class of a bundle p is its first Chern class $c_1(p) \in H^2(B;\mathbb{Z})$. A one-to-one correspondence is provided by the isomorphism

$$H^1(B;\underline{S^1}) \approx \check{H}^2(B;\mathbb{Z}) \tag{1}$$

where $H^1(B;\underline{S^1})$ is the first sheaf cohomology group of B with coefficients in the sheaf $\underline{S^1}$ of germs of S^1-valued functions on B and $\check{H}^2(B;\mathbb{Z})$ is the second Čech cohomology group of B ([2], [1, Sec. 2.1]). A circle bundle over B can be triangulated, i.e., there is a map $\boldsymbol{E} \xrightarrow{\boldsymbol{p}} \boldsymbol{B}$ of simplicial complexes and a pair of homeomorphisms b, h making the following square commute:

$$\begin{tikzcd} |\boldsymbol{E}| \arrow[r, "h"] \arrow[d, "|\boldsymbol{p}|"] & E \arrow[d, "p"] \\ |\boldsymbol{B}| \arrow[r, "b"] & B. \end{tikzcd} \tag{2}$$

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We address the following question: *if the base triangulating complex* $\boldsymbol{B}$ *is fixed, then which circle bundles* p *have triangulations over* $\boldsymbol{B}$*?* The answer is complete and a bit strange-sounding in the case where $\boldsymbol{B}$, $\boldsymbol{E}$ are semi-simplicial sets and $\boldsymbol{p}$ is a singular map of semi-simplicial sets. A singular map of finite semi-simplicial sets is a natural generalization of a map of simplicial complexes to a more flexible combinatorial category which still functorially represents PL maps by geometric realization. A semi-simplicial set has its simplices ordered. The orders create special orientations of simplices, and thus simplicial chain and cochain complexes $C^{\triangle}_{\bullet}(\boldsymbol{B};\mathbb{Z})$, $C^{\bullet}_{\triangle}(\boldsymbol{B};\mathbb{Z})$ computing the integer singular homology and cohomology of B. The answer is as follows.

Theorem 1. *A circle bundle* p *can be semi-simplicially triangulated over the base finite semi-simplicial set* $\boldsymbol{B}$ *if and only if its integer Chern class* $c_1(p) \in H^2(|\boldsymbol{B}|;\mathbb{Z})$ *can be represented by a binary simplicial cocycle in* $Z^2_{\triangle}(\boldsymbol{B};\{0,1\} \subset \mathbb{Z})$ *having values* 0 *and* 1 *on* 2*-simplices. For classical simplicial triangulations, the condition is necessary but not sufficient.*

In particular, we get an effortless & local construction of triangulated circle bundles over a triangulation of a closed surface. In this situation, any binary 2-cochain is a cocycle. When the surface is oriented, the circle bundles are classified by these Chern numbers, and we have the following theorem.

Theorem 2. *Let* $\boldsymbol{T}$ *triangulate an oriented closed surface. Then we can semi-simplicially triangulate over* $\boldsymbol{T}$ *any circle bundle with Chern number* c *such that*

$$|c| \leqslant \frac{1}{2}\#\boldsymbol{T}_2.$$

When the equality holds, the triangulation can be only semi-simplicilal, but not simplicial.

Theorem 1 sounds like a certain discrete relative of another Weil–Kostant theorem, namely, the theorem on the "prequantum bundle" [1, Sec. 2.2], saying that to a simplectic form $\omega \in \Omega^2(M)$ having integer periods on a differential manifold M there corresponds a circle bundle on M with connection form whose curvature is ω. Here, the role of a simplectic form is played by a binary simplicial cocycle, and the role of a connection is played by a certain "minimal" triangulation which can be associated to any triangulation up to choices using our "spindle contraction trick." Such a minimal triangulation has the associated Kontsevich piecewise differential connection form, providing the rational local formulas of [13]. Its curvature symplectic form integrated over the base simplices and shifted by the standard 2-coboundary $\frac{1}{2}$ is exactly the integer binary cocycle.

1.2. Theorem 1 is based on an observation, a trick, a formula, and an experimental fact emphasizing the central role of circular permutations in the subject. We will describe the plan of the paper.

First, we need to collect in Sec. 2 some stuff on semi-simplicial sets, their geometric realizations, and PL topology.

Then, in Sec. 3, we pass to the observation. The observation was central in [13]: the stalk of a triangulation of an oriented circle bundle over an ordered k-simplex can be identified with an oriented necklace whose beads are labeled by the vertices $0, 1, \ldots, k$ of the base simplex. The beads correspond to the maximal $(k+1)$-dimensional simplices in the stalk. Under this correspondence, stalks of the minimal triangulation go to circular permutations of vertices of the base. A minimally triangulated circle bundle corresponds to a local system of circular permutations of the base ordered simplices. These local systems are combinatorial sheaves on the base semi-simplicial complexes, and they have a representing (or classifying) object, the simplicial set $\boldsymbol{SC}$ of all circular permutations.

In Sec. 4, we discuss the "spindle contraction" trick in a triangulation of a circle bundle. The trick is a bundle "simple map" from [16], and in our case it reduces a triangulation of a circle bundle over a fixed simplicial base to a minimal triangulation over the same base.

In Sec. 5, we introduce the universal binary Chern cocycle formula for minimally triangulated circle bundles. It is a form of the local formula from [13].

In Sec. 6, we relate the Huntington cyclic order axioms to the local binary formula for the Chern class. The axiomatic extension condition for a cyclic order appears to be exactly a binary form of the Chern cocycle. A very small calculation unfolds the coincidence.

In Sec. 7, we assemble the proof of Theorem 1. Using the spindle contraction and the formula, we associate with any triangulation of a circle bundle over $\boldsymbol{B}$ a binary Chern 2-cocycle. This provides the "if" direction of the statement. By Huntington's axiomatics, using a binary 2-cocycle, we construct a unique minimally triangulated circle bundle having the cocycle as its Chern cocycle, completing the "only if" direction.

In Sec. 8, the proof of Theorem 2 is assembled.

1.3. It is clear that the subject fits into the context of crossed simplicial groups and generalized orders (see, for example, [4]), but we postpone this aspect to further investigations.

§2. Simplicial and semi-simplicial sets and complexes

2.1. Semi-simplicial sets with singular morphisms added were introduced in [15] under the cryptic name "ndc css" and show up in literature under random names. For example, they are called "trisps" in [9]. Acknowledging the serious historical mess in the terminology, we call them "semi-simplicial complexes," due to their equally good, and in some aspects better, behavior as compared with locally ordered classical simplicial complexes. One can imagine the category of semi-simplicial complexes as a subcategory of simplicial sets which has the best possible behavior of its *core*, the set of nondegenerate simplices relative to maps. They have all finite limits and useful colimits commuting with them. The core of limits has an expression using the Eilenberg–Zilber order product of simplices. Kan's second normal subdivison functor Sd^2 acts functorially, producing classical simplicial complexes with homeomorphic geometric realizations. Therefore, they have an associated functorial PL structure on geometric realizations in the finite case. To summarize: singular morphisms of semi-simplicial complexes can be used to combinatorially encode PL maps of PL polyhedra, for example, PL fiber bundles. The category of semi-simplicial complexes has a natural Grothendieck topology generated by coverages by nondegenerate simplices. Generally, the whole cellular sheaf theory as in [3] works similarly for semi-simplicial complexes. The site structure in the finite case is actually a generalization of the P. S. Alexandroff non-Hausdorff topology on abstract classical simplicial complexes.

2.2. We denote by Δ the category of finite linear orders $[k]=\{0,1,2,...,k\}$ and nondecreasing maps between them called operators. Injective maps are boundary operators, surjective ones are degeneracy operators.

Set-valued presheaves on Δ are simplicial sets. The category of simplicial sets is denoted by $\widehat{\Delta}$. For a simplicial set $\Delta^{\mathrm{op}} \xrightarrow{\boldsymbol{X}} \boldsymbol{\mathcal{S}}\mathbf{ets}$, elements of $\boldsymbol{X}_k$ are called k-simplices. For a boundary operator $[m] \xrightarrow{\mu} [k]$ and a simplex $x \in \boldsymbol{X}_k$, the m-simplex $\mu^*(x) \in \boldsymbol{X}_m$ is called the μth boundary of x. The same for degeneracies.

2.3. The part of the category Δ generated by all injective maps is denoted by $\underline{\Delta}$. Set-valued presheaves on $\underline{\Delta}$ are called semi-simplicial sets. They form the category $\widehat{\underline{\Delta}}$.

One can make a semi-simplicial set out of a simplicial set by forgetting all the degeneracies. This provides a functor $\widehat{\Delta} \xrightarrow{F} \widehat{\underline{\Delta}}$ having the left adjoint functor S. The theory of semi-simplicial complexes is based on the Rourke–Sanderson adjacency $S \dashv F$. The functor S freely adds degeneracies to a semi-simplicial set making it a simplicial set. Completing the image of S to a full subcategory in $\widehat{\Delta}$, we obtain a full subcategory $\tilde{\underline{\Delta}}$ of $\widehat{\Delta}$, the category which we call the *category of semi-simplicial complexes*.

We denote by $\langle m\rangle \xrightarrow{\langle\mu\rangle} \langle k\rangle$ the images of orders and operators under the Yoneda embedding. We imagine them as standard face and degeneracy maps of ordered abstract simplices. The Yoneda images of $\underline{\Delta}, \Delta$ belong to $\tilde{\underline{\Delta}}$.

The category of singular morphisms $\operatorname{Arr}\tilde{\underline{\Delta}}$ is a convenient category for triangulations of bundles by geometric realization.

§3. Triangulations and necklaces

Here we will repeat a few points from [13] in a way convenient for the current exposition.

3.1. Triangulations of circle bundles. Suppose we have a finite semi-simplicial complex $\boldsymbol{B}$ and an oriented circle bundle $S^1 \to E \to |\boldsymbol{B}|$ triangulated over $\boldsymbol{B}$, i.e., a semi-simplical complex $\boldsymbol{B}$ and a singular map $\boldsymbol{E} \xrightarrow{\boldsymbol{p}} \boldsymbol{B}$ of semi-simplicial complexes for which there exists a homeomorphism h making the following diagram commutative:

$$\begin{tikzcd} {|\boldsymbol{E}|} \arrow[rr, "h"] \arrow[dr, "|\boldsymbol{p}|"'] & & E \arrow[dl, "p"] \\ & {|\boldsymbol{B}|.} & \end{tikzcd} \tag{3}$$

The homeomorphism h creates on p a structure of a PL oriented circle bundle. Any two such homeomorphisms create fiberwise PL isomorphic structures. Moreover, over a PL polyhedral base, oriented S^1-bundles understood as principal $U(1)$-bundles or as oriented PL fiber bundles are the same thing. On the total space E, one can always choose an interior flat

Euclidean metric making all the fibers of p to be of constant perimeter (2π, or 1, or whatever makes the formulas nicer). This will miraculously turn an oriented PL S^1-bundle p into a principal U(1)-bundle p in a unique, up to a U(1)-gauge transformation, way. Therefore, if h exists, then the combinatorics of the map $\boldsymbol{p}$ determines the isomorphism class of an S^1-bundle, and hence its Chern class $c_1(p) \in H^2(B;\mathbb{Z})$ in the base, by the Weil–Kostant theorem.

3.2. Simplicial circle bundles. Picking a base k-simplex $\langle k\rangle \xrightarrow{x} \boldsymbol{B}$, we can form the stalk of $\boldsymbol{p}$ over x, which is the pullback $x^*\boldsymbol{E} \xrightarrow{x^*\boldsymbol{p}} \langle k\rangle$ and which we call an *elementary s.c. bundle* over a simplex. The bundle p is oriented. The orientation fixes a generator in the first integer simplicial homology group of the total complex $x^*\boldsymbol{E}$. Simplicial boundary transition maps between the stalks of $\boldsymbol{p}$ send generators to generators, representing the orientation as a constant local system on the base $\boldsymbol{B}$ associated with the triangulation. By a *simplicial circle bundle* (*s.c. bundle*) on a semi-simplicial complex $\boldsymbol{B}$ we mean a local system of oriented elementary s.c. bundles on $\boldsymbol{B}$ and orientation-preserving transition boundary maps. It assembles by colimit in the category $\mathrm{Arr}\,\underline{\tilde{\Delta}}$ of singular morphisms to a map $\boldsymbol{E} \to \boldsymbol{B}$ having a canonical structure of a PL triangulated S^1-bundle on the geometric realization (if $\boldsymbol{B}$ is finite) with a canonical structure of a U(1)-principal bundle. (We are in a simple situation of a stack where elementary s.c. bundles and transition boundary maps are the "descent data.")

3.3. The necklace of an elementary s.c. bundle. Now let $\boldsymbol{R} \xrightarrow{\boldsymbol{e}} \langle k\rangle$ be an elementary s.c. bundle over $\langle k\rangle$ having $n \geqslant k+1$ maximal $(k+1)$-dimensional simplices in the total complex $\boldsymbol{R}$. The semi-simplicial bundle $\boldsymbol{e}$ is determined by an oriented necklace $\mathcal{N}(\boldsymbol{e})$ whose n beads are colored by the vertices of the base simplex, i.e., the numbers $\{0,\dots,k\}$. Figure 3.3 presents a picture of an elementary s.c. bundle over the 1-simplex $\langle 1\rangle$.

To an elementary simplicial circle bundle $\boldsymbol{e}$ over $\langle k\rangle$ having n maximal $(k+1)$-dimensional simplices in the total space, we associate a $(k+1)$-necklace $\mathcal{N}(\boldsymbol{e})$, i.e., a circular word of length n in the ordered alphabet of $k+1$ letters numbered by the vertices of the base simplex. Any $(k+1)$-dimensional simplex of $\boldsymbol{R}$ has a unique edge that shrinks to a vertex i of the base simplex by the Yoneda simplicial degeneration $\langle k+1\rangle \xrightarrow{\langle\sigma_i\rangle} \langle k\rangle, i = 0,1,2,\dots,k$. Take the general fiber of the projection $|\boldsymbol{e}|$. It is a circle broken into n intervals oriented by the orientation of the bundle,

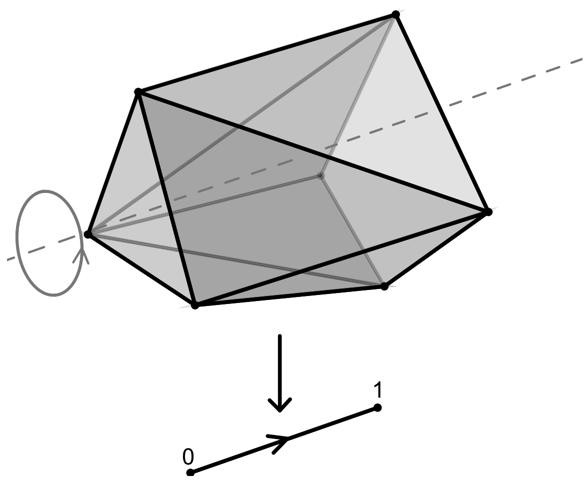

Figure 1. An elementary simplicial circle bundle over the interval.

and every interval on it is the intersection with a maximal $(k+1)$-simplex. The maximal simplex is uniquely named by the vertex of the base where its collapsing edge collapses. This creates a coloring of the n intervals by $k+1$ ordered vertices of the base simplex. Thus, we get a necklace $\mathcal{N}(\boldsymbol{e})$ out of the combinatorics of $\boldsymbol{e}$ ([13, Sec. 16]). The process is illustrated in Fig 2. This process is invertible: having an oriented necklace ϑ whose beads are colored by $[k]$, we can assemble an elementary oriented s.c. bundle $\boldsymbol{E}C(\vartheta) \xrightarrow{\boldsymbol{e}c(\vartheta)} \langle k \rangle$ as the colimit in $\underline{\tilde{\Delta}} \downarrow_{\langle k \rangle}$ (or $\hat{\Delta} \downarrow_{\langle k \rangle}$) of the Yoneda degeneracies $\langle k+1 \rangle \xrightarrow{\langle \sigma_i \rangle} \langle k \rangle$.

3.4. Local systems of oriented necklaces. Let $\boldsymbol{E} \xrightarrow{\boldsymbol{p}} \boldsymbol{B}$ be an s.c. bundle, $\langle k \rangle \xrightarrow{x} \boldsymbol{B}$ be a simplex of the base, $x^*\boldsymbol{E} \xrightarrow{x^*\boldsymbol{p}} \langle k \rangle$ be the corresponding subbundle, and $\langle k-1 \rangle \xrightarrow{d_i x} \boldsymbol{B}$ be the ith boundary of the simplex x. Then, by construction, the necklace $\mathcal{N}(d_i x)^*\boldsymbol{p}$ is obtained from the necklace $\mathcal{N}(x^*\boldsymbol{p})$ by deleting all the beads colored by i. But the face maps between elementary subbundles contain more information, since the elementary bundles and the corresponding necklaces may have combinatorial automorphisms. Therefore, they should be recorded in the descent data of the bundle. After this fix, the bundle $\boldsymbol{p}$ is encoded in the local system

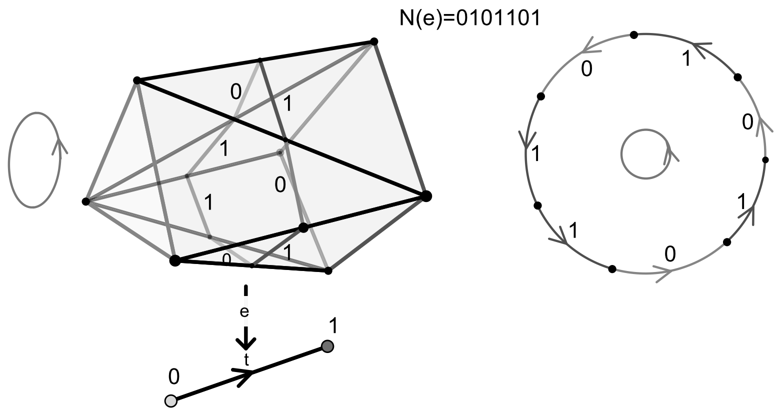

Figure 2.

$\mathcal{N}(\boldsymbol{p})$ of oriented necklaces made of ordered vertices of the base simplices, and, vice versa, a local system Θ of necklaces on the base $\boldsymbol{B}$ encodes the bundle $\boldsymbol{E}C(\Theta) \xrightarrow{\boldsymbol{e}c(\Theta)} \boldsymbol{B}$.

3.5. Classical simplicial vs. semi-simplicial triangulations. Not every oriented necklace ϑ with beads colored by $[k]$ has a classical simplicial complex as $\boldsymbol{E}C(\vartheta)$. We say that ϑ has two colors $i, j \in [k]$ "not mixed" if, after deleting from ϑ all the beads except those colored by i, j, the remaining two sorts of beads stay in two solid blocks (see the proof of [11, Lemma 0.1]).

Proposition 3. *The complex $\boldsymbol{E}C(\vartheta)$ is classically simplicial if and only if*

(i) *ϑ has at least 3 beads of each color and*

(ii) *any two of its colors are "mixed."*

Proof. A semi-simplicial complex is classically simplicial if and only if its one-dimensional skeleton is classically simplicial, i.e, is a graph without loops or multiple edges. The entire one-dimensional skeleton of $\boldsymbol{E}C(\vartheta)$ sits over the one-dimensional skeleton of the base simplex $\langle k \rangle$. Condition (i) guarantees that there are no loops or multiple edges in circles over vertices, conditon (ii) guarantees that there are no multiple edges in the total complex over the base edges (see the proof of [11, Lemma 0.1]). $\square$

If the conditions of Proposition 3 are not satisfied, then the complex $\boldsymbol{E}C(\vartheta)$ is essentially semi-simplicial, and some of its simplices can have glued vertices, or two different 1-simplices can have two vertices in common.

3.6. Circular permutations and minimal elementary s.c. bundles. We arrived at our main objects.

If an oriented necklace ϑ has a single bead of every color from $[k]$, then ϑ is a *circular permutation of* $[k]$. We denote by $\boldsymbol{S}C$ the $\mathbb{N}$-graded set of circular permutations.

Denote by $\boldsymbol{S}$ the $\mathbb{N}$-graded set with $\boldsymbol{S}_k$ being the symmetric group of all permutations of $[k]$. A circular permutation in $\boldsymbol{S}C_k$ is the same thing as the right coset of a permutation $\omega \in \boldsymbol{S}_k$ by the right action of a cyclic subgroup $\boldsymbol{C}_k$ of $\boldsymbol{S}_k$. Thus, we have a map of graded sets $\boldsymbol{S}_k \xrightarrow{\circlearrowright} \boldsymbol{S}C_k$ sending a permutation to its right cyclic coset, i.e., to a circular permutation.

Let us organize the correspondence $\circlearrowright$ in a way comparable with boundaries of necklaces from Sec. 3.4, and also add degeneracies, making $\circlearrowright$ a morpism of simplicial sets.

First, we add boundaries and degeneracies to the graded set $\boldsymbol{S}$. Define a boundary $\boldsymbol{S}_k \xrightarrow{d_i} \boldsymbol{S}_{k-1}$, $i = 0, \ldots, k$, as deleting the element $\omega(i)$ from a permutation ω followed by the monotone reordering. Thus, $\boldsymbol{S}$ becomes a semi-simplicial set.

Define a degeneracy $\boldsymbol{S}_k \xrightarrow{s_i} \boldsymbol{S}_{k+1}$, $i = 0, \ldots, k$, as inserting into a permutation a new element, with the value $\omega(i) + \frac{1}{2}$, right after $\omega(i)$ at place $\omega(i) + 1$ and reordering the values monotonically into positive integers. Now $\boldsymbol{S}$ is a simplicial set. This simplicial set of permutations $\boldsymbol{S}$ is a classical object, called the "symmetric crossed simplicial group," which was introduced independently in [6, 10] and later in [5].

The map $\circlearrowright$ induces a similarly defined simplicial structure on $\boldsymbol{S}C$ making the map $\circlearrowright$ simplicial.

Now we introduce the following definition.

- For a circular permutation $\vartheta \in \boldsymbol{S}C_k$, the elementary s.c. bundle

$$\boldsymbol{E}C(\vartheta) \xrightarrow{\boldsymbol{e}c(\vartheta)} \langle k \rangle$$

  is called *minimal*. We say that a s.c. bundle is minimal if all its stalks over simplices are minimal.

Circular words and corresponding bundles have no automorphisms. Therefore, a bundle having minimal stalks over all simplices is the same thing as a local system of circular permutations of the base simplices and its simplicial boundary maps. It is the same thing as a simplicial map $\boldsymbol{B} \to \boldsymbol{S}C$.

We arrive at the conclusion that the functor on $\underline{\tilde{\Delta}}$ assigning to a semi-simplicial set $\boldsymbol{B}$ the set of all minimally triangulated circle bundles over $\boldsymbol{B}$ is represented by $\boldsymbol{S}C$.

Actually, a minimal elementary s.c. bundle $\boldsymbol{e}c(\vartheta)$ is the stalk of the simplicial map $\boldsymbol{S} \xrightarrow{\circlearrowleft} \boldsymbol{S}C$ over the base simplex $\langle k\rangle \xrightarrow{\vartheta} \boldsymbol{S}C$. Therefore, $\circlearrowleft$ is the universal minimal s.c. bundle over $\boldsymbol{S}C$. We do not prove this fact in this paper.

3.7. The geometry of minimal elementary s.c. bundles. Minimal elementary s.c. bundles are the same thing as the twisted product projection $\boldsymbol{C} \times_\omega \langle k\rangle \to \langle k\rangle$ where $\boldsymbol{C}$ is Connes' cyclic crossed simplicial group, or "simplicial circle," and $\omega \in \boldsymbol{S}_k$ is a permutation of the base vertices.

Now we will describe elementary minimal s.c. bundles geometrically (see Figs. 3 and 4). Let $\omega \in \boldsymbol{S}_k$ be a permutation and $(\omega)_\circlearrowleft \in \boldsymbol{S}C_k$ be the corresponding circular permutation. We construct an elementary s.c. bundle

$$\boldsymbol{E}C(\omega)_\circlearrowleft \xrightarrow{\boldsymbol{e}c(\omega)_\circlearrowleft} \langle k\rangle$$

by the following algorithm. Take the geometric prism $\Delta^k \times \Delta^1 \subset \mathbb{R}^k \times \mathbb{R}^1$ and number its verities by $[k] \times [1]$. Then apply the algorithm: at step 0, take the $(k+1)$-simplex that is the convex hull of the bottom k-simplex with vertices $(0,0),\dots,(0,k)$ and the point $(\omega(0),1)$. The result will have the top k-simplex with vertices

$$(\omega(0),1),(\omega(1),0),\dots,(\omega(k),0).$$

Then iterate, building a pile of $(k+1)$-simplices. At step i, add the $(k+1)$-simplex that is the convex hull of the point $(\omega(i),1)$ and the top k-simplex in the pile already constructed. It is a very simple sort of "shelling" process in simplicial combinatorics. At step k, we will obtain a certain triangulation $\boldsymbol{E}(\omega)$ of the prism $\Delta^k \times \Delta^1$. At the last step of the construction of $\boldsymbol{E}C(\omega)_\circlearrowleft$, i.e., the bundle corresponding to the circular word $(\omega)_\circlearrowleft$, we glue together the very top and the very bottom k-simplices. This can be done only semi-simplicially. The general fiber of the projection intersects the interiors of the $(k+1)$-simplices in the circular order $(\omega)_\circlearrowleft$, and we could start from any cyclic shift of the word ω with the same result.

The most important for us are the circular permutations of $[0],[1],[2],[3]$ and the corresponding minimal elementary s.c. bundles. We have

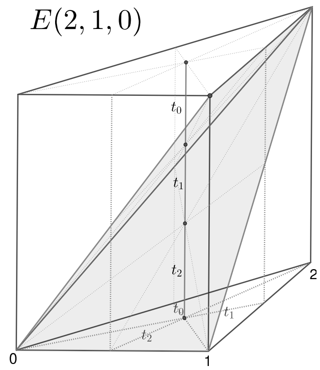

Figure 3.

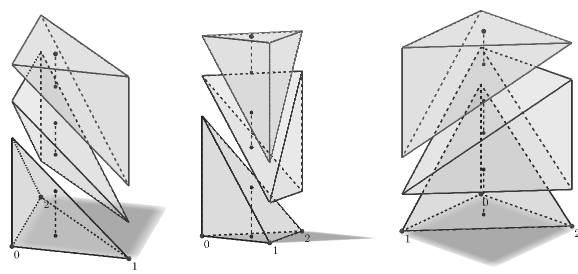

Figure 4. Different views of the triangulation $E(0,2,1)$

- a single circular permutation of one element $(0)_{\circlearrowleft}$;
- a single circular permutation of two elements $(0,1)_{\circlearrowleft}$;
- 2 circular permutations of three elements: even, $(0,1,2)_{\circlearrowleft}$, and odd, $(2,1,0)_{\circlearrowleft}$;
- 6 circular permutations of four elements.

These faces and boundaries form the skeleton $\boldsymbol{SC}(3)$ of $\boldsymbol{SC}$ depicted as a hexagram in Fig. 5.

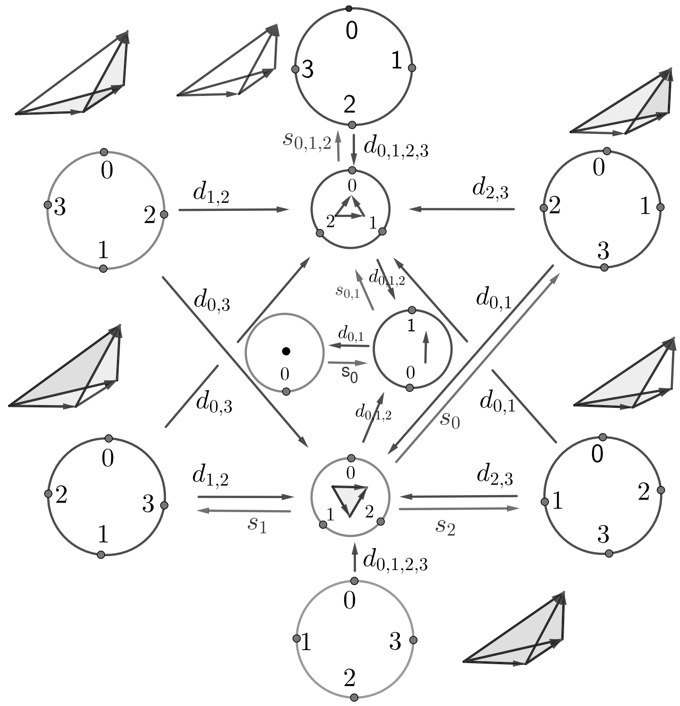

Figure 5. The skeleton $\boldsymbol{SC}(3)$ depicted as a hexagram. All the circles are oriented clockwise.

§4. The spindle contraction trick

Here we will show how, up to a free choice, one can reduce any s.c. bundle to a minimal one preserving the bundle isomorphism class of the geometric realization. In PL topology, concordant fiber bundles are isomorphic, and vice versa. We say that two s.c. bundles $\boldsymbol{p}, \boldsymbol{q}$ on $\boldsymbol{B}$ are strongly concordant if there is a s.c. bundle $\boldsymbol{h}$ on $\boldsymbol{B} \times \langle 1 \rangle$ such that its restrictions to $\boldsymbol{B} \times \langle 0 \rangle$ and $\boldsymbol{B} \times \langle 1 \rangle$ are $\boldsymbol{p}$ and $\boldsymbol{q}$. Geometric realizations of concordant bundles are isomorphic. By *circles of a s.c. bundle* $\boldsymbol{p}$ we mean its 0-stalks over vertices $\boldsymbol{B}_0$ of $\boldsymbol{B}$. A circle of $\boldsymbol{p}$ is a semi-simplicial oriented circle consisting of vertices and oriented arcs contracting to a vertex of the base by the bundle projection $\boldsymbol{p}$.

Proposition 4. *Any s.c. bundle is strongly concordant to a minimal s.c. bundle, uniquely determined by a free choice of a single arc in earch circle of the bundle.*

We will prove the proposition after introducing the spindle contraction trick.

4.1. Suppose that we have a vertex $v \in \boldsymbol{B}_0$, a circle $c(v)$ over v, and an arc $a \in c_1(v)$. Consider the star $\operatorname{st} a \to \boldsymbol{E}$ of a in $\boldsymbol{E}$ and the star $\operatorname{st}(v) \to \boldsymbol{B}$ of v. The projection $\boldsymbol{p}$ induces a subbundle, the "spindle" $\operatorname{sp}(a)$ undersood as a morpism in $\operatorname{Arr} \underline{\tilde{\Delta}}$:

$$\operatorname{sp}(a): \begin{array}{ccc} \operatorname{st}(a) & \longrightarrow & \boldsymbol{E} \\ {\scriptstyle \boldsymbol{p}'}\downarrow & & \downarrow{\scriptstyle \boldsymbol{p}} \\ \operatorname{st}(v) & \longrightarrow & \boldsymbol{B}. \end{array} \tag{4}$$

The projection of the spindle to the base can be understood as a morphism in $\operatorname{Arr} \underline{\tilde{\Delta}}$:

$$\begin{array}{ccc} \operatorname{st}(v) & \xleftarrow{\boldsymbol{p}'} & \operatorname{st}(a) \\ {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle \boldsymbol{p}'} \\ \operatorname{st}(v) & \xleftarrow{\mathrm{Id}} & \operatorname{st}(v). \end{array} : \underline{\operatorname{sp}}(a) \tag{5}$$

The "spindle contraction" $\boldsymbol{b}/a$ is the colimit in $\operatorname{Arr} \underline{\tilde{\Delta}}$ of the diagram

$$\begin{array}{ccc} p' & \xrightarrow{\operatorname{sp}(a)} & \boldsymbol{p} \\ {\scriptstyle \underline{\operatorname{sp}}(a)}\downarrow & & \downarrow \\ \mathrm{Id}_{\operatorname{st}(v)} & \dashrightarrow & \boldsymbol{p}/a. \end{array} \tag{6}$$

Figures 6, 7 illustrate spindle contractions. Figure 7 illustrates the commutation of two-dimensional spindle contractions over a one-dimensional base, but three-dimensional contractions over a two-dimesional base already do not commute.

Lemma 5. *If the circle $c(v)$ has more than one arc and $a \in c_1(v)$ is the chosen arc, then the spindle contraction $\boldsymbol{p}/a$ is a correct s.c. bundle.*

Proof. In the simplicial language, the contraction may look a bit puzzling, but it is obvious in the language of the local system of necklaces encoding

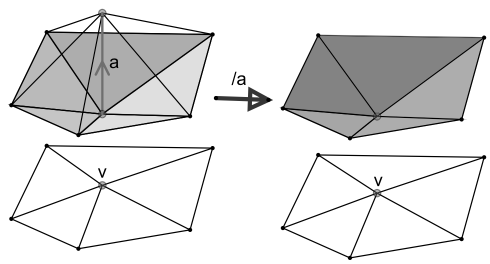

Figure 6.

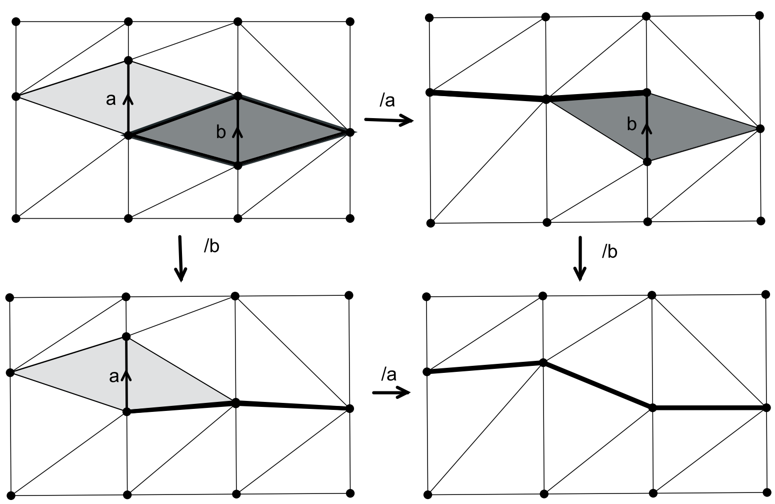

Figure 7.

the bundle $\boldsymbol{p}$. One should take the bead corresponding to the arc a and remove this bead from all the necklaces having it in the boundary. The local system will remain correct if the bead was not the last bead in the necklace $\mathcal{N}(c(v))$ monochromatically colored by v. □

Proof of Proposition 4**.** Take a s.c. bundle $\boldsymbol{E} \times \langle 1 \rangle \xrightarrow{\boldsymbol{p} \times \langle 1 \rangle} \boldsymbol{B} \times \langle 1 \rangle$. It has two copies of the bundle $\boldsymbol{p}$, labeled by 0 and 1. Take the copy labeled by 1. Pick in it the copy a_1 of the arc a over the copy v_1 of the vertex v

and perform the spindle contraction $\boldsymbol{p} \times \langle 1\rangle/a_1$. It coincides with $\boldsymbol{p}/a$ on the 1-side and does not affect the 0-side, because only the star is affected. One can do contractions in any order until some circle of the bundle has more than two arcs. Now, select the single arc in each circle of the bundle and contract all the others by spindle contractions. The resulting strongly concordant minimal s.c. bundle will be completely determined by this selection. $\square$

§5. The local binary formula for the Chern cocycle of minimal triangulations

$\mathbf{c}_{01}$ The local binary formula for the Chern cocycle of minimal s.c. bundles is a universal simpicial Chern 2-cocycle on

$$\mathbf{c}_{01} \in Z^2_{\triangle}(\boldsymbol{S}C; \{0,1\} \subset \mathbb{Z})$$

defined as the parity of the circular permutation of 3 elements:

$$\begin{aligned}\mathbf{c}_{01}(0,1,2)_{\circlearrowright} &= \mathbf{c}_{01}(\boldsymbol{e}c(0,1,2)_{\circlearrowright}) = 0,\\ \mathbf{c}_{01}(2,1,0)_{\circlearrowright} &= \mathbf{c}_{01}(\boldsymbol{e}c(2,1,0)_{\circlearrowright}) = 1.\end{aligned} \tag{7}$$

It is the rational local formula from [13] shifted by the universal 2-coboundary $\frac{1}{2}$. Alternatively, it can be directly obtained from the exponential sequence of sheaves ([1, Sec. 2.1]) using sections related to the top or bottom hats of unique spindles over the Čech (hyper)cover of the base by stars that has the initial semi-simplicial base complex as its Čech nerve. Alternatively, it can be guessed and checked, since we know a geometric triangulation of the Hopf bundle [12].

§6. The Huntington cyclic order axioms, Kan properties of $\boldsymbol{S}C$, and binary Chern cocycle

6.1. A cyclic order. A total cyclic order on a set is a way to inject the set into an oriented circle. We imagine the circle drawn on paper to be oriented clockwise. For a finite set, it is a way to complete the set up to a graph cycle by introducing oriented edges between elements meaning "the next." If the elements of a finite set are numbered by the set $[k]$, then a total cyclic order on this set is $(\omega)_{\circlearrowright} \in \boldsymbol{S}C_k$ for some permutation of its elements $\omega \in \boldsymbol{S}_k$. Abstract total cyclic order relations were introduced by the philosopher E. V. Huntington [7, 8] as one of the fundamental orders of the (Platonic) Universe; another exposition can be found in [14]. It can be defined axiomatically by a ternary relation $HC(a, b, c)$. The meaning of $HC(a,b,c) =$ "True" is that the ordered triple a,b,c sits on the circle clockwise. As Huntington put it: "The arc running from a through b to c in the direction of the arrow is less than one complete circuit." The independent axioms of a total cyclic order are as follows.

(i) Cyclicity: If $HC(a,b,c)$, then $HC(b,c,a)$.
(ii) Asymmetry: If $HC(a,b,c)$, then not $HC(c,b,a)$.
(iii) Transitivity: If $HC(a,b,c)$ and $HC(a,c,d)$, then $HC(a,b,d)$.
(iv) Totality: If a,b, and c are distinct, then either $HC(a,b,c)$ or $HC(c,b,a)$.

For a finite set ordered by $[k]$, we can read Huntington's theory of total cyclic orders as follows.

A set consisting of one or two ordered elements has a unique total cyclic order. Let $k=2$, then axioms (i), (ii), (iv) are applicable, and a Huntington total cyclic order on $[2]$ fixes one of the two cyclic permutations of 3 elements: either even, i.e., $(0,1,2)_{\circlearrowright}$ (if $HC(0,1,2)$ holds), or odd, i.e, $(2,1,0)_{\circlearrowright}$ (if $HC(2,1,0)$ holds). Thus, the two total Huntington cyclic orders on the ordered set $[2]$ fix and are fixed by the function $\mathbf{c}_{01}$ from (7). It is the key observation.

If $k=4$, then the transitivity axiom (iii) starts playing a role and gives a condition when four circular permutations of subtriples of $[3]$ fix a unique circular permutation of the entire set $[3]$.

For all $k \geqslant 4$, the theory states that a system of circular permutations of all subtriples that satisfy the transitivity condition for all subquadruples fixes a unique circular order on the entire set $[k]$.

Note that subsets of circular permutations are simplicial boundaries in $\boldsymbol{SC}$. Now we can translate the above observations into a form of the Kan extension lifting property for circular permutations over simplicial pairs $(\langle k\rangle, \partial\langle k\rangle)$ in the whole range with a single gap in dimension 3.

Proposition 6. *If $k=0,1$ or $k \geqslant 4$, then any map of the $(k-1)$-sphere $\partial\langle k\rangle \xrightarrow{\varphi} \boldsymbol{SC}$ has a unique lift to a map of $\langle k\rangle$, i.e., there exists a unique map $\langle k\rangle \xrightarrow{\tilde\varphi} \boldsymbol{SC}$ such that $\tilde\varphi|_{\partial\langle k\rangle} = \varphi$. If $k=2$, then there exist two different lifts. The dimension $k=3$ is exceptional.*

We can see that $\boldsymbol{SC} \approx K(\mathbb{Z},2)$ by a simplicial homotopy argument, and we observe that it follows directly from its Huntington local axiomatic description above. Proposition 6 states that the simplicial set $\boldsymbol{SC}$ is minimal Kan contractible in all dimensions except 2. Therefore, it has homotopy groups π_i vanishing if $i=0,1$ or $i \geqslant 3$. We need to know π_2. But, by the Hurewicz theorem, this amounts to computing the second homology of $\boldsymbol{SC}$, which is just the homology of the 2-sphere, by the inspection of the hexagram in Fig. 5.

6.2. The Chern binary cocycle and the cyclic order transitivity axiom. In Proposition 6, Huntington's axioms became translated into a simplicial homotopy of $\boldsymbol{SC}$ with a gap in dimension 3, where the transitivity axiom (iii) is not formulated topologically. We can fill the gap a bit miraculously.

Let $f \in C^2_\triangle(\partial\langle 3\rangle; \{0,1\} \subset \mathbb{Z})$ be a $\{0,1\}$-valued integer cochain on the boundary of the ordered 3-simplex $\partial\langle 3\rangle$. We can translate it into a Huntington cyclic relation f^H on ordered subtriples of the set of vertices $[3]$, which is the same thing as fixing either even or odd cyclic permutations of the set of vertices of faces, which is the same thing as a singular map $\partial\langle 3\rangle \xrightarrow{\overline{f}} \boldsymbol{SC}$, which is the same thing as a minimal s.c. bundle $\overline{f}^*$ on $\partial\langle 3\rangle$.

We know that if for the relation f^H the transitivity holds, then the cyclic orders on the triples assemble into a cyclic order on the whole set $[3]$, or, equivalently, $\overline{f}$ has an extension over $\langle 3\rangle$ and the minimal circle bundle $\overline{f}^*$ has an extension to a minimal bundle $\tilde{f}^*$ over $\langle 3\rangle$. For the bundle $\tilde{f}^*$, the cochain f is its Chern binary cocycle $\mathbf{c}_{01}(\tilde{f}^*)$ (see Sec. 5), and, therefore, the transitivity of f^H implies that

$$\sum_{i=0}^{4}(-1)^i f_i = 0. \tag{8}$$

The inverse statement is true.

Proposition 7. *If Eq.* (8) *holds, then the Huntington order f^H is transitive, or, equivalently, the cyclic orders on triples are uniquely extendable to a cyclic order on* $[3]$*, or, equivalently, the corresponding minimal bundle is uniquely extendable.*

Proof. The proof is experimental. It is a pseudoscientific check of cases while meditating over the hexagram in Fig. 5 representing the 3-skeleton $\boldsymbol{SC}(3)$. But the check is very short. In Table 1 we list all $2^4 = 16$ binary 2-cochains f on $\partial\langle 3\rangle$ (in the order of 4-positional binary numbers). They correspond to 16 minimal s.c. bundles on $\partial\langle 3\rangle$. The value

$$\mathbf{c}(f) = \sum_{i=0}^{4}(-1)^i f_i$$

is the Chern number of the bundle $\tilde{f}^*$. It can be equal to 0 (the trivial bundle), ± 1 (the Hopf bundle with opposite orientations), ± 2 (the tangent bundle to the sphere S^2 with opposite orientations). Among them, 6 are cocycles, i.e. minimal trivial bundles on $\partial\langle 3\rangle$. In parallel we list all 6 minimal elementary bundles on the entire set $\langle 3\rangle$ corresponding to the circular permutations of 4 elements. Computing their boundary bundles gives exactly all the 6 minimal trivial bundles on $\partial\Delta^3$, corresponding to the 6 binary cocycles. This provides a one-to-one correspondence between the 6 binary cocycles of $Z^2_\triangle(\partial\langle 3\rangle; \{0,1\} \subset \mathbb{Z})$ and the trivial s.c. bundles on $\partial\langle 3\rangle$ extending to a minimal s.c. bundle on $\langle 3\rangle$. The correspondence is presented in Table 1.

$f_0(123)$	$f_1(023)$	$f_2(013)$	$f_3(012)$	
+	-	+	-	$\mathbf{c}(f)$
$(123)_\circlearrowleft$ 0	$(023)_\circlearrowleft$ 0	$(013)_\circlearrowleft$ 0	$(012)_\circlearrowleft$ 0	$(0123)_\circlearrowleft$ $\underline{0}$
0	0	0	1	-1
0	0	1	0	1
$(()_\circlearrowleft 231)$ 0	$(031)_\circlearrowleft$ 0	$(031)_\circlearrowleft$ 1	$(021)_\circlearrowleft$ 1	$(0231)_\circlearrowleft$ $\underline{0}$
0	1	0	0	-1
0	1	0	1	-2
$(312)_\circlearrowleft$ 0	$(032)_\circlearrowleft$ 1	$(031)_\circlearrowleft$ 1	$(012)_\circlearrowleft$ 0	$(0312)_\circlearrowleft$ $\underline{0}$
0	1	1	1	-1
1	0	0	0	1
$(213)_\circlearrowleft$ 1	$(023)_\circlearrowleft$ 0	$(013)_\circlearrowleft$ 0	$(021)_\circlearrowleft$ 1	$(0213)_\circlearrowleft$ $\underline{0}$
1	0	1	0	2
1	0	1	1	1
$(132)_\circlearrowleft$ 1	$(032)_\circlearrowleft$ 1	$(013)_\circlearrowleft$ 0	$(012)_\circlearrowleft$ 0	$(0132)_\circlearrowleft$ $\underline{0}$
1	1	0	1	-1
1	1	1	0	1
$(321)_\circlearrowleft$ 1	$(032)_\circlearrowleft$ 1	$(031)_\circlearrowleft$ 1	$(021)_\circlearrowleft$ 1	$(0321)_\circlearrowleft$ $\underline{0}$

Table 1.

□

§7. Proof of Theorem 1.

Summarizing the achievements, we get the following.

Proposition 8. *The set of minimal s.c. bundles on the base semi-simplicial set $\boldsymbol{B}$ is in a canonical one-to-one correspondence with the local systems of circular permutations of ordered vertices of base simplices, simplicial maps* $\mathrm{Hom}(\boldsymbol{B}, \boldsymbol{S}C)$*, and simplicial* 2*-cocycles in* $Z^2_\triangle(\boldsymbol{B}; \{0,1\} \subset \mathbb{Z})$.

Proof. The first two statements were discussed in Sec. 3.6. The last statement we know from the general Huntington theory and Proposition 7. A bundle uniquely determines a cocycle.

We need the inverse: a binary 2-cocycle uniquely determines a bundle. Here is why: a binary 2-cocycle uniquely determines a local system of circular permutations on the 2-skeleton; by Proposition 7, the cocycle condition provides the transitivity of the system of cyclic orders, therefore, it is uniquely extendable over the 3-skeleton. Now, by the Kan property of cyclic orders from Proposition 6, the local system of circular permutations on the 3-skeleton is uniquely extendable to the entire set $\boldsymbol{B}$. □

Proof of Theorem 1. By Proposition 4, we know that any semi-simplicial s.c. bundle triangulated over $\boldsymbol{B}$ is concordant to a minimal one and, therefore, its Chern class is representable by a simplicial 2-cochain in the base with binary values. By Proposition 8, the inverse statement is true. For classically simplicial triangulations, the condition is necessary but not sufficient (see [11]). □

§8. Effortless & local assembly of triangulated circle bundles with a prescribed Chern number over a closed oriented triangulated surface

For minimally triangulated circle bundles over triangulated oriented closed surfaces, we get a purely unobstructed free way of constructing triangulated circle bundles with a prescribed Chern number.

8.1. Suppose we have an oriented surface M, triangulated by a semi-simplicial complex $\boldsymbol{T}$, and let $[M] \in Z_2^\triangle(\boldsymbol{T}; \mathbb{Z})$ be the fundamental class of M, fixing the orientation. Then any 2-simplex $x \in \boldsymbol{T}_2$ obtains a relative positive or negative orientation $o(x) \in \mathbb{Z}/2\mathbb{Z}$ according to the value $(-1)^{o(x)}$ of the fundamental class $[M]$ on the simplex x.

Lemma 9. *A semi-simplicial triangulation of an oriented closed surface always has an even number of* 2*-simplices, half of them positively oriented and the other half negatively.*

Proof. Pick the 1-cochain $\mathbf{1}_1 \in C^1_\triangle(\boldsymbol{T};\mathbb{Z})$ having the value 1 on every 1-simplex. Then $d(\mathbf{1}_1) = \mathbf{1}_2$ is a coboundary in $B^2_\triangle(\boldsymbol{T};\mathbb{Z})$ having the value 1 on any 2-simplex. By Stokes' theorem, the pairing is $\langle \mathbf{1}_2, [M]\rangle = 0$. Therefore, the number of positively oriented simplices is equal to the number of negatively oriented ones, and the total number $\#\boldsymbol{T}_2$ is even. $\square$

8.2. Proof of Theorem 2.

Proof of Theorem 2. Take a simplicial 2-cochain

$$u \in C^2_\triangle(\boldsymbol{T}; \{0,1\} \subset \mathbb{Z}).$$

It will be a 2-cocycle, since every 2-cochain in $C^2_\triangle(\boldsymbol{T};\mathbb{Z})$ is a cocycle. This 2-cocycle defines an integer $c(u)$, the element of the second cohomology group, by pairing with the fundamental cocycle:

$$H^2(M;\mathbb{Z}) = \mathbb{Z} \ni c(u) = \langle u, [M]\rangle = \Sigma_{x\in\boldsymbol{T}_2}(-1)^{o(x)}u(x).$$

By Lemma 9, $c(u)$ can be any integer number from the interval

$$[-\frac{1}{2}\#\boldsymbol{T}_2, \dots, -1, 0, 1, \dots, \frac{1}{2}\#\boldsymbol{T}_2].$$

The maximum value $\frac{1}{2}\#\boldsymbol{B}(2)$ of $c(u)$ is obtained by distributing 1's on the positively oriented simplices and 0's on the negatively oriented ones. The minimum value $-\frac{1}{2}\#\boldsymbol{B}(2)$ is obtained by distributing 1's on the negatively oriented simplices and 0's on the positively oriented ones. Picking u, we can put the circular permutation $(0,1,2)_\circlearrowleft$ on every 2-simplex x with $u(x) = 0$ and the circular permutation $(2,0,1)_\circlearrowleft$ on every simplex x with $u(x) = 1$. We effortlessly get a local system of necklaces, since the boundaries of circular permutations of three elements are always the trivial circular permutations $(0,1)_\circlearrowleft, (0)_\circlearrowleft$ and always fit, because they are always of a single type and have no automorphisms. Therefore, replacing circular permutations by elementary s.c. bundles, we obtain a minimal s.c. bundle having u as its Chern cocycle and $c(u)$ as its Chern number. According to Proposition 4, any bundle triangulated over $\boldsymbol{T}$ is concordant to a minimal one and, therefore, has a binary simplicial representative of its Chern cocycle.

By the argument from [11, Lemma 0.1], the bundle with the maximum possible Chern number $\frac{1}{2}\#\boldsymbol{T}_2$ can be only semi-simplically triangulated over $\boldsymbol{T}$. $\square$

It would be interesting to further investigate the case of classically simplicial triangulations. In view of Proposition 3, it seems that the spindle contraction reduces a classically simplicial bundle triangulation to a classically simplicial triangulation with only several possible types of elementary subbundles. So, the analysis can be accessible.

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St.Petersburg Department
of Steklov Institute of Mathematics
and Chebyshev Laboratory,
St.Petersburg State University,
St.Petersburg, Russia

E-mail: `mnev@pdmi.ras.ru`

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