Записки научных
семинаров ПОМИ
Том 479, 2019 г.

I. A. Panin

A SHORT PROOF OF A THEOREM DUE TO O. GABBER

Abstract. A very short proof of *an unpublished result* due to O. Gabber is given. More presicely, let R be a regular local ring, containing *a finite field* k. Let $\mathbf{G}$ be a simply-connected reductive group scheme over k. We prove that a principal $\mathbf{G}$-bundle over R is trivial, if it is trivial over the fraction field of R. This is the mentioned unpublished result due to O. Gabber. We derive this result from a purely geometric one proven in another paper of the author and stated in the Introduction.

§1. Introduction

Let R be a commutative unital ring. Recall that an R-group scheme $\mathbf{G}$ is called *reductive*, if it is affine and smooth as an R-scheme and if, moreover, for each algebraically closed field Ω and for each ring homomorphism $R \to \Omega$ the scalar extension $\mathbf{G}_\Omega$ is a connected reductive algebraic group over Ω. This definition of a reductive R-group scheme coincides with [2, Exp. XIX, Definition 2.7].

Assume that U is a regular scheme, $\mathbf{G}$ is a reductive U-group scheme. Recall that a U-scheme $\mathcal{G}$ with an action of $\mathbf{G}$ is called *a principal* $\mathbf{G}$*-bundle over* U, if $\mathcal{G}$ is faithfully flat and quasi-compact over U and the action is simple transitive, that is, the natural morphism $\mathbf{G} \times_U \mathcal{G} \to \mathcal{G} \times_U \mathcal{G}$ is an isomorphism, see [7, Section 6]. It is well known that such a bundle is trivial locally in étale topology but in general not in Zariski topology. Grothendieck and Serre conjectured that $\mathcal{G}$ is trivial locally in Zariski topology, if it is trivial generically.

More precisely, a well-known conjecture due to J.-P. Serre and A. Grothendieck (see [18, Remarque, p. 31], [5, Remarque 3, p. 26–27], and [6, Remarque 1.11.a]) asserts that given a regular local ring R and its field of fractions K and given a reductive group scheme $\mathbf{G}$ over R, the map

$$H^1_{\text{ét}}(R, \mathbf{G}) \to H^1_{\text{ét}}(K, \mathbf{G}),$$

Key words and phrases: semi-simple algebraic group, principal bundle, Grothendieck–Serre conjecture.

The author acknowledges support of the RFBR grant No. 19-01-00513.

induced by the inclusion of R into K, *has a trivial kernel.* If R contains an infinite field, then the conjecture is proved in [FP]. If R contains a finite field, then the conjecture is proved in the series [12–14]. Thus, the conjecture is true for any regular semi-local domain containing a field.

Particularly, due to works [12–14] the conjecture is true for any semi-local regular domain containing a finite field k and any reductive group $\mathbf{G}$ defined over k. These recovers the unpublished result due to O. Gabber. However this way of getting Gabber's result is very lengthy. If we replace the class of all reductive groups over k with a smaller class of reductive simply-connected groups, then we arrive to Theorem 1.1 stated right below. A proof of this theorem is given already in [12].

The main aim of the present paper is to present *a very short proof* of this theorem. By this way we demonstrate the power of Theorem 1.2 proven in [10, Theorem 1.2].

For a scheme U we denote by $\mathbb{A}^1_U$ the affine line over U and by $\mathbb{P}^1_U$ the projective line over U. Let T be a U-scheme. By a principal $\mathbf{G}$-bundle over T we understand a principal $\mathbf{G} \times_U T$-bundle. We refer to [2, Exp. XXIV, Sec. 5.3] for the definitions of a simple simply-connected group scheme over a scheme and a semi-simple simply-connected group scheme over a scheme.

Theorem 1.1 (O. Gabber)**.** *Let R be a regular semi-local domain containing a finite field k, and let K be its field of fractions. Given a simply-connected reductive group scheme $\mathbf{G}$ over k, the map*

$$H^1_{et}(R, \mathbf{G}) \to H^1_{et}(K, \mathbf{G}),$$

induced by the inclusion R into K, has trivial kernel.

The latter theorem is *the unpublished theorem* due to O. Gabber [4]. Theorem 1.1 is easily derived below from purely geometric result proven in [10, Theorem 1.2].

Theorem 1.2. *Let X be an affine k-smooth irreducible k-variety, and let $x_1, x_2, \dots, x_n$ be closed points in X. Let $U = Spec(\mathcal{O}_{X,\{x_1,x_2,\dots,x_n\}})$ and $f \in k[X]$ be a non-zero function vanishing at each point x_i. Then there is a monic polinomial $h \in O_{X,\{x_1,x_2,\dots,x_n\}}[t]$, a commutative diagram of* *schemes with the irreducible affine U-smooth scheme Y*

$$\begin{array}{ccccc} (\mathbb{A}^1 \times U)_h & \xleftarrow{\tau_h} & Y_h := Y_{\tau^*(h)} & \xrightarrow{(p_X)|_{Y_h}} & X_f \\ \big\downarrow{\scriptstyle \mathrm{inc}} & & \big\downarrow{\scriptstyle \mathrm{inc}} & & \big\downarrow{\scriptstyle \mathrm{inc}} \\ (\mathbb{A}^1 \times U) & \xleftarrow{\tau} & Y & \xrightarrow{p_X} & X \end{array} \tag{1}$$

and a morphism $\delta : U \to Y$ subjecting to the following conditions:

(i) *the left hand side square is an elementary distinguished square in the category of affine U-smooth schemes in the sense of* [8, Defn. 3.1.3]*;*
(ii) *$p_X \circ \delta = can : U \to X$, where can is the canonical morphism;*
(iii) *$\tau \circ \delta = i_0 : U \to \mathbb{A}^1 \times U$ is the zero section of the projection $\mathrm{pr}_U : \mathbb{A}^1 \times U \to U$;*
(iv) *$h(1) \in \mathcal{O}_{X,\{x_1,x_2,\dots,x_n\}}[t]$ is a unit.*

Theorem 1.2 is one of the main geometric result proven in [10]. It is a consequence of a theory of nice triples developed in that paper. The theory of nice triples was invented in [9] and developed in [10]. The theory is inspired by the Voevodsky theory of standard triples and has as an initial aim to get Gersten's type results.

The author thanks A. Suslin, A. Stavrova and D. Orlov for their interest to the topic of the present paper on its various stages. He thanks M.Ojanguren for many inspiring ideas arising from our joint works with him and resulting with the very basic geometric constructions in [10, Sec. 4].

Let us give *a very brief survey* of the history of works regarding the constant group case of the conjecture. In the paper [1] the conjecture was proved in the constant group case, providing that the base field is perfect and infinite. The case of non-perfect base field was done in [16, 17]. The case of finite base field was done in [4] (still unpublished). A good survey on the topic of the present paper can be found in [3] and [11].

§2. A preliminary construction

Let k be a finite field. Let $\mathcal{O}$ be the semi-local ring $\mathcal{O}_{X,\{x_1,x_2,\dots,x_n\}}$ of finitely many *closed points* on a k-smooth irreducible affine k-variety X. Set $U = \operatorname{Spec} \mathcal{O}$. Let $\mathbf{G}$ be a reductive group scheme over k. Let $\mathcal{G}$ be a principal $\mathbf{G}$-bundle over U trivial over the generic point of U. We may and will suppose that the principal **G**-bundle $\mathcal{G}$ is the restriction to U of a principal **G**-bundle $\mathcal{G}'$ on X, and the restriction of $\mathcal{G}'$ to a principal open subset X_{f} is trivial. If $U = \mathrm{Spec}(\mathcal{O})$, then we may and will suppose that f vanishes at each point x_i.

Theorem 1.2 states that there are a monic polynomial $h \in \mathcal{O}[t]$, a commutative diagram (1) of schemes with the irreducible affine U-smooth Y, and a morphism $\delta : U \to Y$ subjecting to conditions (i) to (iv) from Theorem 1.2.

Construct now a principal **G**-bundle $\mathcal{G}_t$ on $\mathbb{A}^1 \times U$ as follows. Take the pull-back $p_X^*(\mathcal{G}')$ of $\mathcal{G}'$ to Y. The restriction of $p_X^*(\mathcal{G}')$ to Y_h is trivial, since the restriction of $\mathcal{G}'$ to X_{f} is trivial. Take now the trivial **G**-bundle over the principal open subset $(\mathbb{A}^1 \times U)_h$ and glue it with $p_X^*(\mathcal{G}')$ via an isomorphism over Y_h. This way we get a principal **G**-bundle $\mathcal{G}_t$ over $\mathbb{A}^1 \times U$. Clearly, the monic polynomial h and the principal **G**-bundle $\mathcal{G}_t$ on $\mathbb{A}^1 \times U$ satisfy the following properties

(i) the **G**-bundle $\mathcal{G}_t$ is trivial over the open subscheme $(\mathbb{A}^1_U)_h$ in $\mathbb{A}^1_U$ given by $h(t) \neq 0$;

(ii) the restriction of $\mathcal{G}_t$ to $\{0\} \times U$ coincides with the original **G**-bundle $\mathcal{G}$;

(iii) $h(1) \in \mathcal{O}$ is a unit.

§3. Simply-connected case of the theorem due to Gabber

The unpublished theorem due to Gabber [4] states particularly that if the base field k is finite, then the Grothendieck–Serre conjecture is true for any reductive group scheme **G** over k. The main aim of the present section is to recover that result in the simply-connected case.

Proof of Theorem 1.1. Using [15] or [19] the case of a general regular semi-local ring containing k is easily reduced to the case, when $\mathcal{O}$ is the semi-local ring $\mathcal{O}_{X,x_1,\dots,x_n}$ of a finitely many closed points $x_1, \dots, x_n$ on an affine k-smooth variety X. Using standard arguments (see for instance [9, Section 11]) we may and will suppose that **G** is a simple simply-connected k-group. Set $U = \mathrm{Spec}\,\mathcal{O}$. Let $\mathcal{G}$ be a principal **G**-bundle over U trivial over the generic point of U. Consider the monic polynomial $h \in \mathcal{O}[t]$ and the principal **G**-bundle $\mathcal{G}_t$ on $\mathbb{A}^1 \times U$ constructed in Section 2.

The k-group scheme **G** is defined over k and k is finite, and **G** is a simple simply-connected k-group. Hence, **G** contains a k-Borel subgroup scheme.

Particularly, **G** is isotropic. So, the group **G**, the **G**-bundle $\mathcal{G}_t$ and the monic polynomial $h \in \mathcal{O}[t]$ satisfy the hypotheses of Theorem [9, Thm.1.3]. Thus the principal **G**-bundle $\mathcal{G}_t$ is trivial. Hence the restriction of $\mathcal{G}_t$ to $\{0\} \times U$ is trivial. From the other side the latter restriction coincides with the original **G**-bundle $\mathcal{G}$ by the property (ii). Hence the original **G**-bundle $\mathcal{G}$ is trivial. □

References

1. J.-L. Colliot-Thélène, M. Ojanguren, *Espaces Principaux Homogènes Localement Triviaux.* — Publ. Math. IHÉS **75**, No. 2 (1992), 97–122.
2. M. Demazure, A. Grothendieck, *Schémas en groupes.* — Lect. Notes Math., **151–153**, Springer-Verlag, Berlin–Heidelberg–New York, 1970.
3. R. Fedorov, I. Panin, *A proof of the Grothendieck–Serre conjecture on principal bundles over regular local rings containing infinite fields.* — Publ. Math. de l'IHES **122**, No. 1 (2015), 169–193.
4. O. Gabber, announced and still unpublished.
5. A. Grothendieck, *Torsion homologique et section rationnalles.* — In: Anneaux de Chow et applications, Séminaire Chevalley, 2-e année, Secrétariat mathématique, Paris, 1958.
6. A. Grothendieck, *Le group de Brauer.* II. In: Dix exposés sur la cohomologique de schémes, Amsterdam, North-Holland, 1968.
7. A. Grothendieck, *Technique de descente et theoremes d'existence en geometrie algebrique: I. Generalites. Descente par morphismes delement plats.* In: Seminaire Bourbaki, Vol. 5, Exp. No. 190, pp. 299–327. Soc. Math. France, Paris, 1995.
8. F. Morel, V. Voevodsky, *A^1-homotopy theory of schemes.* — Publ. Math. IHÉS, **90** (1999), 45–143.
9. I. Panin, A. Stavrova, N. Vavilov, *On Grothendieck–Serre's conjecture concerning principal G-bundles over reductive group schemes:* I. — Compositio Math., **151** (2015), 535–567.
10. I. Panin, *Nice triples and moving lemmas for motivic spaces.* — Izv. RAN, Ser. Mat. **83**, No. 4 (2019), 158–193.
11. I. Panin, *On Grothendieck–Serre conjecture concering principal bundles.* — Proc. International Congress of Mathematics, 2018, Rio de Janeiro, Vol. 1, 201–222.
12. I. Panin, *Nice triples and Grothendieck–Serre's conjecture concerning principal G-bundles over reductive group schemes.* — Duke M. J. **168**, No. 2 (2019), 351–375.
13. I. Panin, *Two purity theorems and Grothendieck–Serre's conjecture concerning principal G-bundles over regular semi-local rings.* `arXiv:1707.01763`
14. I. Panin, *Proof of Grothendieck–Serre conjecture on principal G-bundles over semi-local regular domains containing a finite field.* `arXiv:1707.01767`
15. D. Popescu, *General Néron desingularization and approximation.* — Nagoya Math. J. **104** (1986), 85–115.
16. M. S. Raghunathan, *Principal bundles admitting a rational section.* — Invent. Math. **116**, No. 1–3 (1994), 409–423.

17. M. S. Raghunathan, *Erratum: Principal bundles admitting a rational section.* — Invent. Math. **121**, No. 1 (1995), 223.
18. J.-P. Serre, *Espaces fibrés algébriques.* In: Anneaux de Chow et applications, Séminaire Chevalley, 2-e année, Secrétariat mathématique, Paris, 1958.
19. R. G. Swan, *Néron–Popescu desingularization.* — Algebra and Geometry (Taipei, 1995), Lect. Algebra Geom. 2, Internat. Press, Cambridge, MA, 1998, 135–192.
20. V. Voevodsky, *Cohomological theory of presheaves with transfers.* In: Cycles, Transfers, and Motivic Homology Theories, Ann. Math. Studies, 2000, Princeton University Press.

С.-Петербургское отделение
Математического института
им. В. А. Стеклова РАН
E-mail: panin@pdmi.ras.ru

Поступило 2 октября 2019 г.