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ON SCHURIAN FUSIONS OF THE ASSOCIATION SCHEME OF A GALOIS AFFINE PLANE OF PRIME ORDER

Abstract. The schurian fusions of the association scheme of a Galois affine plane of prime order are completely identified.

§1. Introduction

An *association scheme* $\mathcal{X}$ on a (finite) set Ω can be thought as a special partition S of the Cartesian square Ω^2, that contains the diagonal as one of the classes (for the exact definitions, see Section 2). It is very rare that each coarser partition of Ω^2 with the diagonal as a class is also an association scheme, a *fusion* of $\mathcal{X}$. In [7], it was proved that this is true if $\mathcal{X}$ is the scheme of a finite affine plane $\mathcal{A}$, i.e., Ω is the point set of $\mathcal{A}$ and the nondiagonal classes of S are in one-to-one correspondence with the parallel classes of $\mathcal{A}$. Thus if $\mathcal{A}$ is of order q, then $|\Omega| = q^2$ and $\mathcal{X}$ has exactly $p(n)$ different fusions, where $n = q + 1$ and $p(n)$ is the number of all partitions of the set $\{1, \dots, n\}$.

An association scheme $\mathcal{X}$ on Ω is said to be *schurian* if there exists a group $K \leq \mathrm{Sym}(\Omega)$ such that the classes of the partition S are the orbits of the induced action of K on Ω^2. The schurity problem in a class of association schemes consists in identifying the schurian schemes in the class in question, see [6]. In the present paper, we solve this problem for the class of all schurian fusions of the association scheme of a Galois affine plane of prime order.

Main Theorem. *A schurian fusion of the scheme of a Galois affine plane of prime order p is one of the following:*

(1) *wreath or subtensor product of two trivial schemes of degree p,*
(2) *primitive pseudocyclic scheme,*
(3) *one of the two exceptional schemes,*

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(4) *the involutive fusion of one of the above schemes.*

The first three cases in the Main Theorem are basic. In case (1), the wreath product is unique and schurian, whereas there are non-schurian subtensor products, see example in [11, Theorem 26.4]. The schurian schemes in case (2) are obtained from 3/2-transitive subgroups of $\mathrm{AGL}(2,p)$; again there are many non-schurian primitive pseudocyclic schemes, see [3, Example 2.6.15]. Two exceptional schurian schemes from case (3) correspond to the alternating subgroups $\mathrm{Alt}(4)$ and $\mathrm{Alt}(5)$ of the group $\mathrm{PGL}(2,p)$. For certain values of p, these schemes may be primitive pseudocyclic, see Subsection 5.1.

A fusion of a scheme $\mathcal{X}$ is said to be *involutive* if there exists an algebraic automorphism φ of $\mathcal{X}$ such that each class of the partition associated with this fusion is of the form $s\cup\varphi(s)$, $s\in S$. The class of schemes in case (4) is quite large and can contain schemes occurring in the other three cases. Moreover, many involutive fusions of (even schurian) schemes are non-schurian.

The proof of the Main Theorem is given in Sec. 4; the key ingredients are a classification of 2-closed permutation groups of prime-squared degree [4] and an information on the orbits of subgroups of $\mathrm{PGL}(2,q)$ [2]. In Sec. 2, we cite some standard facts on association schemes. The scheme of an affine plane is defined and studied in Sec. 3. Section 5 contains concluding remarks and open problems.

Notation.

Throughout this paper, Ω is a finite set.

The diagonal of the Cartesian product Ω^2 is denoted by 1_Ω. For a relation $s\subseteq\Omega^2$, we set

$$s^*=\{(\beta,\alpha)\,:\,(\alpha,\beta)\in S\}\quad\text{and}\quad \alpha s=\{\beta\in\Omega\,:\,(\alpha,\beta)\in s\}$$

for all $\alpha\in\Omega$. For $S\subseteq 2^{\Omega^2}$, we denote by $S^\cup$ the set of all unions of the elements of S. We define $S^*=\{s^*\,:\,s\in S\}$, $S^\#=S\setminus\{1_\Omega\}$ and $\alpha S=\cup_{s\in S}\alpha s$, where $\alpha\in\Omega$. By C_p and $\mathbb{F}_q$, we denote the cyclic group of order p and a finite field of order q, respectively. By $\mathrm{Sym}(n)$, $\mathrm{Alt}(n)$, and D_{2n}, we denote the symmetric and alternating group of degree n, and dihedral group of order $2n$, respectively.

§2. Association schemes

In this section, we cite all required concepts on association schemes; the notation, terminology and results are taken from [3], see also [6].

2.1. Definitions. Let Ω be a finite set and S a partition of the Cartesian square Ω^2. A pair $\mathcal{X} = (\Omega, S)$ is called an *association scheme* or *scheme* on Ω if the following conditions are satisfied: $1_\Omega \in S$, $S^* = S$, and given $r, s, t \in S$, the number

$$c_{rs}^t := |\alpha r \cap \beta s^*|$$

does not depend on the choice of $(\alpha, \beta) \in t$. The elements of Ω, S, $S^\cup$, and the numbers c_{rs}^t are called the *points*, *basis relations*, *relations*, and *intersection numbers* of $\mathcal{X}$, respectively. The numbers $|\Omega|$ and $|S|$ are called the *degree* and *rank* of $\mathcal{X}$. A scheme of rank 2 is said to be *trivial.* The set S of all basis relations of $\mathcal{X}$ is denoted by $S(\mathcal{X})$.

2.2. Isomorphisms and schurity. A bijection from the point set of a scheme $\mathcal{X}$ to the point set of a scheme $\mathcal{X}'$ is called an *isomorphism* from $\mathcal{X}$ to $\mathcal{X}'$ if it induces a bijection between their sets of basis relations. The schemes $\mathcal{X}$ and $\mathcal{X}'$ are said to be isomorphic if there exists an isomorphism from $\mathcal{X}$ to $\mathcal{X}'$.

An isomorphism from a scheme $\mathcal{X}$ to itself is called *automorphism* if the induced permutation of the basis relations of $\mathcal{X}$ is the identity. The set

$$\mathrm{Aut}(\mathcal{X}) = \{f \in \mathrm{Sym}(\Omega) \,:\, s^f = s \text{ for all } s \in S\}$$

of all automorphisms of $\mathcal{X}$ is a group with respect to composition. One can easily see that $\mathrm{Aut}(\mathcal{X}) = \mathrm{Sym}(\Omega)$ if and only if the scheme $\mathcal{X}$ is trivial.

Let $K \leq \mathrm{Sym}(\Omega)$ be a transitive permutation group, and let S denote the set of orbits in the induced action of K on Ω^2. Then,

$$\mathrm{Inv}(K) := (\Omega, S)$$

is a scheme; we say that $\mathrm{Inv}(K)$ *is associated* with K. A scheme $\mathcal{X}$ on Ω is said to be *schurian* if it is associated with the group $\mathrm{Aut}(\mathcal{X})$ (or equivalently with a certain transitive permutation group on Ω).

2.3. Algebraic isomorphisms and fusions. Let $\mathcal{X}$ and $\mathcal{X}'$ be schemes. A bijection $\varphi : S \to S',\ r \mapsto r'$ is called an *algebraic isomorphism* from $\mathcal{X}$ to $\mathcal{X}'$ if

$$c_{rs}^t = c_{r's'}^{t'}, \qquad r, s, t \in S. \tag{1}$$

Each isomorphism f from $\mathcal{X}$ onto $\mathcal{X}'$ induces an algebraic isomorphism $s \mapsto s^f$, but not every algebraic isomorphism is induced by an isomorphism. The group of all algebraic automorphisms of $\mathcal{X}$ is denoted by $\mathrm{Aut}_{\mathrm{alg}}(\mathcal{X})$.

Let $K \le \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X})$. Given $s \in S$, denote by s^K the union of all relations s^k, $k \in K$. Then the pair

$$\mathcal{X}^K = (\Omega, S^K)$$

with $S^K = \{s^K : s \in S\}$, is called the *algebraic fusion* of $\mathcal{X}$ with respect to the group K. When the order of K equals 2, the fusion is said to be *involutive*.

2.4. Parabolics. Let $\mathcal{X} = (\Omega, S)$ be a scheme. Following [8], any equivalence relation $e \in S^{\cup}$ is called a *parabolic* of $\mathcal{X}$. Clearly, 1_Ω and Ω^2 are parabolics of $\mathcal{X}$; they are said to be trivial. The scheme $\mathcal{X}$ is said to be *primitive* if they are the only parabolics of $\mathcal{X}$; otherwise, $\mathcal{X}$ is said to be *imprimitive*. The following almost obvious statement is well known.

Proposition 2.1. *For a transitive group K, the scheme* $\mathrm{Inv}(K)$ *is primitive if and only if so is the group K.*

Let e be a parabolic of $\mathcal{X}$. Denote by Ω/e the set of all classes of e. For any $s \in S$, we define $s_{\Omega/e}$ to be the relation on Ω/e that consists of all pairs (Δ, Γ) such that the relation $s_{\Delta,\Gamma} = s \cap (\Delta \times \Gamma)$ is not empty. Then the pairs

$$\mathcal{X}_{\Omega/e} = (\Omega/e, S_{\Omega/e}) \quad \text{and} \quad \mathcal{X}_\Delta = (\Delta, S_\Delta),$$

where $S_{\Omega/e}$ and S_Δ are the sets of all nonempty relations of the form $s_{\Omega/e}$ and $s_{\Delta,\Delta}$, respectively, are schemes; here, s runs over S, and $\Delta \in \Omega/e$ is fixed.

If $\mathcal{X}$ is schurian, then $\mathcal{X}_{\Omega/e}$ is the scheme associated with the group induced by the action of $\mathrm{Aut}(\mathcal{X})$ on Ω/e, whereas $\mathcal{X}_\Delta$ is the scheme induced by the action of the setwise stabilizer of Δ in $\mathrm{Aut}(\mathcal{X})$ on Δ.

2.5. Wreath and subtensor products. Let Ω_1 and Ω_2 be sets and $\Omega = \Omega_1 \times \Omega_2$. Denote by e_1 and e_2 the equivalence relations on Ω such that

$$\Omega/e_1 = \{\{\alpha\} \times \Omega_2 : \ \alpha \in \Omega_1\} \quad \text{and} \quad \Omega/e_2 = \{\Omega_1 \times \{\alpha\} : \ \alpha \in \Omega_2\}.$$

In what follows, the set Ω_i is canonically identified both with Ω/e_i and with a class of the equivalence relation e_{3-i}, $i = 1, 2$.

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be schemes on Ω_1 and Ω_2, respectively. The *wreath product* of $\mathcal{X}_1$ and $\mathcal{X}_2$ is defined to be the scheme on Ω that has the smallest rank among the schemes $\mathcal{X}$ having a parabolic $e = e_2$ and such that

$$\mathcal{X}_{\Omega_1} = \mathcal{X}_1 \quad \text{and} \quad \mathcal{X}_{\Omega/e_2} = \mathcal{X}_2,$$

where Ω_1 on the left-hand side is treated as a class of e (in particular, $\mathcal{X}_\Delta = \mathcal{X}_1$ for all $\Delta \in \Omega/e_1$). The basis relations of the wreath product can be found explicitly, see [3, Subsection 3.4.1].

A *subtensor product* of $\mathcal{X}_1$ and $\mathcal{X}_2$ is defined to be a scheme $\mathcal{X} = (\Omega, S)$ such that e_1 and e_2 are parabolics of $\mathcal{X}$,

$$\mathcal{X}_{\Omega/e_1} = \mathcal{X}_1 \quad \text{and} \quad \mathcal{X}_{\Omega/e_2} = \mathcal{X}_2,$$

and each relation of $\mathcal{X}$ is contained in the product

$$s_1 \otimes s_2 = \{((\alpha_1, \alpha_2), (\beta_1, \beta_2)) \in \Omega \times \Omega : \ (\alpha_1, \alpha_2) \in s_1, \ (\beta_1, \beta_2) \in s_2\},$$

where s_1 and s_2 are basis relations of $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively. Such a scheme is not unique and coincides with the tensor product of $\mathcal{X}_1$ and $\mathcal{X}_2$ if the rank of $\mathcal{X}$ equals the product of the ranks of $\mathcal{X}_1$ and $\mathcal{X}_2$, see [3, Subsection 3.2.2].

Proposition 2.2. *Let $K_1 \leq \mathrm{Sym}(\Omega_1)$ and $K_2 \leq \mathrm{Sym}(\Omega_2)$ be transitive. Then*

(1) *the scheme of the wreath product $K_1 \wr K_2$ in the imprimitive action equals the wreath product of* $\mathrm{Inv}(K_1)$ *and* $\mathrm{Inv}(K_2)$*,*
(2) *the scheme of the subdirect product $K_1 \sqcup K_2$ in the product action equals the subtensor product of* $\mathrm{Inv}(K_1)$ *and* $\mathrm{Inv}(K_2)$*.*

Proof. Follows from [3, Theorem 3.4.6] and [3, Subsection 3.2.21]. □

2.6. Pseudocyclic schemes. Let $\mathcal{X} = (\Omega, S)$ be a scheme, and let s be a basis relation of $\mathcal{X}$. The numbers

$$n_s = c_{ss^*}^{1_\Omega} \quad \text{and} \quad c(s) = \sum_{r \in S} c_{rr^*}^{s}$$

are called the *valency* and *indistinguishing number* of s, respectively. The scheme $\mathcal{X}$ is said to be *pseudocyclic* if there exists a positive integer k such that

$$n_s = k = c(s) + 1$$

for all $s \in S^{\#}$ (another but equivalent definition is given in [9, Theorem 3.2]). It is known that the scheme of any Frobenius group is pseudocyclic, and the converse statement is true whenever $|\Omega|$ is much greater than k.

§3. Affine schemes and their fusions

Let $\mathcal{A}$ be a finite affine plane with point set Ω. Then the set $\Omega^2 \setminus 1_\Omega$ can be partitioned into the classes according to parallelism: two pairs (α, β) and (α', β') of points are in one class if and only if

$$\alpha\beta = \alpha'\beta' \quad \text{or} \quad \alpha\beta \parallel \alpha'\beta',$$

where $\alpha\beta$ and $\alpha'\beta'$ are the lines through α and β, and α' and β', respectively.

The obtained classes together with 1_Ω form a partition of Ω^2; denote it by $S_{\mathcal{A}}$. Then the pair

$$\mathcal{X}_{\mathcal{A}} = (\Omega, S_{\mathcal{A}})$$

is an association scheme [7]. It is called the scheme of $\mathcal{A}$ [7]. The basic properties of this scheme are straightforward and given in the lemma below, see also [7, 10].

Lemma 3.1. *In the above notation, let q be the order of $\mathcal{A}$, $\mathcal{X} = \mathcal{X}_{\mathcal{A}}$, and $S = S_{\mathcal{A}}$. The following statements hold:*

(1) $|\Omega| = q^2$ *and* $|S^{\#}| = q + 1$,
(2) *any $s \in S^{\#}$ is the disjoint union of q complete graphs of order q; in particular, $n_s = q - 1$,*
(3) $\mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}) = \mathrm{Sym}(S)_{1_\Omega}$;[1] *in particular, the scheme $\mathcal{X}$ is pseudocyclic.*

Corollary 3.2. *Let $\mathcal{X}$ be a fusion of the scheme $\mathcal{X}_{\mathcal{A}}$. Then given a parabolic e of $\mathcal{X}$ and $\Delta \in \Omega/e$, the schemes $\mathcal{X}_\Delta$ and $\mathcal{X}_{\Omega/e}$ are trivial.*

Let $\mathcal{X}$ be a fusion of the scheme $\mathcal{X}_{\mathcal{A}}$. From statement (2) of Lemma 3.1, it follows that the valency of any irreflexive basis relation of $\mathcal{X}$ is a multiple of $q - 1$. Set

$$\Lambda(\mathcal{X}) = \Big\{ \frac{n_s}{q-1} : \ s \in S(\mathcal{X})^{\#} \Big\}.$$

Clearly, this set contains at most $q + 1$ positive integers each of which is less than or equal to $q + 1$.

[1]Here, $\mathrm{Sym}(S)_{1_\Omega}$ is the point stabilizer of 1_Ω in $\mathrm{Sym}(S)$.

Lemma 3.3. *In the above notation, set* $\Lambda = \Lambda(\mathcal{X})$. *Then*

(1) $\mathcal{X}$ *is imprimitive if and only if* $1 \in \Lambda$,

(2) $\mathcal{X}$ *is pseudocyclic if and only if* $|\Lambda| = 1$.

Proof. The "if" part of statement (1) immediately follows from statement (2) of Lemma 3.1. To prove the "only if" part, assume that the scheme $\mathcal{X}$ is imprimitive. Then there is a nontrivial parabolic e of $\mathcal{X}$. Denote by a the number of irreflexive basis relations of $\mathcal{X}$ contained in e. By statements (1) and (2) of Lemma 3.1, we have

$$1 \le a < q+1 \quad \text{and} \quad 1 + a(q-1) \text{ divides } q^2.$$

Consequently, $a = 1$. It follows that $e = 1_\Omega \cup s$ for some $s \in S(\mathcal{X})^\#$. Thus, $\Lambda(\mathcal{X})$ contains the number $\frac{n_s}{q-1} = 1$.

The "only if" part of statement (2) immediately follows from the definition of pseudocyclic scheme. To prove the "if" part, assume that $\Lambda(\mathcal{X}) = \{d\}$ for some positive integer $d \le q+1$. Then each irreflexive basis relation of $\mathcal{X}$ is a union of exactly d relations belonging $S_{\mathcal{A}}^\#$. By statement (3) of Lemma 3.1, this implies that there exists a cyclic group

$$K \le \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}_{\mathcal{A}})$$

of order d that fixes 1_Ω, acts semiregularly on $S_{\mathcal{A}}^\#$. Thus in accordance with [9, Theorem 3.4], the scheme $\mathcal{X}$ is pseudocyclic. □

Let $\mathcal{A}$ be a Galois affine plane of order q. It is easily seen that the group $\mathrm{Aut}(\mathcal{X}_{\mathcal{A}})$ contains the center of $\mathrm{GL}(2,q)$. Now if $\mathcal{X}$ is a fusion of $\mathcal{X}_{\mathcal{A}}$, then $\mathrm{Aut}(\mathcal{X})$ contains $\mathrm{Aut}(\mathcal{X}_{\mathcal{A}})$, and hence

$$Z(\mathrm{GL}(2,q)) \le \mathrm{Aut}(\mathcal{X}). \tag{2}$$

From now on assume that $\mathcal{X}$ is schurian and, in addition,

$$\mathrm{Aut}(\mathcal{X}) \le \mathrm{AGL}(2,p). \tag{3}$$

Then the group $\mathrm{Aut}(\mathcal{X})$ preserves the parallelism in $\mathcal{A}$ and hence acts on the parallel classes of $\mathcal{A}$. Since the parallel classes are in one-to-one correspondence with the relations of $\mathcal{S}_{\mathcal{A}}$, this action induces a group $K \le \mathrm{Sym}(\mathcal{S}_{\mathcal{A}})$ leaving the relation 1_Ω fixed. By statement (3) of Lemma 3.1, this implies that

$$K \le \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}_{\mathcal{A}}).$$

Since K is induced by the automorphism group of $\mathcal{X}$, this scheme is the algebraic fusion of $\mathcal{X}_{\mathcal{A}}$ with respect to K. On the other hand, in view of (2) and (3) the group K can be identified with a subgroup of $\mathrm{PGL}(2,q)$

acting on $q+1$ points of the underlying projective line. Thus, the following statement holds.

Theorem 3.4. *Let $\mathcal{A}$ be a Galois affine plane of order q and $\mathcal{X}$ a schurian fusion of $\mathcal{X}_{\mathcal{A}}$. Assume that condition* (3) *holds. Then there is $K \le \mathrm{PGL}(2,q)$ such that*

$$\mathcal{X} = (\mathcal{X}_{\mathcal{A}})^K.$$

In particular, $\Lambda(\mathcal{X})$ equals the set $N(K)$ of cardinalities of the orbits of K.

§4. The proof of the Main Theorem

By the hypothesis of the theorem, $\mathcal{X}$ is the scheme of $\mathrm{Aut}(\mathcal{X})$; in particular, $\mathcal{X}$ is primitive (respectively, imprimitive) if and only if $\mathrm{Aut}(\mathcal{X})$ is primitive (respectively, imprimitive) (Proposition 2.1). The proof is divided into two parts depending on whether or not the group scheme $\mathcal{X}$ is imprimitive.

The imprimitive case corresponds to statement (1) of the Main Theorem; here we use a characterization of the 2-closed subgroups of $\mathrm{Sym}(p^2)$ given in [4]. Statements (2), (3), and (4) of the Main Theorem arise in the primitive case; here our tool is the information on the subgroups of $\mathrm{PGL}(2,q)$ given in [2].

4.1. The scheme $\mathcal{X}$ is imprimitive. The group $\mathrm{Aut}(\mathcal{X})$ being the automorphism group of a scheme is 2-closed in the sense of [12]. Therefore, we make use of the following statement which is an immediate consequence of [4, Theorem 14].

Lemma 4.1. *Let $K \le \mathrm{Sym}(p^2)$ be a 2-closed group with a regular subgroup $C_p \times C_p$. Then one of the following statements holds.*

- (i) *K is primitive, and $K \le \mathrm{AGL}(2,p)$, or $K = \mathrm{Sym}(p) \wr \mathrm{Sym}(2)$ or $\mathrm{Sym}(p^2)$,*
- (ii) *K is imprimitive, and one of the following statements holds:*
  - (ii1) *$K = \mathrm{Sym}(p) \times K'$, where $K' \le \mathrm{Sym}(p)$,*
  - (ii2) *$K < \mathrm{AGL}(1,p) \times \mathrm{AGL}(1,p)$,*
  - (ii3) *$K = K_1 \wr K_2$, where $K_1, K_2 \le \mathrm{Sym}(p)$ are 2-closed groups.*

By Lemma 4.1 for $K = \mathrm{Aut}(\mathcal{X})$, we have two cases: the first one is formed by statements (ii1) and (ii2), whereas the second one consists of just statement (ii3). In the former case, K is subdirect product of two groups. Therefore the scheme $\mathcal{X}$ is the subtensor product of two schemes of degree

p (statement (2) of Proposition 2.2), and both of them are trivial (Corollary 3.2). In the latter case, $\mathcal{X}$ is the wreath product $\mathrm{Inv}(K_1) \wr \mathrm{Inv}(K_2)$ (statement (1) of Proposition 2.2), and again both of them are trivial (Corollary 3.2). Thus if $\mathcal{X}$ is imprimitive, then statement (1) of the Main Theorem holds.

4.2. The scheme $\mathcal{X}$ is primitive. Without loss of generality, we may assume that (a) $\mathcal{X}$ is not trivial, for otherwise statement (2) of the Main Theorem holds and (b) the relation

$$1 \notin \Lambda(\mathcal{X}) \tag{4}$$

holds, for otherwise $\mathcal{X}$ is imprimitive by statement (1) of Lemma 3.3. Then p is odd and the following statement is a special case of the results proved in [2, Theorem 2 and Sec. 4].

Lemma 4.2. *Let $K \leq \mathrm{PGL}(2,p)$ be an intransitive permutation group acting on $p+1$ points of the underlying projective line, and $N = N(K)$. Then one of the following statements holds:*

(1) $K = C_d$ *and* $N \subseteq \{1, d\}$, $d \geq 1$,
(2) $K = D_{2d}$ *and* $N \subseteq \{2, d, 2d\}$, $d \geq 2$,
(3) $K = C_p \rtimes C_d$ *and* $N \subseteq \{1, p\}$, $d \mid p-1$,
(4) $K = \mathrm{Alt}(4)$, $\mathrm{Alt}(5)$, *or* $\mathrm{Sym}(4)$.

By Theorem 3.4 for $q = p$, there exists a group K satisfying the hypothesis of Lemma 4.2 and such that

$$\mathcal{X} = (\mathcal{X}_{\mathcal{A}})^K \quad \text{and} \quad \Lambda = N,$$

where $\mathcal{A}$ is a Galois affine plane of order p and $\Lambda = \Lambda(\mathcal{X})$. Note that this group is intransitive, because the scheme $\mathcal{X}$ is nontrivial. To complete the proof we will verify that in each of the four cases of Lemma 4.2, the conclusion of the Main Theorem holds.

In the case (1), assumption (4) implies that $N = \{d\}$. It follows that $|\Lambda| = 1$. Thus the scheme $\mathcal{X}$ is pseudocyclic by statement (2) of Lemma 3.3.

In the case (2), one can see as above that the scheme $\mathcal{X}$ is pseudocyclic whenever $2 \notin N$ and $d \notin N$. Assume first that $2 \in N$. Denote by K' the kernel of the action of K on an orbit of size 2. Then K' is a subgroup of index 2 and $1 \in N(K')$. It follows that if

$$\mathcal{X}' = (\mathcal{X}_{\mathcal{A}})^{K'}, \tag{5}$$

then $\mathcal{X}$ is an involutive fusion of $\mathcal{X}'$ and $1 \in N(K') = \Lambda(\mathcal{X}')$. The scheme $\mathcal{X}'$ is imprimitive by statement (1) of Lemma 3.3. By the first part of the proof (the imprimitive case), this implies that statement (1) of the Main Theorem holds for $\mathcal{X}'$, and we are done.

Remaining in the case (2), we may assume that $N = \{d, 2d\}$. Then K has a subgroup K' of index 2 such that

$$N(K') = \{d\}. \tag{6}$$

Indeed, the action of K on an orbit of cardinality d is permutation isomorphic to the action of K on the right cosets of a subgroup generated by an involution $k \in K$. Depending on whether or not k lies in the center of K, one can take as K' a subgroup of K isomorphic to D_d or C_d. Now, in view of (6), the scheme $\mathcal{X}'$ defined by formula (5) is pseudocyclic (statement (2) of Lemma 3.3). Therefore statement (2) of the Main Theorem holds for $\mathcal{X}'$. Since $\mathcal{X}$ is an involutive fusion of $\mathcal{X}'$, we are done.

To complete the proof, it suffices to note that in the case (3) the scheme $\mathcal{X}$ is pseudocyclic by assumption (4), whereas in the case (4) the scheme $\mathcal{X}$ is either exceptional ($K = \mathrm{Alt}(4)$ or $\mathrm{Alt}(5)$), or an involutive fusion of the scheme (5) with $K' = \mathrm{Alt}(4)$ for $K = \mathrm{Sym}(4)$.

§5. Concluding remarks

In what follows, $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$ denote the classes of schemes in statements (1), (2), (3), and (4) of the Main Theorem, respectively.

5.1. Interrelation between the classes from the Main Theorem. In view of the remarks made after the Main Theorem, we are interested in the interrelation between the classes $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$. The schemes in $\mathcal{C}_1$ are imprimitive, whereas those in $\mathcal{C}_2$ and $\mathcal{C}_3$ are not. Therefore,

$$\mathcal{C}_1 \cap \mathcal{C}_2 = \mathcal{C}_1 \cap \mathcal{C}_3 = \varnothing.$$

The classes $\mathcal{C}_2$ and $\mathcal{C}_3$ have nontrivial intersection. This follows from the information on the orbit lengths of the groups $\mathrm{Alt}(4), \mathrm{Alt}(5) \le \mathrm{PGL}(2,p)$ obtained in [2, Lemmas 9,11]. Indeed, the exceptional schemes associated with groups $\mathrm{Alt}(4)$ and $\mathrm{Alt}(5)$ are primitive pseudocyclic if, e.g.,

$$p = -1 \pmod{a}, \quad a = 3 \text{ or } 5.$$

5.2. The automorphism groups. In principle, all the information of the automorphism group of the scheme $\mathcal{X}$ in the Main Theorem can be extracted from Lemma 4.1. In the most cases, we have

$$\mathrm{Aut}(\mathcal{X}) \le \mathrm{AGL}(2,p),$$

i.e., $\mathcal{X}$ is isomorphic to a normal Cayley scheme over $C_p \times C_p$ in the sense of [5]. Apart from this case, the only possibility for the group $\mathrm{Aut}(\mathcal{X})$ are the following:

$$\mathrm{Sym}(p) \times \mathrm{Sym}(p), \quad \mathrm{Sym}(p) \wr \mathrm{Sym}(p), \quad \mathrm{Sym}(p) \wr \mathrm{Sym}(2), \quad \mathrm{Sym}(p^2). \quad (7)$$

The first two groups appear in statements (ii1) and (ii3) of Lemma 4.1 and the schemes of these groups are in the class $\mathcal{C}_1$, whereas the second two groups appear in statement (i) and the schemes of these groups are the Hamming scheme $H(2,p)$ and trivial scheme lying in the classes $\mathcal{C}_4$ and $\mathcal{C}_2$, respectively.

5.3. Further research. The first natural problem is to generalize the Main Theorem to the p-powers q, i.e., to find a compact description of schurian fusions of a Galois affine plane of order q. In this way, one can still use the results of [2] where they were established arbitrary q. However, to the author knowledge, there is no generalization of Lemma 4.1.

The class $\mathcal{C}_2$ contains the cyclotomic schemes over near-fields of order p^2 [1] and the schemes of Frobenius groups. It would be interesting to find other schemes in $\mathcal{C}_2$ (if they are).

From the algorithmic point of view, one of the problem in the above context is how to recognize the schemes $\mathcal{X}$ from the Main Theorem in the class of all association schemes efficiently. Definitely, this can easily be done if $\mathrm{Aut}(\mathcal{X})$ is one of the groups (7). For the other schemes, the problem can efficiently be reduced to recognizing schemes belonging to the classes $\mathcal{C}_2$ and $\mathcal{C}_4$.

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